

The Loewner–Kufarev Energy and Foliations by Weil–Petersson Quasicircles

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Abstract

We study foliations by chord-arc Jordan curves of the twice punctured Riemann sphere $\mathbb{C} \setminus \{0\}$ using the Loewner–Kufarev equation. We associate to such a foliation a function on the plane that describes the “local winding” along each leaf. Our main theorem is that this function has finite Dirichlet energy if and only if the Loewner driving measure ρ has finite Loewner–Kufarev energy, defined by

$$S(\rho) = \frac{1}{2} \iint_{S^1 \times \mathbb{R}} \nu'_t(\theta)^2 d\theta dt$$

whenever ρ is of the form $\nu_t(\theta)^2 d\theta dt$, and set to ∞ otherwise. Moreover, if either of these two energies is finite they are equal up to a constant factor, and in this case, the foliation leaves are Weil–Petersson quasicircles.

This duality between energies has several consequences. The first is that the Loewner–Kufarev energy is reversible, that is, invariant under inversion and time-reversal of the foliation. Furthermore, the Loewner energy of a Jordan curve can be expressed using the minimal Loewner–Kufarev energy of those measures that generate the curve as a leaf. This provides a new and quantitative characterization of Weil–Petersson quasicircles. Finally, we consider conformal distortion of the foliation and show that the Loewner–Kufarev energy satisfies an exact transformation law involving the Schwarzian derivative. The proof of our main theorem uses an isometry between the Dirichlet energy space on the unit disc and $L^2(2\rho)$ that we construct using Hadamard’s variational formula expressed by means of the Loewner–Kufarev equation. Our results are related to κ -parameter duality and large deviations of Schramm-Loewner evolutions coupled with Gaussian random fields.

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1 Introduction

Methods involving monotone evolution of simply connected domains in the plane have been successfully applied to a range of problems in complex analysis and beyond. In recent years notably in the context of probabilistic lattice models, random conformal geometry, and physical growth processes [11, 33, 38, 52, 53, 59]. The Riemann mapping theorem and the notion of Carathéodory convergence allow to encode the dynamics by a continuous family of conformal maps from a reference domain, such as the unit disk \mathbb{D} , onto the evolving domains. Motivated by the Bieberbach conjecture, Loewner proved in 1923 [36] that the family of maps generated by progressively slitting a domain along a simple curve satisfies a differential equation with a real-valued control function. Loewner’s equation was ultimately instrumental in its resolution by de Branges about sixty years later [13]. This method was extended by Kufarev [31] and further developed by Pommerenke [44] to cover general evolution families beyond slitted domains. In this case, the dynamics is described by the Loewner–Kufarev equation which is controlled by a family of measures. It is a fundamental problem to relate geometric properties of the evolving domains to analytic properties of the family of driving measures.

In this paper, we study the Loewner–Kufarev equation driven by members of the class of measures having finite *Loewner–Kufarev energy*, a quantity that arises as a large deviation rate function for Schramm–Loewner evolutions, SLE_κ when $\kappa \rightarrow \infty$ [5]. We show that the Loewner–Kufarev energy can be considered dual to the Loewner energy of a single Jordan curve, a quantity that also appears in the context of large deviations for SLE_κ , but this time when $\kappa \rightarrow 0+$ [49, 63, 64]. (Our methods and conclusions are entirely deterministic, however, and no results from probability are needed in this paper.) More precisely, the boundaries of the evolving domains are Weil–Petersson quasicircles (equivalent to having finite Loewner energy) that foliate the twice punctured Riemann sphere $\mathbb{C} \setminus \{0\}$. We prove that Loewner–Kufarev energy, now viewed as attached to the foliation, equals the Dirichlet energy of a geometric object — the winding function.

This energy duality leads to a complete characterization of measures generating Weil–Petersson quasicircles. We are not aware of any other case where there is an equivalent description of the generated non-smooth Jordan curve interfaces in terms of the driving measure. As further consequences of energy duality, we will see that the induced foliation and its energy exhibit several remarkable features and symmetries.

Besides the introduction and analysis of this natural class of non-smooth foliations, and the implications to the theory of planar growth processes, another purpose of this paper is to provide large deviation counterparts to recent results from the theory of SLE processes and Liouville quantum gravity, including SLE duality and the mating-of-trees theorem. We will discuss links to random conformal geometry and open problems related to further connections at the end of the paper.

1.1 Loewner energy and Weil–Petersson quasicircles

Let γ be a Jordan curve in \mathbb{C} which separates 0 from ∞ and write D and D^* for the bounded and unbounded component, respectively, and set $\mathbb{D}^* = \mathbb{C} \setminus \overline{D}$. If $f : \mathbb{D} \rightarrow D$ and $h : \mathbb{D}^* \rightarrow D^*$ are conformal maps fixing 0 and ∞ respectively, then the Möbius invariant

Loewner energy of γ can be defined as

$$I^L(\gamma) = \mathcal{D}_{\mathbb{D}}(\log |f'|) + \mathcal{D}_{\mathbb{D}^*}(\log |h'|) + 4 \log |f'(0)/h'(\infty)|, \quad (1.1)$$

where

$$\mathcal{D}_D(u) = \frac{1}{\pi} \int_D |\nabla u|^2 dA$$

is the Dirichlet integral and $h'(\infty) = \lim_{z \rightarrow \infty} h'(z)$. Here and below we write dA for two-dimensional Lebesgue measure. The quantity I^L was originally introduced in a different form [49, 63], and the identity (1.1) was established in [64]¹. See also Section 10.

A Jordan curve γ has finite Loewner energy if and only if it belongs to the set of Weil–Petersson quasicircles [61], a class of non-smooth chord-arc Jordan curves that has a number of equivalent characterizations from various different perspectives, see, e.g., [8, 12, 28, 56, 57, 61, 65]. One way to characterize Weil–Petersson quasicircles is to say that their welding homeomorphisms $h^{-1} \circ f|_{S^1}$ (more precisely, the equivalence class modulo left-action by the group of Möbius transformations preserving S^1) belong to the Weil–Petersson Teichmüller space $T_0(1)$, defined as the completion of $\text{Möb}(S^1) \setminus \text{Diff}^\infty(S^1)$ using its unique homogeneous Kähler metric. Weil–Petersson Teichmüller space appears in several different contexts [9, 26, 41, 42, 47, 55, 67], and carries an infinite dimensional Kähler–Einstein manifold structure. In fact, the Loewner energy itself is a Kähler potential for the Weil–Petersson metric on $T_0(1)$, see [61] for definitions and a thorough discussion.

1.2 Loewner–Kufarev energy

Let \mathcal{N} be the set of Borel measures ρ on the cylinder $S^1 \times \mathbb{R}$ with the property that $\rho(S^1 \times I)$ equals $|I|$ for any interval I . Each $\rho \in \mathcal{N}$ can be disintegrated into a measurable family of probability measures $(\rho_t)_{t \in \mathbb{R}}$ on S^1 .

Let $(D_t)_{t \in \mathbb{R}}$ be a family of simply connected domains such that $0 \in D_t \subset D_s$ for all $s \leq t$. We assume that for each t , the conformal radius of D_t at 0 equals e^{-t} which implies $\bigcup_{t \in \mathbb{R}} D_t = \mathbb{C}$. Let $(f_t : \mathbb{D} \rightarrow D_t)_{t \in \mathbb{R}}$ be the associated family of conformal maps normalized so that $f_t(0) = 0$ and $f_t'(0) = e^{-t}$. (Here and below we write $'$ for ∂_z .) By a result of Pommerenke [44], there exists $\rho \in \mathcal{N}$ such that $(f_t)_{t \geq s}$ satisfies the *Loewner–Kufarev equation*

$$\partial_t f_t(z) = -z f_t'(z) H_t(z), \quad H_t(z) = \int_{S^1} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\rho_t(\theta).$$

The Herglotz integral H_t is holomorphic in \mathbb{D} with positive real part. The equation is interpreted in the sense that $t \mapsto f_t(z)$ is absolutely continuous. Conversely, for any $\rho \in \mathcal{N}$ the monotone family of simply connected domains $(D_t)_{t \in \mathbb{R}}$ and the corresponding family of conformal maps $(f_t)_{t \in \mathbb{R}}$ can be recovered via the Loewner–Kufarev equation. We say that $(f_t)_{t \in \mathbb{R}}$ is the *Loewner chain* driven by ρ . It is sometimes convenient to work

¹The right-hand side of (1.1) was introduced in [61] and there referred to as the universal Liouville action (up to a constant factor π). Since we will discuss connections to random conformal geometry, we choose to use the term Loewner energy.

with the family of inverse maps $(g_t = f_t^{-1} : D_t \rightarrow \mathbb{D})_{t \in \mathbb{R}}$, referred to as the *uniformizing Loewner chain*. See Sections 2.2 and 7.1 for more details.

In order to provide intuition, let us discuss a few examples.

- The classical definition of Loewner chain [36] is the one generated by a continuous *driving function* $t \mapsto W_t \in S^1$. This corresponds to ρ_t being the Dirac measure at W_t . When $t \mapsto W_t$ is sufficiently regular, the complement of $\bigcap_{t \in \mathbb{R}} D_t$ is a simple curve in \mathbb{C} connecting ∞ with 0. The Schramm-Loewner evolution, SLE_κ , is the Loewner chain obtained when the driving function is chosen as $W_t = e^{i\sqrt{\kappa}B_t}$ where (B_t) is one-dimensional Brownian motion on the real line.
- If $\rho_t = (2\pi)^{-1}d\theta$ for all $t \in \mathbb{R}$, where $d\theta$ denotes the Lebesgue measure on S^1 , then $(D_t = e^{-t}\mathbb{D})_{t \in \mathbb{R}}$ is the family of concentric disks centered at 0.
- If $\rho \ll dt \times d\theta$ with smooth density, then writing ρ_t as $\rho_t(\theta)d\theta$, we have that the normal velocity (in the direction into D_t) of the interface ∂D_t at the point $f_t(e^{i\theta})$ equals $2\pi|f_t'(e^{i\theta})|\rho_t(\theta)$. See (5.3).
- (Becker's condition) If there exists $k \in [0, 1)$ such that H_t takes values in the same compact set $\{z \in \mathbb{C} : |z - 1| \leq k|z + 1|\}$ for a.e. $t \geq 0$ and (say) ρ_t is uniform for all $t < 0$, then the interfaces $(\partial D_t)_{t \geq 0}$ are all k -quasicircles [7]. However, not all k -quasicircles can be generated in this way [24, Thm. 3].

For $\rho \in \mathcal{N}$, the *Loewner–Kufarev energy* $S(\rho)$ introduced in [5] is defined by

$$S(\rho) = \int_{\mathbb{R}} L(\rho_t) dt, \quad \text{where} \quad L(\rho_t) = \frac{1}{2} \int_{S^1} \nu_t'(\theta)^2 d\theta,$$

whenever $d\rho_t(\theta) = \nu_t^2(\theta)d\theta$ and $L(\rho_t)$ is set to ∞ otherwise. Notice that for fixed t , $L(\rho_t)$ is the Dirichlet energy of ν_t on S^1 , which is finite for a relatively regular measure. We refer to measures with finite Loewner–Kufarev energy simply as *finite energy measures*. We will discuss how the Loewner–Kufarev energy relates to conformally invariant random systems in Section 10.

1.3 Main results

We call a monotone and continuous² family $(\gamma_t)_{t \in \mathbb{R}}$ of chord-arc Jordan curves whose union covers $\mathbb{C} \setminus \{0\}$ (or $\mathbb{D} \setminus \{0\}$) a *foliation*, see Section 2.3. Individual curves in a foliation are called *leaves*. Our definition allows more than one leaf passing through a given point $z \in \mathbb{C} \setminus \{0\}$. If the family of interfaces $(\gamma_t = \partial D_t)_{t \in \mathbb{R}}$ arising from the Loewner–Kufarev equation driven by the measure ρ form a foliation, we say that ρ *generates a foliation*.

Theorem 1.1 (Weil–Petersson leaves). *If $S(\rho) < \infty$, then ρ generates a foliation of $\mathbb{C} \setminus \{0\}$ in which every leaf is a Weil–Petersson quasicircle.*

See Corollary 5.12 and Section 7.2. Theorem 1.1 shows that any ρ with $S(\rho) < \infty$ gives rise to a dynamical process in $T_0(1)$, see Section 10. Moreover, we can view $S(\rho)$ as the

²Monotone is meant in the following sense: if $s < t$ then $D_t \subset D_s$ where D_t is the bounded connected component of $\mathbb{C} \setminus \gamma_t$. Continuity is understood with respect to the supremum norm modulo increasing reparametrization.

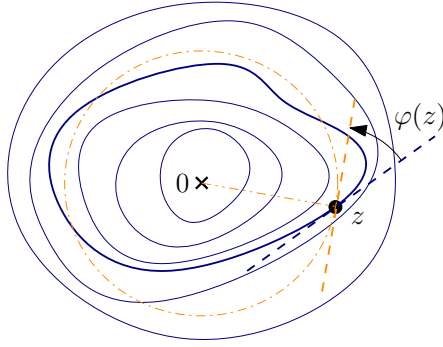


Figure 1.1: Illustration of the winding function φ .

energy of the generated foliation. Later in Theorem 1.4 we prove that all Weil–Peterson quasicircles can be generated by a finite energy measure after proper normalization.

The next result is our main theorem. It shows that the foliation energy can, in fact, be expressed using a geometrically defined quantity, without reference to the Loewner chain.

To state the result, we associate to a foliation $(\gamma_t)_{t \in \mathbb{R}}$ a real-valued function φ as follows. Given the leaf γ_t , let g_t be the conformal map of the bounded component of $\hat{\mathbb{C}} \setminus \gamma_t$ onto \mathbb{D} , fixing 0 and with positive derivative there. Given z at which γ_t has a tangent, we define $\varphi(z)$ to be the non-tangential limit at z of the function $w \mapsto \arg[wg'_t(w)/g_t(w)]$. (We choose the continuous branch that equals 0 at 0.) This defines φ arclength-a.e. on γ_t . Monotonicity of $(\gamma_t)_{t \in \mathbb{R}}$ implies that there is no ambiguity in the definition of $\varphi(z)$ if $z \in \gamma_t \cap \gamma_s$. See Section 2.3 for more details. Modulo 2π , $\varphi(z)$ equals the difference of the argument of the tangent of γ_t at z and that of the tangent to the circle centered at 0 passing through z . See Figure 1.1. We call φ the *winding function* associated with a foliation $(\gamma_t)_{t \in \mathbb{R}}$. The simplest example of winding function is when the measure ρ has zero energy, namely when ρ_t is the uniform probability measure on S^1 for a.e. $t \in \mathbb{R}$. In this case, the associated foliation is the family of concentric circles centered at 0, and the winding function is identically 0. We discuss additional examples in Section 4.3.

In the present non-smooth foliation setting, a function defined on each leaf arclength-a.e. is not necessarily defined Lebesgue-a.e. in \mathbb{C} , see Section 2.3. However, we prove that if it is possible to extend the winding function φ to an element in $W_{\text{loc}}^{1,2}$, then the extension is unique. (This is one main reason why we choose to work in the setting of chord-arc curves.) See Proposition 3.6. Statements about the Dirichlet energy of φ will be understood in terms of this extension whose existence is implicitly part of any such statement.

Theorem 1.2 (Energy duality). *Assume that $\rho \in \mathcal{N}$ generates a foliation and let φ be the associated winding function on \mathbb{C} . Then $\mathcal{D}_{\mathbb{C}}(\varphi) < \infty$ if and only if $S(\rho) < \infty$ and*

$$\mathcal{D}_{\mathbb{C}}(\varphi) = 16 S(\rho).$$

The proof of Theorem 1.2 is completed in Section 7.2.

Remark. The factor 16 in Theorem 1.2 is consistent with the SLE duality relation $\kappa \leftrightarrow 16/\kappa$ [17, 68]. See Section 10.

1.4 Consequences of energy duality

Theorem 1.2 has several implications. The first is the reversibility of the Loewner–Kufarev energy. Consider $\rho \in \mathcal{N}$ and the corresponding evolution family of domains $(D_t)_{t \in \mathbb{R}}$. Applying $z \mapsto 1/z$ to the complementary domains $\hat{\mathbb{C}} \setminus D_t$, we obtain an evolution family of domains $(\tilde{D}_t)_{t \in \mathbb{R}}$ upon time-reversal and reparametrization, which may be described by the Loewner–Kufarev equation with an associated driving measure $\tilde{\rho}$. While there is no known simple description of $\tilde{\rho}$ in terms of ρ , energy duality implies remarkably that the Loewner–Kufarev energy is invariant under this transformation.

Theorem 1.3 (Energy reversibility). *We have $S(\rho) = S(\tilde{\rho})$.*

See Theorem 8.2. Notice that although the Loewner–Kufarev energy is preserved, the reversal transformation redistributes the energy in a highly non-local manner: for instance, suppose that ρ_t is the uniform measure when $t \notin [0, 1]$. Then $L(\rho_t)$ is zero when $t \notin [0, 1]$, yet it is not hard to see that the corresponding reversed foliation may have non-zero $L(\tilde{\rho}_t)$ for a.e. $t \leq 0$ if $D_1 \neq e^{-1}\mathbb{D}$.

By Theorem 1.1, every leaf in the foliation generated by a measure with $S(\rho) < \infty$ satisfies $I^L(\gamma) < \infty$. Conversely, the next result shows that if $I^L(\gamma) < \infty$, then γ can always be realized as a leaf in a foliation generated by Loewner evolution driven by a measure with $S(\rho) < \infty$ and $I^L(\gamma)$ is controlled by $S(\rho)$. We obtain a new and quantitative characterization of Weil–Petersson quasicircles.

Theorem 1.4 (Characterization of Weil–Petersson quasicircles). *A Jordan curve γ separating 0 from ∞ is a Weil–Petersson quasicircle if and only if γ can be realized as a leaf in the foliation generated by a measure ρ with $S(\rho) < \infty$. Moreover, the Loewner energy of γ satisfies the identity*

$$I^L(\gamma) = 16 \inf_{\rho} S(\rho) + 2 \log |f'(0)/h'(\infty)|,$$

where the infimum, which is attained, is taken over all $\rho \in \mathcal{N}$ such that γ is a leaf of the generated foliation. The minimal energy foliation is the one obtained by equipotentials on both sides of γ .

See Theorem 8.6 and Corollary 8.8. By equipotential we mean the image of a circle centered at 0 under the Riemann map from \mathbb{D} to a component of $\mathbb{C} \setminus \gamma$ fixing 0 or taking 0 to ∞ . In this case, the winding function is harmonic in $\mathbb{C} \setminus \gamma$, see Section 8.2. This minimum is zero if and only if γ is a circle centered at 0, whereas $I^L(\gamma)$ is zero for all circles. This explains the presence of the derivative terms. Corollary 8.9 of Theorem 1.4 shows that the Loewner energies of the leaves in a foliation generated by ρ with $S(\rho) < \infty$ are uniformly bounded by $16 S(\rho)$.

We also prove the following identity that simultaneously expresses the interplay between Dirichlet energies under “welding” and “flow-line” operations along a *bounded* curve in a similar spirit as [62, Cor. 1.6]. In our setup, given a chord-arc curve γ separating 0 and ∞ , we define a winding function on γ arclength-a.e. as above. We say that a Weil–Petersson quasicircle γ is *compatible* with $\varphi \in W_{\text{loc}}^{1,2}$, if the winding function of γ coincides with the trace $\varphi|_{\gamma}$ arclength-a.e.

Proposition 1.5 (Complex identity). *Let ψ be a complex valued function on \mathbb{C} with $\mathcal{D}_{\mathbb{C}}(\psi) = \mathcal{D}_{\mathbb{C}}(\operatorname{Re} \psi) + \mathcal{D}_{\mathbb{C}}(\operatorname{Im} \psi) < \infty$ and γ a Weil–Petersson quasicycle separating 0 from ∞ compatible with $\operatorname{Im} \psi$. Let*

$$\zeta(z) := \psi \circ f(z) + \log \frac{f'(z)z}{f(z)}; \quad \xi(z) := \psi \circ h(z) + \log \frac{h'(z)z}{h(z)}. \quad (1.2)$$

Then $\mathcal{D}_{\mathbb{C}}(\psi) = \mathcal{D}_{\mathbb{D}}(\zeta) + \mathcal{D}_{\mathbb{D}^*}(\xi)$.

See Section 8.3.

Finally, we study the change of the Loewner–Kufarev energy under conformal transformation of the foliation. We consider the following setup. Let $\rho \in \mathcal{N}$ be a measure such that ρ_t is the uniform measure for $t < 0$ and

$$S_{[0,1]}(\rho) := \int_0^1 L(\rho_t) dt < \infty.$$

(The choice of the upper bound $T = 1$ is only for notational simplicity and the result is easily generalized to other bounded time intervals.)

We have $D_0 = \mathbb{D}$ and write $K_t = \overline{\mathbb{D}} \setminus D_t$. Let ψ be a conformal map from a neighborhood U of K_1 in $\overline{\mathbb{D}}$ to a neighborhood \tilde{U} of \tilde{K}_1 , another compact hull in \mathbb{D} , such that $\psi(K_1) = \tilde{K}_1$. The family of compact hulls $(\tilde{K}_t := \psi(K_t))_{t \in [0,1]}$ can be generated by a measure $\tilde{\rho}$ with the associated uniformizing Loewner chain $\tilde{g}_t : \mathbb{D} \setminus \tilde{K}_t \rightarrow \mathbb{D}$.

Theorem 1.6 (Conformal distortion). *We have the formulas*

$$L(\tilde{\rho}_t) - L(\rho_t) = \frac{1}{4} \int_{S^1} e^{2i\theta} \mathcal{S}\psi_t(e^{i\theta}) \rho_t(\theta) d\theta + \frac{1}{8} (|\tilde{\rho}_t| - |\rho_t|)$$

and

$$S_{[0,1]}(\tilde{\rho}) - S_{[0,1]}(\rho) = \frac{1}{4} \int_0^1 \int_{S^1} e^{2i\theta} \mathcal{S}\psi_t(e^{i\theta}) \rho_t(\theta) d\theta dt + \frac{1}{8} (\log \tilde{g}'_1(0) - \log g'_1(0)),$$

where $\psi_t = \tilde{g}_t \circ \psi \circ g_t^{-1}$, $\mathcal{S}\psi = (\psi''/\psi')' - (\psi''/\psi')^2/2$ is the Schwarzian derivative, and $|\tilde{\rho}_t| = \tilde{\rho}_t(S^1)$.

The proof is given in Section 9. Theorem 1.6 is related in spirit to conformal restriction formulas [34]. We will connect the expressions in Theorem 1.6 to Brownian loop measures (see [34, 35]) in a forthcoming paper with Lawler.

1.5 Core argument for energy duality (Theorem 1.2)

We first derive a version of Theorem 1.2 for the unit disk. Let \mathcal{N}_+ be defined analogously to \mathcal{N} but considering measures on $S^1 \times \mathbb{R}_+$ and we define the corresponding Loewner–Kufarev energy by $S_+(\rho) = \int_0^\infty L(\rho_t) dt$. The Loewner–Kufarev equation with the initial condition $f_0(z) = z$ generates a Loewner chain $(f_t : \mathbb{D} \rightarrow D_t)_{t \geq 0}$, where $(D_t)_{t \geq 0}$ is a monotone family of simply connected domains in $\mathbb{D} = D_0$. It can be viewed as a special case of the whole-plane Loewner chain by extending the measure ρ to a measure on $S^1 \times \mathbb{R}$, where ρ_t is the uniform probability measure on S^1 when $t < 0$, which implies $D_t = e^{-t}\mathbb{D}$ for all $t \leq 0$.

Theorem 1.7 (Disk energy duality). *Assume that $\rho \in \mathcal{N}_+$ generates a foliation of $\mathbb{D} \setminus \{0\}$ and let φ be the associated winding function. Then $\mathcal{D}_{\mathbb{D}}(\varphi) < \infty$ if and only if $S_+(\rho) < \infty$ and $\mathcal{D}_{\mathbb{D}}(\varphi) = 16 S_+(\rho)$.*

The proof of Theorem 1.7 is completed in Section 6.

The starting point of the proof is Hadamard’s classical formula for the variation of the Dirichlet Green’s function. We express Hadamard’s formula using the Loewner–Kufarev equation:

$$-\partial_t G_{D_t}(z, w) = \int_{S^1} P_{\mathbb{D}}(g_t(z), e^{i\theta}) P_{\mathbb{D}}(g_t(w), e^{i\theta}) d\rho_t(\theta),$$

where $P_{\mathbb{D}}$ is the Poisson kernel for \mathbb{D} and $\rho \in \mathcal{N}_+$, see Lemma 3.1 and also [51, Thm. 4] for a similar result with additional smoothness assumptions. The Sobolev space $\mathcal{E}_0(\mathbb{D}) = W_0^{1,2}(\mathbb{D})$ is a Hilbert space when endowed with the Dirichlet inner product. Hadamard’s formula and the orthogonal decomposition with respect to the Dirichlet inner product along the Loewner evolution lead to a correspondence between $\mathcal{E}_0(\mathbb{D})$ and L^2 -integrable functions on the cylinder $S^1 \times \mathbb{R}_+$. More precisely, setting $L^2(2\rho) := L^2(S^1 \times \mathbb{R}_+, 2\rho)$, we define an operator

$$\iota : C_c^\infty(\mathbb{D}) \rightarrow L^2(2\rho), \quad \phi \mapsto \frac{1}{2\pi} \int_{\mathbb{D}} \Delta(\phi \circ f_t)(z) P_{\mathbb{D}}(z, e^{i\theta}) dA(z). \quad (1.3)$$

We prove the following theorem which is an important step in the proof of our main result and which we also believe to be of independent interest.

Theorem 1.8 (Foliation disintegration isometry). *If $\rho \in \mathcal{N}_+$ generates a foliation of $\mathbb{D} \setminus \{0\}$, then (1.3) extends to a bijective isometry $\iota : \mathcal{E}_0(\mathbb{D}) \rightarrow L^2(2\rho)$ with the inverse mapping $\varkappa : L^2(2\rho) \rightarrow \mathcal{E}_0(\mathbb{D})$*

$$\varkappa[u](w) = 2\pi \int_0^{\tau(w)} P_{\mathbb{D}}[u_t \rho_t](g_t(w)) dt, \quad u_t(\cdot) := u(\cdot, t).$$

See Lemma 3.2, Proposition 3.3, and Theorem 3.5. We show that $P_{\mathbb{D}}[u_t \rho_t]$ is in the harmonic Hardy space \mathfrak{h}^1 on \mathbb{D} , and Theorem 1.8 can be interpreted as a disintegration of finite Dirichlet energy functions into \mathfrak{h}^1 functions. This implies the formula

$$\phi_t^0 \circ g_t = \varkappa[\iota[\phi] \mathbf{1}_{S^1 \times [t, \infty)}] \quad (1.4)$$

where $\phi_t^0 \in \mathcal{E}_0(\mathbb{D})$ is the zero-trace part of the function $\phi_t = \phi \circ f_t$ (so that $\phi_t - \phi_t^0$ is harmonic). See Corollary 3.4.

Remark. If ρ is smooth and t is fixed, the function $\theta \mapsto \iota[\phi](\theta, t)$ can be interpreted as the inward pointing normal derivative in \mathbb{D} at $e^{i\theta}$ applied to ϕ_t^0 . The foliation disintegration isometry is closely related to the “Hadamard operators” considered in [25] in a C^2 -smooth and strictly monotone setting, see Section 3 for further discussion. Here, we consider chord-arc foliations which in general have leaves that are not C^1 , and are not even locally Lipschitz graphs. Moreover, $t \mapsto \gamma_t$ is only continuous and monotone but not strictly monotone. Theorem 1.8 allows us to work under weak regularity assumptions and this level of generality is needed in order to include Weil–Peterson quasicircles and to obtain the equivalence in Theorem 1.2 and Theorem 1.7.

Remark. By considering the Gaussian measures associated to the Hilbert spaces $\mathcal{E}_0(\mathbb{D})$ and $L^2(2\rho)$, Theorem 1.8 immediately entails a decomposition of the Dirichlet Gaussian free field on \mathbb{D} into white-noise on the cylinder $S^1 \times \mathbb{R}_+$ weighted by 2ρ , generalizing the main result of [25].

The next step is to prove that $\rho \in \mathcal{N}_+$ with $S_+(\rho) < \infty$ generates a foliation so that Theorem 1.8 can be applied. For this, we first derive a preliminary, weak energy duality result: under the assumption that ρ is piecewise constant in time and ρ_t is strictly positive and smooth, we prove in Proposition 5.3 that the winding function is defined, continuous and piecewise smooth in $\overline{\mathbb{D}}$. By essentially explicit computation, the identity $16S_+(\rho) = \mathcal{D}_{\mathbb{D}}(\varphi)$ follows. This result combined with an approximation argument is then used to prove that ρ with $S_+(\rho) < \infty$ generates a foliation by Weil–Petersson quasicircles, see Corollary 5.12. At this point the proof of Theorem 1.1 is completed in the case when $\rho \in \mathcal{N}_+$. Then in Section 6 we prove disk energy duality in full generality. From the work in Section 5 we know that ρ with $S_+(\rho) < \infty$ generates a foliation. Using the inverse operator \varkappa we prove that the winding function $\varphi = -\varkappa[2\nu'_t/\nu_t]$ from which it follows immediately that $\varphi \in \mathcal{E}_0(\mathbb{D})$ and that $\mathcal{D}_{\mathbb{D}}(\varphi) = 16S_+(\rho)$.

The final step assumes that $\rho \in \mathcal{N}_+$ generates a foliation whose winding function can be extended so that $\mathcal{D}_{\mathbb{D}}(\varphi) < \infty$, and we want to show that $S_+(\rho) < \infty$. To point to the difficulty, at this stage we do not know that ρ_t is absolutely continuous. However, applying Theorem 1.8, we have that $\iota[\varphi] \in L^2(2\rho)$. Using the integrability information we deduce that H'_t is in the Hardy space \mathcal{H}^1 and this implies that ρ_t is absolutely continuous and that the density is differentiable a.e. It then follows that $S_+(\rho) < \infty$ using Theorem 1.8.

Structure of the paper. We recall basic definitions in Section 2. The foliation disintegration isometry, Theorem 1.8, is established in Section 3. In Section 4 we define the Loewner–Kufarev energy, derive a few basic properties, and discuss examples of finite energy measures and associated foliations. Section 5 is devoted to the proof of energy duality in the disk under strong regularity assumptions and here we obtain the important *a priori* result that finite energy measures generate foliations whose leaves are Weil–Petersson quasicircles, see Corollary 5.12. The general energy duality result for the unit disk, Theorem 1.7, is proved in Section 6. We work in the set-up of the Loewner–Kufarev equation in the unit disk in Sections 2-6 and we introduce the whole-plane Loewner–Kufarev equation only in Section 7, where we prove whole-plane energy duality, see Theorems 1.1 and 1.2. In Section 8 we derive the consequences of energy duality, see Theorem 1.3, 1.4, and Proposition 1.5. Section 9 is devoted to the proof of the conformal distortion formula, Theorem 1.6. Section 10 collects further comments, including a discussion of additional interpretations of our results as well as open problems.

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compared to our previous argument. We thank David Jerison and Greg Lawler for discussions and helpful input, Paul Laurain and Andrea Seppi for discussions on the implication to minimal surfaces, Yan Luo, Curt McMullen, Steffen Rohde, and Wendelin Werner for useful comments on an earlier version of our paper. F.V. acknowledges support by the Knut and Alice Wallenberg Foundation, the Swedish Research Council, and the Ruth and Nils-Erik Stenbäck Foundation. Y.W. acknowledges support by NSF grant DMS-1953945.

2 Preliminaries

2.1 Basic definitions

For a bounded simply connected domain $D \subset \mathbb{C}$, we write $G_D(z, w)$ for the Green's function associated to the positive Laplacian $\Delta = -(\partial_{xx} + \partial_{yy})$ with Dirichlet boundary condition, where $z = x + iy$. It is convenient to normalize the Green's function so that $G_D(z, w) - \log|z - w|$ is harmonic in each variable, that is, so that $G_D = 2\pi\Delta^{-1}$.

For sufficiently regular domains D , the Poisson kernel is defined for $z \in D, \zeta \in \partial D$, by

$$P_D(z, \zeta) := \partial_{n(\zeta)} G_D(z, \zeta),$$

where $\partial_{n(\zeta)}$ is the inward normal derivative at ζ . If $D = \mathbb{D}$ we have

$$G_{\mathbb{D}}(z, w) = -\log \left| \frac{z - w}{1 - \bar{w}z} \right|, \quad P_{\mathbb{D}}(z, e^{i\theta}) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2}.$$

If σ is a finite measure on the unit circle S^1 , identified with $[0, 2\pi]_{/0 \sim 2\pi}$, we write

$$P_{\mathbb{D}}[\sigma](z) = \frac{1}{2\pi} \int_0^{2\pi} P_{\mathbb{D}}(z, e^{i\theta}) d\sigma(\theta)$$

for its Poisson integral. If $d\sigma = u d\theta$ for $u \in L^1(S^1)$ we write simply $P_{\mathbb{D}}[u](z)$. By Fatou's theorem, $P_{\mathbb{D}}[u]$ has non-tangential limit u Lebesgue-a.e. on S^1 .

If D is an open set, the Dirichlet energy of an almost everywhere defined function $\phi : D \rightarrow \mathbb{R}$ is given by

$$\mathcal{D}_D(\phi) := \langle \phi, \phi \rangle_{\nabla} := \frac{1}{\pi} \int_D |\nabla \phi|^2 dA(z)$$

whenever ϕ has distributional first derivatives in $L^2(D)$. Let $W^{1,2}(D)$ be the Sobolev space of real-valued functions ϕ such that both ϕ and its weak first derivatives are in $L^2(D)$ with norm $\|\phi\|_{W^{1,2}(D)} = \|\phi\|_{L^2(D)} + \|\nabla \phi\|_{L^2(D)}$ and write $W_0^{1,2}(D)$ for the closure of smooth and compactly supported functions in D , $C_c^\infty(D)$, in $W^{1,2}(D)$.

When D is bounded, $\mathcal{D}_D^{1/2}$ is a norm equivalent to $\|\cdot\|_{W^{1,2}}$ on $W_0^{1,2}(D)$ by the Poincaré inequality. In this case, we write $\mathcal{E}_0(D)$ for $W_0^{1,2}(D)$ equipped with the norm $\mathcal{D}_D^{1/2}$ which is a Hilbert space endowed with the Dirichlet inner product $\langle \cdot, \cdot \rangle_{\nabla}$. Every ϕ with $\mathcal{D}_{\mathbb{C}}(\phi) < \infty$ has a unique decomposition with respect to D

$$\phi = \phi^0 + \phi^h \quad \text{and} \quad \langle \phi^0, \phi^h \rangle_{\nabla} = 0 \tag{2.1}$$

such that $\phi^0 \in \mathcal{E}_0(D)$ and ϕ^h with $\mathcal{D}_{\mathbb{C}}(\phi^h) < \infty$ is harmonic in D , see, e.g., [1, P. 77] and [62, Appx. B].

The Hardy space \mathcal{H}^p (resp. harmonic Hardy space \mathfrak{h}^p) for $p > 0$ consists of functions f holomorphic (resp. harmonic) in \mathbb{D} satisfying

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Let γ be a Jordan curve in \mathbb{C} , and write D and D^* for the connected components of $\mathbb{C} \setminus \gamma$. Let f be a conformal map from \mathbb{D} onto the bounded component D , and h a conformal map from \mathbb{D}^* onto the unbounded component D^* fixing ∞ . The Loewner energy of γ is defined as

$$I^L(\gamma) = \mathcal{D}_{\mathbb{D}}(\log |f'|) + \mathcal{D}_{\mathbb{D}^*}(\log |h'|) + 4 \log |f'(0)| - 4 \log |h'(\infty)|, \quad (2.2)$$

where $h'(\infty) := \lim_{z \rightarrow \infty} h'(z) = \tilde{h}'(0)^{-1}$ and $\tilde{h}(z) := 1/h(1/z)$. The Loewner energy I^L is finite if and only if either Dirichlet integral on the right-hand side in (2.2) is finite. We summarize this as a lemma.

Lemma 2.1 (See [61, Thm. II.1.12]). *Suppose γ is a bounded Jordan curve. Then the following statements are equivalent:*

1. $I^L(\gamma) < \infty$;
2. $\mathcal{D}_{\mathbb{D}}(\log |f'|) = \frac{1}{\pi} \int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 dA(z) < \infty$;
3. $\mathcal{D}_{\mathbb{D}^*}(\log |h'|) < \infty$.

There are several additional equivalent ways to define the Loewner energy, and the class of Jordan curves with finite Loewner energy coincides with the class of Weil–Petersson quasicircles, see [64]. Henceforth we will refer to Jordan curves with finite Loewner energy as Weil–Petersson quasicircles.

2.2 The Loewner–Kufarev equation

We consider first the version for the unit disc and then later in Section 7 the whole-plane version. Let $\mathcal{M}(\Omega)$ (resp. $\mathcal{M}_1(\Omega)$) be the space of Borel measures (resp. probability measures) on Ω endowed with the topology induced by weak convergence on compact subsets. Let

$$\mathcal{N}_+ = \{\rho \in \mathcal{M}(S^1 \times \mathbb{R}_+) : \rho(S^1 \times I) = |I| \text{ for any interval } I\}.$$

Any $\rho \in \mathcal{N}_+$ can be disintegrated into a family of measures $(\rho_t)_{t \geq 0}$ measurable in t , such that $\rho(d\theta dt) = \rho_t(d\theta)dt$ and $\rho_t \in \mathcal{M}_1(S^1)$. The disintegration is unique in the sense that any two disintegrations $(\rho_t), (\tilde{\rho}_t)$ of ρ must satisfy $\rho_t = \tilde{\rho}_t$ for a.e. t . See, e.g., [14, Sec. 70]. We will work with the Loewner–Kufarev equation driven by measures $\rho \in \mathcal{N}_+$ and we now review some of the relevant definitions and facts. See, e.g., [50, Ch. 8] for proofs and a detailed discussion in this general setting.

Given $\rho \in \mathcal{N}_+$, consider the associated Herglotz integrals

$$H_t(z) := H[\rho_t](z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\rho_t(\theta). \quad (2.3)$$

The mapping $z \mapsto H_t(z)$ is a holomorphic function in \mathbb{D} with positive real part and $t \mapsto H_t(z)$ is measurable. In this setting, the Loewner–Kufarev partial differential equation for \mathbb{D} reads

$$\partial_t f_t(z) = -z f_t'(z) H_t(z), \quad f_0(z) = z. \quad (2.4)$$

The equation (2.4) is understood in the sense that $t \mapsto f_t(z)$ is absolutely continuous, and (2.4) is required to hold for a.e. $t \geq 0$. The exceptional set can be taken to be the same for all $z \in \mathbb{D}$. Throughout the paper, all differential equations are interpreted in a similar manner. The unique solution to (2.4) gives rise to a family of conformal maps $(f_t)_{t \geq 0}$ fixing 0 such that for each t , $f_t : \mathbb{D} \rightarrow D_t = f_t(\mathbb{D})$ where if $s < t$, then $D_t \subset D_s$. In the present case where the disintegrated measures are probability measures we have $f_t'(0) = e^{-t}$. (Indeed, $\partial_t \log f_t'(0) = -H_t(0) = -|\rho_t| \equiv -1$.) We refer to the family $(f_t)_{t \geq 0}$ as the *Loewner chain* driven by ρ . We call the family of compact sets $(K_t = \overline{\mathbb{D}} \setminus D_t)_{t \geq 0}$ the *hulls* associated to the Loewner chain.

A converse statement is also true. Consider a monotone family of simply connected domains $(D_t)_{t \geq 0}$ containing 0: $D_0 = \mathbb{D}$, $0 \in D_t$ for all t , and if $s < t$ then $D_t \subset D_s$. Let $f_t : \mathbb{D} \rightarrow D_t$ be the conformal map normalized by $f_t(0) = 0$, $f_t'(0) > 0$. According to a theorem of Pommerenke [44, Satz 4] (see also [45, Thm. 6.2] and [50]), if $t \mapsto f_t'(0)$ is a decreasing homeomorphism of $\mathbb{R}_+ \rightarrow (0, 1]$, one can reparametrize $(D_t)_{t \geq 0}$ so that $f_t'(0) = e^{-t}$, and there exists a measurable family of holomorphic functions $(H_t)_{t \geq 0}$ in \mathbb{D} , uniquely defined for a.e. $t \geq 0$, with positive real part such that (2.5) holds. The measurable family of probability measures $(\rho_t)_{t \geq 0}$ is obtained from the Herglotz-Riesz representation of $(H_t)_{t \geq 0}$.

The upshot of the discussion is that the measure $\rho \in \mathcal{N}_+$, Loewner chain $(f_t)_{t \geq 0}$, *uniformizing Loewner chain* $(g_t := f_t^{-1} : D_t \rightarrow \mathbb{D})_{t \geq 0}$, the monotone family of simply connected domains $(D_t)_{t \geq 0}$, and the increasing family of hulls $(K_t)_{t \geq 0}$ each determine the others.

Let \mathcal{L}_+ be the set of Loewner chains $(f_t)_{t \geq 0}$, and change notation by writing $f(z, t) = f_t(z)$. We endow \mathcal{L}_+ with the topology of uniform convergence of f on compact subsets of $\mathbb{D} \times \mathbb{R}_+$. In this setting, the continuity of the Loewner transform is well-known.

Lemma 2.2 (See e.g., [38, Prop. 6.1] and [29]). *The Loewner transform $\rho \in \mathcal{N}_+ \mapsto f \in \mathcal{L}_+$ is a homeomorphism.*

The uniformizing Loewner chain $(g_t = f_t^{-1} : D_t \rightarrow \mathbb{D})_{t \geq 0}$ driven by ρ satisfies a similar equation

$$\partial_t g_t(z) = g_t(z) H_t(g_t(z)), \quad g_0(z) = z, \quad (2.5)$$

for a.e. $t < \tau(z)$ where for $z \in \mathbb{D}$,

$$\begin{aligned} \tau(z) &= \sup\{t \geq 0, \text{ the solution of (2.5) exists for all } s < t\} \\ &= \sup\{t \geq 0 : z \in D_t\}, \end{aligned} \quad (2.6)$$

and we set by convention $\tau(z) = 0$ for $z \in S^1$. Further, we can write

$$D_t = \{z \in \mathbb{D}, \tau(z) > t\}.$$

An important property which follows immediately from (2.5) is the *domain Markov property* of the Loewner transform.

Lemma 2.3. *Let $\rho \in \mathcal{N}_+$ and $(g_t)_{t \geq 0}$ the associated uniformizing Loewner chain. Fix $T \geq 0$, then $(\tilde{g}_s := g_{s+T} \circ g_T^{-1})_{s \geq 0}$ is the uniformizing Loewner chain driven by the measure $\tilde{\rho} \in \mathcal{N}_+$ with disintegration $\tilde{\rho}_s = \rho_{s+T}$.*

2.3 Chord-arc foliations and the winding function

Let γ be a rectifiable Jordan curve. We say that γ is chord-arc if there exists a constant $A < \infty$ such that for all $z, w \in \gamma$, we have the inequality $|\gamma^{z,w}| \leq A|z - w|$, where $\gamma^{z,w}$ is the subarc of $\gamma \setminus \{z, w\}$ of smaller length.

Consider a monotone family of simply connected domains $(D_t)_{t \geq 0}$ that can be described by Loewner evolution as in Section 2.2. We call the family $(\gamma_t := \partial D_t)_{t \geq 0}$ a *non-injective chord-arc foliation* of $\overline{\mathbb{D}} \setminus \{0\}$ if

1. For all $t \geq 0$, γ_t is a chord-arc Jordan curve.
2. It is possible to parametrize each curve $\gamma_t, t \geq 0$, by S^1 so that the mapping $t \mapsto \gamma_t$ is continuous in the supremum norm.
3. For all $z \in \overline{\mathbb{D}} \setminus \{0\}$, $\tau(z) < \infty$, where τ is defined in (2.6).

For convenience we shall say simply foliation in what follows, but we stress that the chord-arc assumption is always in effect. We refer to the Jordan curves γ_t as *leaves*. Non-injective here means that we do not require that there is a unique leaf that passes through each point in $\overline{\mathbb{D}}$. The following lemma shows that a foliation in the above sense indeed foliates the punctured unit disk.

Lemma 2.4. *Assume that $(\gamma_t)_{t \geq 0}$ is a foliation of $\overline{\mathbb{D}} \setminus \{0\}$. Then for all $z \in \overline{\mathbb{D}} \setminus \{0\}$, we have $z \in \gamma_{\tau(z)}$. In particular, $\bigcup_{t \geq 0} \gamma_t = \overline{\mathbb{D}} \setminus \{0\}$.*

This lemma shows that the definitions of foliation in $\overline{\mathbb{D}} \setminus \{0\}$ and τ coincide with those given in Section 1.3.

Proof. From the monotonicity of (D_t) and the continuity of $t \mapsto \gamma_t$, we have $\bigcup_{t > \tau(z)} D_t = D_{\tau(z)}$. Since $z \in D_t$ implies $\tau(z) > t$, $z \notin D_{\tau(z)}$. Similarly, $z \notin \overline{D}_t$ implies $t > \tau(z)$ and it follows that $z \in \overline{D_{\tau(z)}} \setminus D_{\tau(z)} = \partial D_{\tau(z)}$. This completes the proof. \square

If $\rho \in \mathcal{N}_+$ and the family of interfaces $(\gamma_t = \partial D_t)_{t \geq 0}$ produced by Loewner evolution forms a foliation of $\overline{\mathbb{D}} \setminus \{0\}$, we say that $\rho \in \mathcal{N}_+$ *generates a foliation*. We will often choose the conformal parametrization for each γ_t , obtained by continuously extending $f_t : \mathbb{D} \rightarrow D_t$ to $\overline{\mathbb{D}}$ by Carathéodory's theorem and then restricting to S^1 .

We will now define the *winding function* φ associated with a foliation $(\gamma_t = \partial D_t)_{t \geq 0}$. It will be convenient to use the notation

$$\vartheta[f](z) = \arg \frac{zf'(z)}{f(z)} = \int_0^z d \arg \frac{f(w) - f(z)}{w - z}, \quad (2.7)$$

when f is a conformal map (defined on a simply connected domain) fixing 0. Here and elsewhere, the continuous branch of $z \mapsto \arg(zf'(z)/f(z))$ is chosen to equal 0 at the origin. The integral is taken along any smooth path from 0 to z . The function ϑ satisfies a chain rule

$$\vartheta[f \circ g] = \vartheta[f] \circ g + \vartheta[g] \quad (2.8)$$

if $f, g, f \circ g$ are all conformal maps defined on simply connected neighborhoods of the origin which f, g fix. Given a curve γ with some parametrization $\gamma(s)$, we say that γ has a tangent at $\gamma(s_0)$ if there exists α such that

$$\lim_{s \rightarrow s_0 \pm} \frac{\gamma(s) - \gamma(s_0)}{|\gamma(s) - \gamma(s_0)|} = \pm e^{i\alpha}.$$

The existence of a tangent does not depend on the chosen parametrization. For any Jordan curve γ we write $\mathcal{T}_\gamma \subseteq \gamma$ for the differentiable points on γ , that is, set of points $z \in \gamma$ at which a tangent exists.

Lemma 2.5. *Suppose $\gamma = \partial D$ is a chord-arc curve where D contains 0 and let $f : \mathbb{D} \rightarrow D$ be the conformal map such that $f(0) = 0, f'(0) > 0$. Then $|\gamma \setminus \mathcal{T}_\gamma| = 0$ and the following statements are equivalent.*

- (i) $f(e^{i\theta}) \in \mathcal{T}_\gamma$.
- (ii) The unrestricted limit of $\arg(f(z) - f(e^{i\theta})) / (z - e^{i\theta})$ in \mathbb{D} exists at $e^{i\theta}$ and is finite.
- (iii) The non-tangential limit of $\arg(f(z) - f(e^{i\theta})) / (z - e^{i\theta})$ exists at $e^{i\theta}$ and is finite.
- (iv) The non-tangential limit of $\arg f'(z)$ exists at $e^{i\theta}$ and is finite.

In this case, the limits are equal modulo 2π .

Proof. The equivalence of (i) and (ii) is the first part of [22, Thm. II.4.2] and this equivalence holds without the chord-arc assumption. Next, (ii) trivially implies (iii) and the converse implication is given in [46, Thm. 5.5] since γ is chord-arc so in particular D is a John domain. The equivalence of (iii) and (iv) is [46, Prop. 4.11]. Finally, since γ is rectifiable $|\gamma \setminus \mathcal{T}_\gamma| = 0$ by [46, Thm. 6.8]. \square

Remark. In the setting of Lemma 2.5, assuming γ is a general Jordan curve (not necessarily chord-arc), (i) still implies that (ii), (iii), and (iv) hold.

Remark. Note that non-tangential approach regions are preserved under conformal mapping between \mathbb{D} and a quasidisk, see [27, Prop. 1.1].

Lemma 2.6. *Suppose $f : \mathbb{D} \rightarrow D \subset \mathbb{D}$ is a conformal map satisfying $f(0) = 0$. If $z \in \partial\mathbb{D}$ is such that the non-tangential limit $\arg f'(z)$ exists and $f(z) \in \partial D$, then $\vartheta[f](z) = 0$.*

Proof. If $z, f(z) \in \partial\mathbb{D}$, then $w \mapsto (f(w) - f(z))/(w - z)$ takes values in the slit domain $\mathbb{C} \setminus \{-tf(z)/z, t \geq 0\}$. Indeed, if this were not true, then there would exist $t \geq 0$ and $w \in \mathbb{D}$ such that $f(w) + twf(z)/z = f(z)(1 + t)$. By Schwarz lemma this implies

$$1 + t = |f(w) + twf(z)/z| \leq |f(w)| + t|w| \leq |w|(1 + t),$$

a contradiction since $|w| < 1$. We can therefore define $w \mapsto \arg(f(w) - f(z))/(w - z)$ continuously and unambiguously for $w \in \mathbb{D}$ by choosing a branch of \arg on this slit domain. Consider the path $s \mapsto sz$, $s \in [0, 1]$ and set

$$v(s) = \arg \frac{f(sz) - f(z)}{sz - z} - \arg \frac{f(z)}{z}, \quad s \in [0, 1).$$

This is a continuous function such that $v(0) = 0$. Equation (2.7) and Lemma 2.5 show that

$$\vartheta[f](z) = \lim_{s \rightarrow 1^-} v(s).$$

To compute the value, we note that the tangent of $\partial\mathbb{D}$ at z , oriented counterclockwise, has argument $\arg z + \pi/2$ modulo 2π . Moreover, if $f(z) \in \partial\mathbb{D}$ is a differentiable point of ∂D then since $D \subset \mathbb{D}$ it follows that ∂D is tangent to $\partial\mathbb{D}$ at $f(z)$ and this tangent has argument $\arg f(z) + \pi/2$ modulo 2π . Using Lemma 2.5 we have that $\lim_{s \rightarrow 1^-} \arg(f(sz) - f(z))/(sz - z)$ coincides with the non-tangential limit of $\arg f'(z)$ modulo 2π , and the latter is equal to the difference of the arguments of the tangents at $f(z)$ and at z , namely $\arg f(z)/z$ modulo 2π . This implies that $\vartheta[f](z) = 0$ modulo 2π .

Since the continuous function $w \mapsto (f(w) - f(z))/(w - z)$ takes values in the slit plane $\mathbb{C} \setminus \{-tf(z)/z, t \geq 0\}$, the function $s \mapsto v(s)$ does not take any two values differing by a non-zero multiple of 2π . Since $v(0) = 0$ it follows that $\vartheta[f](z) = 0$. \square

Let $\mathcal{T}_t = \mathcal{T}_{\gamma_t}$ be the set of $z \in \gamma_t$ at which γ_t has a tangent. By Lemma 2.5 and the remark following it, $\gamma_t \setminus \mathcal{T}_t$ has arclength 0 and if $z \in \mathcal{T}_t$, the harmonic function $\vartheta[g_t](w)$ has a non-tangential limit $\vartheta[g_t](z)$ as w tends to z inside D_t . We define the *winding function* of γ_t by

$$\varphi_{\gamma_t}(z) := \vartheta[g_t](z), \quad z \in \mathcal{T}_t, \quad t \geq 0. \quad (2.9)$$

Remark. Geometrically, modulo 2π , the winding function at $z \in \mathcal{T}_t$ is the angle between the tangent to γ_t and that of the circle passing through z centered at 0, as described in Section 1.3. In Lemma 8.4 we discuss an equivalent geometric definition of the winding function, suggested to us by a referee, at a differentiable point on a Jordan curve using the turning of a smooth curve approaching the point normally.

Lemma 2.7. *Suppose $s < t$ and $z \in \mathcal{T}_s \cap \mathcal{T}_t$. Then $\varphi_{\gamma_s}(z) = \varphi_{\gamma_t}(z)$.*

Proof. By the chain rule (2.8) and Lemma 2.3, it is enough to assume that $s = 0$ and verify that $\varphi_{\gamma_t}(z) = 0$ if $z \in S^1$. This follows from Lemma 2.6 which shows that $\vartheta[g_t](z) = -\vartheta[f_t](g_t(z)) = 0$. \square

This observation allows us to define the winding function of a foliation.

Definition 2.8. *Let $\mathcal{T} := \cup_{t \geq 0} \mathcal{T}_t \subset \overline{\mathbb{D}}$. We define the winding function of a foliation, $\varphi : \mathcal{T} \rightarrow \mathbb{R}$, by forgetting the leaf, namely $\varphi(z) := \varphi_{\gamma_t}(z)$ if $z \in \mathcal{T}_t$. We set by convention $\varphi(0) = 0$.*

Remark. If the leaves are all C^1 , φ is defined everywhere since $\mathcal{T} = \overline{\mathbb{D}} \setminus \{0\}$ by Lemma 2.4. In the general case of a chord-arc foliation, a function defined arclength-a.e. on all leaves need not be defined on $\overline{\mathbb{D}} \setminus \{0\}$ Lebesgue-a.e. To illustrate the subtlety, it is possible to

construct a subset X of \mathbb{D} of full Lebesgue measure and a foliation of piecewise smooth curves such that each leaf intersects X in exactly one point, see, e.g., [40]. On the other hand, we shall prove in Proposition 3.6 that a function in $\mathcal{E}_0(\mathbb{D})$ is determined by its “values” (interpreted appropriately) on the leaves if the curves are chord-arc.

Lemma 2.9. *Suppose $\rho \in \mathcal{N}_+$ generates a foliation and let $T \geq 0$. If ρ_t is the uniform measure for all $t \geq T$, then for all $z \in D_T$,*

$$\varphi(z) = \vartheta[g_T](z). \quad (2.10)$$

In particular, φ is defined and harmonic in D_T . Moreover, $(\gamma_t)_{t \geq T}$ are the equipotentials in D_T .

Proof. If $T = 0$, ρ_t is the uniform measure for all $t \geq 0$ which implies that $g_t(z) = e^t z$ and $\varphi = 0 = \vartheta[g_0]$. Now let $t > T \geq 0$. By Lemma 2.3, we have $g_t \circ f_T(z) = e^{t-T} z$. It follows from $g_t = (g_t \circ f_T) \circ g_T$, (2.8) and $\vartheta[z \mapsto e^{t-T} z] = 0$ that

$$\varphi(z) = \vartheta[g_t](z) = \vartheta[g_T](z) \quad \forall z \in \partial D_t,$$

and we have $(\gamma_t = f_T(e^{T-t} S^1))_{t > T}$ which is by definition the family of equipotentials in D_T . We conclude the proof of (2.10) using $\cup_{t > T} \gamma_t = \cup_{t > T} g_T^{-1}(e^{T-t} S^1) = g_T^{-1}(\mathbb{D} \setminus \{0\}) = D_T \setminus \{0\}$ (and by convention, $\varphi(0) = 0 = \vartheta[g_T](0)$). \square

Lemma 2.10. *Suppose $\rho \in \mathcal{N}_+$ generates a foliation and let φ be the corresponding winding function. For every $T \geq 0$, let $\varphi^{(T)}$ denote the winding function associated to the truncated measure $\rho^{(T)}$ defined by $\rho_t^{(T)} := \rho_t$ for $t \leq T$ and $\rho_t^{(T)}$ is uniform for $t > T$. Then $(\varphi - \varphi^{(T)}) \circ f_T$ is the winding function generated by the measures $(\tilde{\rho}_s := \rho_{T+s})_{s \geq 0}$.*

Proof. From Lemma 2.3, the winding function $\tilde{\varphi}$ associated to $\tilde{\rho}$ is obtained from

$$\vartheta[\tilde{g}_s] = \vartheta[g_{s+T} \circ f_T] = \vartheta[g_{s+T}] \circ f_T - \vartheta[g_T] \circ f_T$$

where $(\tilde{g}_s)_{s \geq 0}$ is the Loewner chain of $\tilde{\rho}$. We conclude the proof using Lemma 2.9. \square

In other words, decomposing the driving measure ρ into the truncated (and uniformly extended) measure on $S^1 \times [0, T]$, and the measure $\tilde{\rho} : s \mapsto \rho_{T+s}$ amounts to decomposing the winding function into $\varphi^{(T)}$ which is defined and harmonic in D_T and $\varphi - \varphi^{(T)}$ which vanishes (arclength-a.e.) on every γ_t for $t \leq T$.

3 The foliation disintegration isometry

3.1 Hadamard’s formula

In this section we use the Loewner–Kufarev equation to derive a formula for the dynamics of the Green’s function. The resulting expression is a version of Hadamard’s classical formula in a smooth setting. We fix $\rho \in \mathcal{N}_+$ and let $(g_t : D_t \rightarrow \mathbb{D})_{t \geq 0}$ be the associated uniformizing Loewner chain.

For each $t \geq 0$, let $G_t(z, w)$ be the Green's function for D_t with Dirichlet boundary condition. For z, w fixed, we extend the definition of $G_t(z, w)$ to $t \geq \min\{\tau(z), \tau(w)\}$ by 0. We will show that with this definition, $t \mapsto G_t(z, w)$ is absolutely continuous under the assumption that ρ generates a foliation. Note that $t \mapsto G_t(z, w)$ is always absolutely continuous for $t \in [0, \min\{\tau(z), \tau(w)\})$ but need not be continuous at $\min\{\tau(z), \tau(w)\}$, e.g., in the case when ∂D_t disconnects an open set containing z, w from 0 as $t \rightarrow \tau = \tau(z) = \tau(w)$.

Lemma 3.1. *Consider the Loewner–Kufarev equation driven by $\rho \in \mathcal{N}_+$. Then for a.e. $t < \min\{\tau(z), \tau(w)\}$ and all $z, w \in D_t$,*

$$-\partial_t G_{D_t}(z, w) = \int_0^{2\pi} P_{\mathbb{D}}(g_t(z), e^{i\theta}) P_{\mathbb{D}}(g_t(w), e^{i\theta}) d\rho_t(\theta).$$

If in addition ρ is assumed to generate a foliation, then $t \mapsto G_t(z, w)$ is absolutely continuous for all t .

See also [51, Thm. 4] for a similar result.

Proof. First note that for $z, w \in D_t$, by conformal invariance of the Green's function,

$$G_t(z, w) = G_{D_t}(z, w) = -\log \left| \frac{g_t(z) - g_t(w)}{1 - \overline{g_t(w)}g_t(z)} \right|. \quad (3.1)$$

Setting $z_t = g_t(z), w_t = g_t(w)$, the Loewner–Kufarev equation (2.5) implies that for a.e. $t < \min\{\tau(z), \tau(w)\}$, $\partial_t z_t = z_t H_t(z_t)$, $\partial_t w_t = w_t H_t(w_t)$ and so

$$\begin{aligned} -\partial_t G_t(z, w) &= \operatorname{Re} \partial_t \log \frac{z_t - w_t}{1 - \overline{w_t}z_t} \\ &= \operatorname{Re} \frac{\overline{w_t}z_t (H_t(z_t) + \overline{H_t(w_t)})}{1 - \overline{w_t}z_t} + \operatorname{Re} \frac{z_t H_t(z_t) - w_t H_t(w_t)}{z_t - w_t}. \end{aligned}$$

Next, note that

$$\begin{aligned} \operatorname{Re} \frac{\overline{w_t}z_t (H_t(z_t) + \overline{H_t(w_t)})}{1 - \overline{w_t}z_t} &= \int_0^{2\pi} \operatorname{Re} \frac{\overline{w_t}z_t \cdot 2(1 - z_t \overline{w_t})}{(1 - z_t \overline{w_t})(e^{i\theta} - z_t)(e^{-i\theta} - \overline{w_t})} d\rho_t(\theta) \\ &= \int_0^{2\pi} \operatorname{Re} \frac{2z_t \overline{w_t}}{(e^{i\theta} - z_t)(e^{-i\theta} - \overline{w_t})} d\rho_t(\theta) \\ &= \int_0^{2\pi} \operatorname{Re} \frac{2z_t \overline{w_t} (e^{-i\theta} - \overline{z_t})(e^{i\theta} - w_t)}{|e^{i\theta} - z_t|^2 |e^{i\theta} - w_t|^2} d\rho_t(\theta). \end{aligned} \quad (3.2)$$

Moreover,

$$\begin{aligned} \operatorname{Re} \frac{z_t H_t(z_t) - w_t H_t(w_t)}{z_t - w_t} &= \operatorname{Re} \int_0^{2\pi} \left(\frac{e^{i\theta} + z_t}{e^{i\theta} - z_t} + \frac{2w_t e^{i\theta}}{(e^{i\theta} - z_t)(e^{i\theta} - w_t)} \right) d\rho_t(\theta) \\ &= \operatorname{Re} \int_0^{2\pi} \frac{(e^{i\theta} + z_t)(e^{-i\theta} - \overline{z_t})|e^{i\theta} - w_t|^2}{|e^{i\theta} - z_t|^2 |e^{i\theta} - w_t|^2} d\rho_t(\theta) \\ &\quad + \operatorname{Re} \int_0^{2\pi} \frac{2w_t e^{i\theta} (e^{-i\theta} - \overline{z_t})(e^{-i\theta} - \overline{w_t})}{|e^{i\theta} - z_t|^2 |e^{i\theta} - w_t|^2} d\rho_t(\theta). \end{aligned} \quad (3.3)$$

Adding (3.2) and (3.3) and simplifying,

$$-\partial_t G_t(z, w) = \int_0^{2\pi} \frac{1 - |z_t|^2}{|e^{i\theta} - z_t|^2} \cdot \frac{1 - |w_t|^2}{|e^{i\theta} - w_t|^2} d\rho_t(\theta) = \int_0^{2\pi} P_{\mathbb{D}}(z_t, e^{i\theta}) P_{\mathbb{D}}(w_t, e^{i\theta}) d\rho_t(\theta),$$

as claimed.

Let us now assume $\rho \in \mathcal{N}_+$ generates a foliation and prove that $t \mapsto G_t(z, w)$ is absolutely continuous for all $t \geq 0$. For all $z, w \in \mathbb{D}$, $z \neq w$, we need to show that $G_t(z, w) \rightarrow 0$ as $t \nearrow \min\{\tau(z), \tau(w)\}$. Without loss of generality, we assume that $\tau(z) \leq \tau(w)$. If $\tau(z) < \tau(w)$, it is clear from (3.1) that $G_t(z, w) \rightarrow 0$ as $t \rightarrow \tau(z)$ since $|g_t(w)|$ is bounded away from 1 for $t \in [0, \tau(z)]$. Assume now that $\tau = \tau(z) = \tau(w)$. Lemma 2.4 implies that $z, w \in \partial D_\tau$. Since $g_\tau : D_\tau \rightarrow \mathbb{D}$ is a conformal map between two Jordan domains, g_τ extends to a homeomorphism $\overline{D}_\tau \rightarrow \overline{\mathbb{D}}$ and $g_\tau(z) \neq g_\tau(w) \in S^1$. We claim that

$$\lim_{t \rightarrow \tau^-} g_t(z) = g_\tau(z).$$

Assuming this, it follows that $|g_t(z) - g_t(w)|$ is bounded away from 0 on $[0, \tau]$. Therefore $G_t(z, w) \rightarrow 0$ as $t \rightarrow \tau^-$ using (3.1). To prove the claim, let $t_n, n = 1, 2, \dots$, be any sequence such that $t_n \nearrow \tau$ and such that $\zeta := \lim_{n \rightarrow \infty} g_{t_n}(z)$ exists. Pick $\varepsilon > 0$ and let $v \in D_\tau$ be such that the interior distance between z and v in D_τ is at most ε . By the Beurling estimate there is a constant C that does not depend on z, v such that $|g_\tau(z) - g_\tau(v)| \leq C\sqrt{\varepsilon}$ and by monotonicity of the domains (Schwarz' lemma) the same estimate holds with τ replaced by any $t_n < \tau$. Let n_0 be large such that if $n > n_0$ then $|g_{t_n}(v) - g_\tau(v)| + |g_{t_n}(z) - \zeta| < \varepsilon$. Such n_0 exists by definition of ζ and since $t \mapsto g_t(v)$ is continuous on $[0, \tau(v))$ and $\tau(v) > \tau(z)$. Combining these estimates with the triangle inequality it follows that $|g_\tau(z) - \zeta| \leq 2C\sqrt{\varepsilon}$. \square

3.2 Disintegration isometry

In this section we assume that $\rho \in \mathcal{N}_+$ generates a foliation of $\overline{\mathbb{D}} \setminus \{0\}$. We will write

$$L^2(2\rho) := L^2(S^1 \times \mathbb{R}_+, 2\rho)$$

and $\langle \cdot, \cdot \rangle_{L^2(2\rho)}$ for the corresponding inner product. Recall also that we write $\mathcal{E}_0(\mathbb{D})$ for $W_0^{1,2}(\mathbb{D})$ equipped with the Dirichlet energy norm.

Lemma 3.2. *Suppose $\rho \in \mathcal{N}_+$ generates a foliation and that $\phi \in C_c^\infty(\mathbb{D})$. Then*

$$\iota[\phi](\theta, t) := \frac{1}{2\pi} \int_{\mathbb{D}} \Delta(\phi \circ f_t)(z) P_{\mathbb{D}}(z, e^{i\theta}) dA(z) \quad (3.4)$$

is an element of $L^2(2\rho)$. Moreover, for all $T > 0$,

$$\mathcal{D}_{\mathbb{D}}(\phi) - \mathcal{D}_{\mathbb{D}}(\phi_T^0) = \|\iota[\phi] \mathbf{1}_{S^1 \times [0, T]}\|_{L^2(2\rho)}^2$$

where $\phi_T^0 \in \mathcal{E}_0(\mathbb{D})$ is the zero-trace part of $\phi \circ f_T \in W^{1,2}(\mathbb{D})$ as in (2.1).

In particular, the operator $\iota : \mathcal{E}_0(\mathbb{D}) \cap C_c^\infty(\mathbb{D}) \rightarrow L^2(2\rho)$ is norm-preserving.

Proof. Let $\phi \in C_c^\infty(\mathbb{D})$ set $\mu = \Delta\phi$ and write $G_0 = 2\pi\Delta^{-1}$, where we recall that we consider the positive Laplacian. (We write G_0 for both the operator and function.) For $z \in \mathbb{D}$,

$$\phi(z) = \frac{1}{2\pi}G_0\mu(z) = \frac{1}{2\pi}\int_{\mathbb{D}}G_0(w, z)\mu(w) dA(w).$$

Using integration by parts, since ϕ is smooth and has compact support,

$$\begin{aligned}\int_{\mathbb{D}}|\nabla\phi(z)|^2 dA(z) &= \int_{\mathbb{D}}\phi(z)\Delta\phi(z) dA(z) \\ &= \frac{1}{2\pi}\int_{\mathbb{D}}\int_{\mathbb{D}}G_0(w, z)\mu(w)\mu(z) dA(w) dA(z).\end{aligned}$$

As before we set $G_t(z, w) = G_{D_t}(z, w) = G_0(z_t, w_t)$ for $z, w \in D_t$ (recall that $z_t = g_t(z)$ and $w_t = g_t(w)$) and 0 otherwise. Let

$$\phi_t := \phi \circ f_t, \quad \mu_t := \Delta\phi_t = |f_t'|^2\mu \circ f_t$$

and note that the zero-trace part of ϕ_t satisfies $\phi_t^0 = G_0\mu_t/2\pi$. Moreover, performing a change of variable,

$$\begin{aligned}\int_{\mathbb{D}}|\nabla\phi_T^0(z)|^2 dA(z) &= \frac{1}{2\pi}\int_{\mathbb{D}}\int_{\mathbb{D}}G_0(w, z)\mu_T(w)\mu_T(z) dA(w) dA(z) \\ &= \frac{1}{2\pi}\int_{\mathbb{D}}\int_{\mathbb{D}}G_0(w, z)\mu \circ f_T(w)\mu \circ f_T(z)|f_T'(z)|^2|f_T'(w)|^2 dA(w) dA(z) \\ &= \frac{1}{2\pi}\int_{D_T}\int_{D_T}G_T(w, z)\mu(w)\mu(z) dA(w) dA(z).\end{aligned}\tag{3.5}$$

Next, note that by Lemma 3.1

$$-\partial_t G_t(z, w) = \int_0^{2\pi} P_{\mathbb{D}}(g_t(z), e^{i\theta})P_{\mathbb{D}}(g_t(w), e^{i\theta})d\rho_t(\theta) \geq 0$$

for $z, w \in \mathbb{D}$ and that $P_{\mathbb{D}}(g_t(w), e^{i\theta}) \geq 0$. Since the singularity of the Green's function is logarithmic (and therefore integrable), we obtain

$$\begin{aligned}\int_0^T \int_{D_t} \int_{D_t} -\partial_t G_t(z, w) dA(z) dA(w) dt &= \int_0^T \int_{\mathbb{D}} \int_{\mathbb{D}} -\partial_t G_t(z, w) dA(z) dA(w) dt \\ &= \int_{\mathbb{D}} \int_{\mathbb{D}} \int_0^T -\partial_t G_t(z, w) dt dA(z) dA(w) \\ &= \int_{\mathbb{D}} \int_{\mathbb{D}} G_0(z, w) - G_T(z, w) dA(z) dA(w) < \infty.\end{aligned}$$

Therefore, using the smoothness of h , we can apply Fubini's theorem (repeatedly) and (3.5) to compute

$$\begin{aligned}2\pi \left(\int_{\mathbb{D}} |\nabla\phi(z)|^2 dA(z) - \int_{\mathbb{D}} |\nabla\phi_T^0(z)|^2 dA(z) \right) \\ = \int_{\mathbb{D}} \int_{\mathbb{D}} (G_0(z, w) - G_T(z, w)) \mu(z)\mu(w) dA(z) dA(w)\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{D}} \int_{\mathbb{D}} \int_0^T -\partial_t G_t(z, w) \mu(z) \mu(w) dt dA(w) dA(z) \\
&= \int_0^T \int_{D_t} \int_{D_t} -\partial_t G_t(z, w) \mu(z) \mu(w) dA(w) dA(z) dt \\
&= \int_0^T \int_{D_t} \int_{D_t} \mu(z) \mu(w) \int_0^{2\pi} P_{\mathbb{D}}(g_t(z), e^{i\theta}) P_{\mathbb{D}}(g_t(w), e^{i\theta}) d\rho_t(\theta) dA(w) dA(z) dt \\
&= \int_0^T \int_0^{2\pi} \int_{D_t} \int_{D_t} \mu(z) \mu(w) P_{\mathbb{D}}(g_t(z), e^{i\theta}) P_{\mathbb{D}}(g_t(w), e^{i\theta}) dA(w) dA(z) d\rho_t(\theta) dt \\
&= \iint_{S^1 \times [0, T]} \left(\int_{\mathbb{D}} \mu_t(z) P_{\mathbb{D}}(z, e^{i\theta}) dA(z) \right) \left(\int_{\mathbb{D}} \mu_t(w) P_{\mathbb{D}}(w, e^{i\theta}) dA(w) \right) d\rho(\theta, t) \\
&= (2\pi)^2 \iint_{S^1 \times [0, T]} |\iota[\phi](\theta, t)|^2 d\rho(\theta, t),
\end{aligned}$$

as claimed. Letting $T \rightarrow \infty$, by monotone convergence, we obtain that the operator $\iota : \mathcal{E}_0(\mathbb{D}) \cap C_c^\infty(\mathbb{D}) \rightarrow L^2(2\rho)$ is norm-preserving. \square

Remark. Given a signed measure μ supported on $\overline{\mathbb{D}}$, the *balayage* of μ to $\partial\mathbb{D}$ is a measure ν on $\partial\mathbb{D}$ such that the logarithmic potentials of μ and ν agree on \mathbb{D}^* . Viewing μ as a charge density, ν represents the optimal way (with respect to logarithmic energy) to redistribute charge to $\partial\mathbb{D}$ while keeping the potential fixed in the complementary domain. In our setting, for t fixed, one can see that $\iota[\phi](\theta, t)$ in fact equals the (density of) the balayage of the “charge density” $\Delta\phi_t dA$ to $\partial\mathbb{D}$. See, e.g., [32, Ch. IV].

Remark. The disintegration isometry is closely related to the general approach of [25], see in particular Section 3 of that paper. Hedenmalm and Nieminen consider C^2 -smooth, strictly monotone deformations of domains, and do not employ the Loewner equation. In the smooth, strictly monotone case, a version of our Lemma 3.2 can be deduced from results of [25] via a “polar coordinates” change of variable which also relies on strong regularity assumptions of the interfaces.

Since $C_c^\infty(\mathbb{D})$ is dense in $\mathcal{E}_0(\mathbb{D})$, the operator ι extends to $\mathcal{E}_0(\mathbb{D})$ and we have

$$\mathcal{D}_{\mathbb{D}}(\phi) = \|\iota[\phi]\|_{L^2(2\rho)}^2, \quad \phi \in \mathcal{E}_0(\mathbb{D}).$$

We will now construct an inverse operator $\varkappa : L^2(2\rho) \rightarrow \mathcal{E}_0(\mathbb{D})$ as follows. For $u \in L^2(2\rho)$ and $\phi \in C_c^\infty(\mathbb{D})$, we consider

$$\Phi_u : \quad \phi \mapsto 2 \iint_{\mathbb{R}_+ \times S^1} u(t, \theta) \iota[\phi](\theta, t) d\rho(t, \theta) = \langle u, \iota[\phi] \rangle_{L^2(2\rho)}$$

which extends to a bounded linear operator $\mathcal{E}_0(\mathbb{D}) \rightarrow \mathbb{R}$. Indeed, the Cauchy-Schwarz inequality and Lemma 3.2 show that

$$\left| \langle u, \iota[\phi] \rangle_{L^2(2\rho)} \right| \leq \|u\|_{L^2(2\rho)} \|\iota[\phi]\|_{L^2(2\rho)} = \|u\|_{L^2(2\rho)} \mathcal{D}_{\mathbb{D}}(\phi)^{1/2}. \quad (3.6)$$

By the Riesz representation theorem there exists a unique $\varkappa[u] \in \mathcal{E}_0(\mathbb{D})$ with the property that for all $\phi \in C_c^\infty(\mathbb{D})$,

$$\frac{1}{\pi} \int_{\mathbb{D}} \varkappa[u] \Delta\phi dA(z) = \frac{1}{\pi} \int_{\mathbb{D}} \nabla \varkappa[u] \nabla \phi dA(z) = \Phi_u(\phi). \quad (3.7)$$

It follows immediately that $\varkappa \circ \iota = \text{Id}_{\mathcal{E}_0(\mathbb{D})}$. We now give the explicit formula for \varkappa . In this statement and below we use the notation $u_t(\cdot) := u(\cdot, t)$ for $u \in L^2(2\rho)$.

Proposition 3.3. *Suppose $\rho \in \mathcal{N}_+$ generates a foliation. Let $u \in L^2(2\rho)$. Then for a.e. $w \in \mathbb{D}$,*

$$\varkappa[u](w) = 2\pi \int_0^{\tau(w)} P_{\mathbb{D}}[u_t \rho_t](g_t(w)) dt,$$

and $t \mapsto P_{\mathbb{D}}[u_t \rho_t](g_t(w)) \in L^1([0, \tau(w)], dt)$.

Proof. By linearity and splitting $u = u^+ - u^-$ with $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$, it suffices to prove the proposition when $u \geq 0$. We let

$$\tilde{\varkappa}[u](w) := 2\pi \int_0^{\tau(w)} P_{\mathbb{D}}[u_t \rho_t](g_t(w)) dt = \int_0^{\tau(w)} \int_{S^1} P_{\mathbb{D}}(g_t(w), e^{i\theta}) u(\theta, t) d\rho_t(\theta) dt.$$

We first prove that for all $\phi \in C_c^\infty(\mathbb{D})$,

$$2 \int_0^\infty \int_{S^1} u(\theta, t) \iota[\phi](\theta, t) d\rho_t(\theta) dt = \frac{1}{\pi} \int_{\mathbb{D}} \Delta\phi(w) \tilde{\varkappa}[u](w) dA(w). \quad (3.8)$$

For this, we will repeatedly interchange the order of integration in the following computation. To justify this, we let

$$\phi^+ := \Delta^{-1} \max\{\Delta\phi, 0\} \quad \text{and} \quad \phi^- := \Delta^{-1} \max\{-\Delta\phi, 0\}.$$

We have $\phi = \phi^+ - \phi^-$. Since ϕ is smooth, $\max\{\Delta\phi, 0\}$ is Lipschitz continuous. By elliptic regularity (see, e.g., [23, Sec. 4.3]), $\phi^\pm, \phi^- \in \mathcal{E}_0(\mathbb{D}) \cap C^{2,\alpha}(\mathbb{D})$ for any $\alpha < 1$ and clearly $\Delta\phi^\pm \geq 0$. Moreover, recalling that $\phi_t^\pm = \phi^\pm \circ f_t$, we have $\Delta\phi_t^\pm = \Delta(\phi^\pm \circ f_t) = |f_t'|^2 (\Delta\phi^\pm) \circ f_t \geq 0$. So it follows that

$$\iota[\phi^\pm](\theta, t) = \frac{1}{2\pi} \int_{\mathbb{D}} \Delta\phi_t^\pm(z) P_{\mathbb{D}}(z, e^{i\theta}) dA(z) \geq 0.$$

Since $u, \iota[\phi^\pm] \in L^2(2\rho)$, the Cauchy-Schwarz inequality shows that $u \cdot \iota[\phi^\pm] \in L^1(2\rho)$. Using Fubini's theorem, this implies that $u(\theta, t) \Delta\phi_t(z) P_{\mathbb{D}}(z, e^{i\theta})$ is in $L^1(\mathbb{D} \times S^1 \times \mathbb{R}_+, dA \times d\rho_t(\theta) \times dt)$. Therefore the left-hand side of (3.8) equals

$$\begin{aligned} & 2 \int_0^\infty \int_{S^1} u(\theta, t) \left[\frac{1}{2\pi} \int_{\mathbb{D}} \Delta\phi_t(z) P_{\mathbb{D}}(z, e^{i\theta}) dA(z) \right] d\rho_t(\theta) dt \\ &= \frac{1}{\pi} \int_0^\infty \int_{\mathbb{D}} \Delta\phi_t(z) \int_{S^1} u(\theta, t) P_{\mathbb{D}}(z, e^{i\theta}) d\rho_t(\theta) dA(z) dt \\ &= \frac{1}{\pi} \int_0^\infty \int_{\mathbb{D}} (\Delta\phi) \circ f_t(z) |f_t'(z)|^2 \int_{S^1} u(\theta, t) P_{\mathbb{D}}(z, e^{i\theta}) d\rho_t(\theta) dA(z) dt \\ &= \frac{1}{\pi} \int_0^\infty \int_{D_t} \Delta\phi(w) \int_{S^1} u(\theta, t) P_{\mathbb{D}}(g_t(w), e^{i\theta}) d\rho_t(\theta) dA(w) dt \\ &= \frac{1}{\pi} \int_{\mathbb{D}} \Delta\phi(w) \int_0^{\tau(w)} \int_{S^1} u(\theta, t) P_{\mathbb{D}}(g_t(w), e^{i\theta}) d\rho_t(\theta) dt dA(w) \end{aligned}$$

which proves (3.8). This implies that $\tilde{\varkappa}[u] \in \mathcal{E}(\mathbb{D})$ since for all $\phi \in C_c^\infty(\mathbb{D})$, we have by Cauchy-Schwarz inequality

$$\left| \langle \Delta\phi, \tilde{\varkappa}[u] \rangle_{L^2(\mathbb{D})} \right| \leq C \|u\|_{L^2(\rho)} \|\nabla\phi\|_{L^2(\mathbb{D})}$$

for some universal constant C .

Given this, (3.7) implies that $\varkappa[u] - \tilde{\varkappa}[u]$ is weakly harmonic, and therefore, by Weyl's lemma, it is harmonic in \mathbb{D} . Since we know that $\varkappa[u] \in \mathcal{E}_0(\mathbb{D})$, to complete the proof, it is therefore enough to show that $\tilde{\varkappa}[u] \in \mathcal{E}_0(\mathbb{D})$ (namely, that it has zero trace). For this, we show that $\tilde{\varkappa}[u]$ can be extended to a function $\psi \in W^{1,2}(e\mathbb{D})$ where $\psi(w) = 0$ for all $1 < |w| < e$ (see, e.g., [10, Prop. 9.18]). To construct the extension, we define a measure $\hat{\rho} \in \mathcal{N}_+$ with $\hat{\rho}_t = \rho_{t-1}$ for $t \geq 1$ and uniform for $t \in [0, 1)$. Similarly, let $\hat{u}(\theta, t) := u(\theta, t-1)$ for $t \geq 1$, and $\hat{u}(\theta, t) = 0$ for $t \in [0, 1)$.

We clearly have $\hat{u} \in L^2(2\hat{\rho})$, and $\psi(w) := \tilde{\varkappa}[\hat{u}](e^{-1}w) = 0$ if $1 < |w| < e$. By construction, we have $\psi \in \mathcal{E}_0(e\mathbb{D})$, hence $W^{1,2}(e\mathbb{D})$, from the proof above. Moreover, if \hat{g}_t is the uniformizing Loewner chain of $\hat{\rho}$, then for $t \in [0, 1]$ we have $\hat{g}_t(z) = e^t z$ and for $t \geq 1$, $\hat{g}_t(z) = g_{t-1}(ez)$. It follows that $\tilde{\varkappa}[u](w) = \tilde{\varkappa}[\hat{u}](e^{-1}w) = \psi(w)$ for $w \in \mathbb{D}$. This shows that $\varkappa = \tilde{\varkappa}$ in $\mathcal{E}_0(\mathbb{D})$.

Finally, since $u \geq 0$, it follows that $P_{\mathbb{D}}[u_t \rho_t](g_t(w)) \in L^1([0, \tau(w)], dt)$ for a.e. $w \in \mathbb{D}$. \square

Corollary 3.4. *Let $\phi \in \mathcal{E}_0(\mathbb{D})$ and $T \geq 0$. The orthogonal decomposition of ϕ into a function in $\mathcal{E}_0(D_T)$ and a function in $\mathcal{E}_0(\mathbb{D})$ harmonic in D_T (2.1) is given by*

$$\phi^{0,T} := \varkappa[\iota[\phi] \mathbf{1}_{S^1 \times [T, \infty)}] \in \mathcal{E}_0(D_T) \quad \text{and} \quad \phi^{h,T} := \varkappa[\iota[\phi] \mathbf{1}_{S^1 \times [0, T)}].$$

Remark. Using the notation of Lemma 3.2, a consequence of the corollary is that $\phi^{0,T} = \phi_T^0 \circ g_T$.

Proof. We define

$$\phi^{0,T} := \varkappa[\iota[\phi] \mathbf{1}_{S^1 \times [T, \infty)}] \quad \text{and} \quad \phi^{h,T} := \varkappa[\iota[\phi] \mathbf{1}_{S^1 \times [0, T)}].$$

Since $\varkappa \circ \iota[\phi] = \phi$, we have $\phi^{0,T} + \phi^{h,T} = \phi$.

We now show that $\phi^{0,T} \in \mathcal{E}_0(D_T)$. For this, consider the Loewner chain driven by $(\hat{\rho}_t := \rho_{T+t})_{t \geq 0}$ and the associated operator $\hat{\varkappa}$. We obtain from the domain Markov property Lemma 2.3 that

$$\phi^{0,T} \circ f_T = \hat{\varkappa}[\iota[\phi](\cdot, \cdot + T)].$$

Then Proposition 3.3 shows that $\phi^{0,T} \circ f_T \in \mathcal{E}_0(\mathbb{D})$. Using conformal invariance, this yields $\phi^{0,T} \in \mathcal{E}_0(D_T)$.

We show that $\phi^{h,T}$ is harmonic in D_T . In fact, for $z \in D_T$, by definition,

$$\phi^{h,T}(z) = 2\pi \int_0^T P_{\mathbb{D}}[\iota[\phi]_t \rho_t](g_t(z)) dt.$$

Note that $P_{\mathbb{D}}[\iota[\phi]_t \rho_t](g_t(\cdot))$ is harmonic in D_T for $t < T$. Therefore, by the characterization of harmonic functions in terms of the mean value property and Fubini's theorem we obtain that $\phi^{h,T}$ is harmonic in D_T . \square

Theorem 3.5. *The operator $\iota : \mathcal{E}_0(\mathbb{D}) \rightarrow L^2(2\rho)$ is a (bijective) isometry.*

Proof. We already know that $\varkappa \circ \iota = \text{Id}_{\mathcal{E}_0(\mathbb{D})}$. To check that $\iota \circ \varkappa = \text{Id}_{L^2(2\rho)}$, it suffices to show that $\text{Ker}(\varkappa) = \{0\}$ in $L^2(2\rho)$. Indeed, for all $u \in L^2(2\rho)$, we have $\varkappa[u] = \varkappa \circ \iota \circ \varkappa[u]$. Then $\text{Ker}(\varkappa) = \{0\}$ implies that $u = \iota \circ \varkappa[u]$.

Now let $u \in \text{Ker}(\varkappa)$. Fix $T \geq 0$. By Corollary 3.4, $\varkappa[u\mathbf{1}_{S^1 \times [0, T]}] \in \mathcal{E}_0(\mathbb{D})$ and $\varkappa[u\mathbf{1}_{S^1 \times [T, \infty)}]$ give the orthogonal decomposition (2.1) of $\varkappa[u] = 0$ with respect to D_T . Therefore, for all $w \in D_T$,

$$\varkappa[u\mathbf{1}_{S^1 \times [0, T]}](w) = 2\pi \int_0^T P_{\mathbb{D}}[u_t \rho_t](g_t(w)) dt = 0.$$

Taking the derivative in T , we obtain that for a.e. $T \geq 0$ the function $P_{\mathbb{D}}[u_T \rho_T]$ vanishes in \mathbb{D} . (One can choose a dense and countable family J of points in \mathbb{D} , we have for a.e. $T \geq 0$ and $w \in D_T \cap J$, $P_{\mathbb{D}}[u_T \rho_T](g_T(w)) = 0$. Since the latter integral is harmonic thus continuous, it vanishes for a.e. $T \geq 0$ and all $w \in D_T$.) But this implies that the measure $u_T \rho_T$ on S^1 is the zero measure. Therefore, $u(\theta, T) = 0$ for ρ_T -a.e. $\theta \in S^1$. It follows that

$$\iint_{S^1 \times \mathbb{R}_+} |u(\theta, t)|^2 d\rho(\theta, t) = 0,$$

which proves $\text{Ker}(\varkappa) = \{0\}$ and this concludes the proof. \square

We will now discuss the relation between $\mathcal{E}_0(\mathbb{D})$ and functions defined on the leaves of a foliation. We will consider a generalization of the standard trace operator for $W^{1,2}$ on Lipschitz domains to chord-arc domains, see [30] and [62, Appx. A]. Suppose $\phi \in W_{\text{loc}}^{1,2}(\mathbb{C})$ and γ is a chord-arc curve in \mathbb{C} . The *Jonsson–Wallin trace* of ϕ on γ is defined for arclength-a.e. $z \in \gamma$ by the following limit of averages on balls $B(z, r) = \{w : |w - z| < r\}$

$$\phi|_{\gamma}(z) := \lim_{r \rightarrow 0^+} \frac{1}{|B(z, r)|} \int_{B(z, r)} \phi dA. \quad (3.9)$$

A function φ defined arclength-a.e. on all leaves of a foliation ($\gamma_t = \partial D_t$) is said to have an extension ϕ in $\mathcal{E}_0(\mathbb{D})$ if for all t , the Jonsson–Wallin trace of ϕ on γ_t (denoted $\phi|_{\gamma_t}$) coincides with φ arclength-a.e. (Here and below we identify $\phi \in \mathcal{E}_0(\mathbb{D})$ with the function in $W^{1,2}(\mathbb{C})$ that is equal to ϕ in \mathbb{D} and 0 in \mathbb{D}^* .)

Proposition 3.6. *If a function φ defined arclength-a.e. on each leaf of a foliation of \mathbb{D} has an extension to $\mathcal{E}_0(\mathbb{D})$, then the extension is unique.*

Proof. Let ϕ be an extension of φ in $\mathcal{E}_0(\mathbb{D})$. For a fixed $T \geq 0$, the orthogonal decomposition $\phi = \phi^{0, T} + \phi^{h, T}$ where $\phi^{0, T} \in \mathcal{E}_0(D_T)$ and $\phi^{h, T} \in \mathcal{E}_0(\mathbb{D})$ is harmonic in D_T is given by

$$\phi^{0, T} = \varkappa[\iota[\phi]\mathbf{1}_{S^1 \times [T, \infty)}] \quad \text{and} \quad \phi^{h, T} = \varkappa[\iota[\phi]\mathbf{1}_{S^1 \times [0, T]}],$$

by Corollary 3.4. We have that $\phi|_{\partial D_T} = \phi^{h, T}|_{\partial D_T}$ arclength-a.e. since they coincide in $\mathbb{C} \setminus D_T$, see [62, Lem. A.2]. Hence, φ determines $\phi^{h, T}|_{D_T}$ since arclength and harmonic measure are mutually absolutely continuous on chord-arc curves. Assume that $\tilde{\phi}$ also extends φ . Then we have for all $w \in D_T$,

$$\begin{aligned} 0 &= \phi^{h, T}(w) - \tilde{\phi}^{h, T}(w) = \varkappa \left[(\iota[\phi] - \iota[\tilde{\phi}])\mathbf{1}_{S^1 \times [0, T]} \right] (w) \\ &= 2\pi \int_0^T P_{\mathbb{D}}[\iota[\phi - \tilde{\phi}]_t \rho_t](g_t(w)) dt. \end{aligned}$$

The proof of Theorem 3.5 shows that $\iota[\phi] = \iota[\tilde{\phi}]$ in $L^2(2\rho)$. Therefore $\phi = \tilde{\phi}$ in $\mathcal{E}_0(\mathbb{D})$ and this completes the proof. \square

4 Loewner–Kufarev energy

4.1 Definitions

For each measure $\sigma \in \mathcal{M}(S^1)$ we define

$$L(\sigma) = \frac{1}{2} \int_{S^1} \nu'(\theta)^2 d\theta \quad (4.1)$$

if $d\sigma(\theta) = \nu^2(\theta) d\theta$ with the non-negative square-root density $\nu \in W^{1,2}(S^1)$ and, by convention, $L(\sigma) = \infty$ otherwise. With this definition, $L(\sigma)$ is the usual Dirichlet energy of ν on S^1 .

Remark. Note that $L(\sigma) = 0$ if and only if σ is the uniform measure on S^1 .

Then for $\rho \in \mathcal{N}_+$ (see Section 2.2), we define

$$S_+(\rho) = \int_0^\infty L(\rho_t) dt, \quad (4.2)$$

where $(\rho_t)_{t \geq 0}$ is a disintegration of ρ . We call $S_+(\rho)$ the *Loewner–Kufarev energy* of the measure ρ . When $L(\rho_t) < \infty$, we write $d\rho_t = \rho_t(\theta) d\theta = \nu_t^2(\theta) d\theta$.

It is also useful to define

$$S_{[a,b]}(\rho) = \int_a^b L(\rho_t) dt.$$

4.2 First properties

We now record a few simple properties that will be used in our proofs.

Lemma 4.1. *Suppose $S_+(\rho) < \infty$ and define the time-changed family of measures $(\tilde{\rho}_s = t'(s)\rho_{t(s)})_{s \geq 0}$, where*

$$t(s) = \int_0^s |\tilde{\rho}_u| du.$$

Then $S_+(\tilde{\rho}) = S_+(\rho)$. Moreover, if $(g_t^\rho)_{t \geq 0}$ and $(g_s^{\tilde{\rho}})_{s \geq 0}$ are the corresponding uniformizing Loewner chains, then $g_s^{\tilde{\rho}} = g_{t(s)}^\rho$.

Proof. We have

$$S_+(\tilde{\rho}) = \int_0^\infty L(\tilde{\rho}_t) dt = \int_0^\infty L(t'(s)\rho_t) ds = \int_0^\infty L(\rho_t) t'(s) ds = S_+(\rho).$$

On the other hand, the Loewner flow driven by $(\tilde{\rho}_s)$ is the solution $s \mapsto g_s^{\tilde{\rho}}(z)$ to

$$\partial_s g_s^{\tilde{\rho}}(z) = g_s^{\tilde{\rho}}(z) H[\tilde{\rho}_s](g_s^{\tilde{\rho}}(z)) \quad g_0^{\tilde{\rho}}(z) = z.$$

Using the definition of $\tilde{\rho}$, we therefore get

$$\partial_t g_s^{\tilde{\rho}}(z) = g_s^{\tilde{\rho}}(z) H[\rho_t](g_s^{\tilde{\rho}}(z))$$

and this implies $g_s^{\tilde{\rho}} = g_{t(s)}^\rho$. □

One should thus view $\tilde{\rho}$ as a time-reparametrization of ρ and the solution to the Loewner equation associated with the measure $\tilde{\rho}$ is a reparametrization of the solution associated to ρ . This invariance property of the Loewner–Kufarev energy suggests that the energy is intrinsic to the foliation (once we know that finite energy measures generate foliations) and does not depend on the particular time-parametrization. This is further reflected in the energy duality Theorem 1.7 expressing S_+ in terms of the winding function.

Lemma 4.2. *If $S_+(\rho) < \infty$, then for every $z \in \overline{\mathbb{D}} \setminus \{0\}$ we have $\tau(z) < \infty$.*

Proof. We claim that there exists $\varepsilon > 0$ such that for all $t \geq 0$, $L(\rho_t) < \varepsilon$ implies $\min_{\theta} \nu_t^2(\theta) > 1/4\pi$. Indeed, suppose without loss in generality that $\nu_t^2(0) \leq 1/4\pi$. Then by the mean value theorem and that $\rho_t \in \mathcal{M}_1(S^1)$, there is θ_0 such that $\nu_t^2(\theta_0) \geq 1/2\pi$ and we obtain easily by interpolating linearly ν_t on $[0, \theta_0]$ that $L(\rho_t) \geq (3 - 2\sqrt{2})/16\pi^2 =: \varepsilon$. Let $E := \{t \geq 0 : L(\rho_t) > \varepsilon\}$. For $t \notin E$ the Poisson integral of ρ_t is continuous on $\overline{\mathbb{D}}$ and by the maximum principle, $P_{\mathbb{D}}[\rho_t] \geq 1/4\pi$.

Now fix $z \in \overline{\mathbb{D}} \setminus \{0\}$. Using (2.5), for $t < \tau(z)$,

$$\begin{aligned} \log |g_t(z)/z| &= \int_0^t \partial_s \operatorname{Re} \log g_s(z) \, ds = \int_0^t \operatorname{Re} H_s(g_s(z)) \, ds \\ &= 2\pi \int_0^t P_{\mathbb{D}}[\rho_s](g_s(z)) \, ds \geq |[0, t] \setminus E|/2 \end{aligned}$$

since the Poisson integral is non-negative. Since $\log |g_t(z)/z| \leq -\log |z|$ it follows that

$$\tau(z) \leq -2 \log |z| + |E| \leq -2 \log |z| + S_+(\rho)/\varepsilon < \infty$$

where we also used Markov’s inequality and that $S_+(\rho) = \int_0^\infty L(\rho_t) dt$. \square

4.3 Examples and energy minimizers

We discussed the simple but important example of a Loewner chain driven by the constant zero-energy measure in the introduction and Section 2.2. Here we give another example of a time-homogeneous driving measure, this time vanishing at $\theta = 0$, and minimizing the Loewner–Kufarev energy among all such measures. We compute explicitly the winding function of the corresponding non-injective foliation and verify energy duality by hand. Even for this simple example, while the computation is straightforward it is not entirely trivial.

Let $T > 0$ and $\nu \in W^{1,2}([0, 2\pi])$ with $\nu(0) = \nu(2\pi) = 0$ and $\int_{S^1} \nu^2(\theta) \, d\theta = 1$. Consider the measure $\rho \in \mathcal{N}_+$ such that $d\rho_t(\theta) = \nu^2(\theta) \, d\theta$ for $t \in [0, T]$, and that is equal to the uniform measure on S^1 otherwise. Among all such measures, $\nu^2(\theta) \, d\theta$ minimizes L if and only if ν is the first eigenfunction of the Dirichlet Laplacian on $[0, 2\pi]$, namely, $\nu^2(\theta) = \sin^2(\theta/2)/\pi$. In this case, we have $L(\rho_t) \equiv 1/8$ so $S_+(\rho) = \int_0^T L(\rho_t) dt = T/8$. The corresponding Herglotz integral can be evaluated explicitly and the Loewner–Kufarev equation in this case is simply

$$\partial_t f_t(z) = -z f_t'(z) (1 - z), \quad z \in \mathbb{D},$$

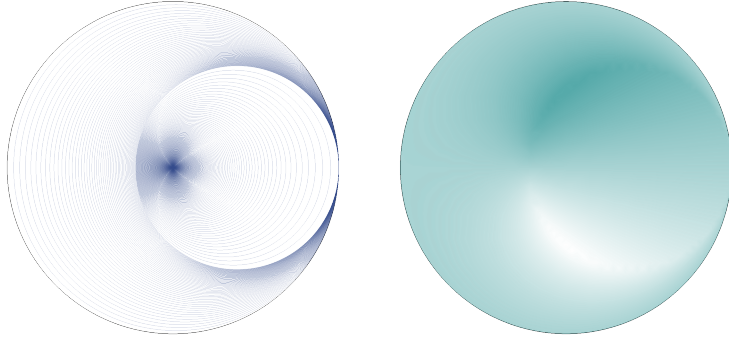


Figure 4.1: *Left*: Leaves of the foliation corresponding to the measure that equals $\pi^{-1} \sin^2(\theta/2) d\theta dt$ for $0 \leq t < 1$ and is uniform for $t \geq 1$, drawn at equidistant times. *Right*: Density plot of the winding function corresponding to the same measure, where lighter color represents a larger value. The winding function is harmonic, but non-zero, in the part foliated after time 1.

see, e.g., [60] where solutions to Loewner equations of this form were studied. We have

$$f_t(z) = \frac{e^{-t}z}{1 - (1 - e^{-t})z}.$$

The leaf ∂D_t is a circle of radius $1/(2 - e^{-t})$ centered at $(1 - e^{-t})/(2 - e^{-t})$ and the hull K_t at time t is a crescent-shaped compact set, see Figure 4.1. In particular, $z = 0$ and $z = 1$ are fixed points of the evolution. We now compute the corresponding winding function. Suppose $z \in K_T$ and let $t = \tau(z)$. Then

$$\varphi(z) = \arg \frac{zg'_t(z)}{g_t(z)} = -\arg(z(1 - e^{-t}) + e^{-t}) = \arg \frac{z}{(z - 1)^2} + \pi.$$

Moreover, for $z \in D_T$, we have

$$\varphi(z) = -\arg(z(1 - e^{-T}) + e^{-T}),$$

which is harmonic in D_T . From these formulas one can verify directly that $\mathcal{D}_{\mathbb{D}}(\varphi) = 2T = 16S_+(\rho)$, which is the statement of energy duality in this case. We note that the computation of $\mathcal{D}_{\mathbb{D}}(\varphi)$ is slightly technical as $\nabla\varphi$ has a singularity at 1 (that is L^2 -integrable).

Remark. Conjugating f_t by $z \mapsto z^m$, it is also possible carry out similar explicit computations for measures of the form $\pi^{-1} \sin^2(m\theta/2)d\theta dt$ that correspond to Laplace eigenfunctions on S^1 corresponding to higher eigenvalues.

5 Weak energy duality and foliation by Weil–Peterson quasicircles

In this section we prove energy duality in the disk, for measures satisfying a strong smoothness assumption. We then use this result and an approximation argument to

prove that any ρ with $S_+(\rho) < \infty$, produces interfaces that are Weil–Peterson quasicircles which form a foliation of $\overline{\mathbb{D}} \setminus \{0\}$.

5.1 Weak energy duality

Let \mathcal{N}_+^∞ denote the set of $\rho \in \mathcal{N}_+$ such that:

1. There exists $T < \infty$, such that for $t > T$, $\rho_t = d\theta/2\pi$ is the uniform measure on S^1 .
2. The mapping $t \mapsto \rho_t$ is piecewise constant (on finitely many time intervals).
3. For all $t \geq 0$, ρ_t has a C^∞ -smooth and strictly positive density with respect to $d\theta$.

Lemma 5.1. *Any $\rho \in \mathcal{N}_+^\infty$ generates a foliation of $\overline{\mathbb{D}} \setminus \{0\}$ in which each leaf is a smooth Jordan curve. The corresponding winding function φ is continuous and piecewise smooth on $\overline{\mathbb{D}}$.*

Proof. We assume first that $\rho \in \mathcal{N}_+^\infty$ and that ρ_t is constant on $[0, T]$. Let $H := H[\rho_t]$ which is constant for $t \leq T$. The family $(f_t)_{t \in [0, T]}$ solves the equation $\partial_t f_t(z) = -z f_t'(z) H(z)$ with $f_0(z) = z$ and forms a semigroup of conformal maps with fixed point 0. (See Lemma 2.3.) The solution therefore enjoys the following representation: there exists a starlike³ domain D and a conformal map $\psi : \mathbb{D} \rightarrow D$ fixing 0 such that

$$f_t(z) = \psi^{-1}(e^{-t}\psi(z)), \quad \psi(z) = z \exp\left(\int_0^z \frac{1}{w} \left(\frac{1}{H(w)} - 1\right) dw\right).$$

See, e.g., [58, Sec. 3] and the references therein. Since H is the Herglotz integral of a smooth, positive function, the maximum principle implies that it is non-zero on $\overline{\mathbb{D}}$ and the smoothness implies that all derivatives exist and extend continuously to $\overline{\mathbb{D}}$. (See, e.g., [22, Cor. II.3.3].) It follows that ψ is smooth on $\overline{\mathbb{D}}$. The foliation generated by ρ is given by the family $(\gamma_t = \psi^{-1}(e^{-t}\psi(S_1)))$ for $t \leq T$, and for $t > T$, the leaves γ_t are equipotential curves in D_T by Lemma 2.9. We now verify the smoothness of the winding function $z \mapsto \varphi(z)$ in K_T . For this, consider $(\theta, t) \in S^1 \times [0, T]$ and set $z(\theta, t) = f_t(e^{i\theta})$. Then $\varphi(z(\theta, t)) = -\arg e^{i\theta} f_t'(e^{i\theta})/f_t(e^{i\theta})$. We have

$$\begin{aligned} \partial_\theta z &= i e^{i\theta} f_t'(e^{i\theta}) \\ \partial_t z &= \partial_t f_t(e^{i\theta}) = e^{i\theta} f_t'(e^{i\theta}) H(e^{i\theta}). \end{aligned}$$

Since $\operatorname{Re} H > 0$ the Jacobian is non-zero. Further differentiation using the Loewner equation shows that $z(\theta, t)$ is smooth and by the inverse function theorem, the inverse is also smooth in K_T , and so is φ . Since $\varphi|_{D_T} = \vartheta(g_T)$ by Lemma 2.9 and is continuous on $\overline{D_T}$, we obtain that φ is continuous on $\overline{\mathbb{D}}$ and smooth in $\overline{\mathbb{D}} \setminus \gamma_T$.

Using Lemma 2.3 and the fact the smoothness is preserved under composition of smooth conformal mappings, the statement immediately generalizes to arbitrary $\rho \in \mathcal{N}_+^\infty$. \square

Lemma 5.2. *If $\rho \in \mathcal{N}_+$ and $z \in \mathbb{D}$, then $-\partial_t \vartheta[f_t](z) |_{t=0} = \operatorname{Im} z H_0'(z)$.*

³Starlike here means that D contains 0 and for every $z \in D$, the line segment $[0, z] \subset D$.

Proof. Using (2.4), we have for $t \geq 0$,

$$\partial_t \log f_t(z) = -z \frac{f'_t(z)}{f_t(z)} H_t(z).$$

and

$$\partial_t \log f'_t(z) = -\frac{\partial_z(z f'_t(z) H_t(z))}{f'_t(z)} = -\left(1 + z \frac{f''_t(z)}{f'_t(z)}\right) H_t(z) - z H'_t(z).$$

Hence,

$$\partial_t \log \frac{z f'_t(z)}{f_t(z)} = -\left(1 + z \frac{f''_t(z)}{f'_t(z)} - z \frac{f'_t(z)}{f_t(z)}\right) H_t(z) - z H'_t(z).$$

Since $f_0(z) = z$, we have $f''_0(z) = 0$ and the claimed expression when evaluated at $t = 0$ follows by taking the imaginary part. \square

Proposition 5.3 (Weak energy duality). *Suppose $\rho \in \mathcal{N}_+^\infty$ and let φ be the associated winding function. Then $\mathcal{D}_\mathbb{D}(\varphi) = 16S_+(\rho)$.*

Proof. By Lemma 5.1 the winding function φ is continuous and piecewise smooth on \mathbb{D} . Moreover, $\varphi|_{S^1} = 0$ and therefore $\varphi \in \mathcal{E}_0(\mathbb{D})$ [10, Thm. 9.17] and we may apply the foliation disintegration isometry to φ . We claim that for $t \in [0, T]$, $\iota[\varphi](\theta, t) = -2\nu'(\theta)/\nu(\theta)$. Given this, we can apply Theorem 3.5 to conclude the proof. To prove the claim, set $\varphi_t = \varphi \circ f_t$ and recall that by definition

$$\begin{aligned} \iota[\varphi](\theta, t) &= \frac{1}{2\pi} \int_{\mathbb{D}} \Delta \varphi_t(z) P_{\mathbb{D}}(z, e^{i\theta}) \, dA(z) \\ &= \frac{1}{2\pi} \int_{\mathbb{D}} \Delta \varphi_t(z) \partial_{n(e^{i\theta})} G_{\mathbb{D}}(z, e^{i\theta}) \, dA(z). \end{aligned} \quad (5.1)$$

On the other hand, by Lemma 5.1, $\Delta \varphi_t$ is smooth, so the last term in (5.1) equals

$$\frac{1}{2\pi} \partial_n \int_{\mathbb{D}} \Delta \varphi_t(z) G_{\mathbb{D}}(z, e^{i\theta}) \, dA(z) = \partial_n \varphi_t^0(e^{i\theta}).$$

Recall from Lemma 2.9 and Lemma 2.10 that $z \mapsto \varphi_t^0(z)$ is the winding function generated by the measures $s \mapsto \rho_{t+s}$ for $s \geq 0$. Hence, it suffices to consider $t = 0$ and show that for a.e. $\theta \in [0, 2\pi)$,

$$\partial_n \varphi_0^0(e^{i\theta}) = \partial_n \varphi_0(e^{i\theta}) = -2\nu'(\theta)/\nu_0(\theta). \quad (5.2)$$

We know that ∂D_t is C^∞ for all t , and all (complex) derivatives of f_t are continuous on $\overline{\mathbb{D}}$ and since the Herglotz integral H_t is also continuous on $\overline{\mathbb{D}}$, so is $\partial_t f_t(z) = -z f'_t(z) H_t(z)$. It follows that the normal velocity (with respect to ∂D_t) of the interface at the point $f_t(e^{i\theta})$ can be written

$$\begin{aligned} \text{vel}_n(t, \theta) &= -\text{Re} \left(\frac{\partial_t f_t(e^{i\theta}) \overline{e^{i\theta} f'_t(e^{i\theta})}}{|f'_t(e^{i\theta})|} \right) \\ &= \text{Re} \left(\frac{e^{i\theta} f'_t(e^{i\theta}) H_t(e^{i\theta}) \overline{e^{i\theta} f'_t(e^{i\theta})}}{|f'_t(e^{i\theta})|} \right) \\ &= |f'_t(e^{i\theta})| \text{Re} H_t(e^{i\theta}) = 2\pi |f'_t(e^{i\theta})| \nu_t(\theta)^2 > 0. \end{aligned} \quad (5.3)$$

In particular, at $t = 0$ the normal velocity of the interface at $e^{i\theta}$ equals $2\pi\nu_0(\theta)^2$. On the other hand, using the chain rule (2.8),

$$\begin{aligned} 2\pi\nu_0(\theta)^2\partial_n\varphi_0(e^{i\theta}) &= \partial_t[\varphi_0(f_t(e^{i\theta}))]|_{t=0} = \partial_t[\vartheta[g_t](f_t(e^{i\theta}))]|_{t=0} \\ &= -\partial_t\left(\vartheta[f_t](e^{i\theta})\right)|_{t=0}. \end{aligned}$$

It follows from Lemma 5.2 that

$$-\partial_t\left(\vartheta[f_t](e^{i\theta})\right)|_{t=0} = \operatorname{Im}(e^{i\theta}H'_0(e^{i\theta})) = -4\pi\nu_0(\theta)\nu'_0(\theta).$$

The last identity is not hard to verify by hand, using integration by parts and the smoothness of ν_0 , see also Lemma 6.2. This proves (5.2) and concludes the proof. \square

We will use the following special case of the generalized Grunsky inequality, which provides a useful bound to control $\mathcal{D}_{\mathbb{D}}(\arg f'_t)$ in terms of $\mathcal{D}_{\mathbb{D}}(\vartheta[f_t])$ (see Corollary 5.6).

Lemma 5.4 (See [61, P. 70-71]). *Suppose that $f : \mathbb{D} \rightarrow \mathbb{C}$ and $h : \mathbb{D}^* \rightarrow \mathbb{C}$ are univalent functions on \mathbb{D} and \mathbb{D}^* such that $f(0) = 0$ and $h(\infty) = \infty$, and $f(\mathbb{D}) \cap h(\mathbb{D}^*) = \emptyset$. Then we have*

$$\int_{\mathbb{D}} \left| \frac{f'(z)}{f(z)} - \frac{1}{z} \right|^2 dA(z) + \int_{\mathbb{D}^*} \left| \frac{h'(z)}{h(z)} - \frac{1}{z} \right|^2 dA(z) \leq 2\pi \log \left| \frac{h'(\infty)}{f'(0)} \right|.$$

Equality holds if the omitted set $\mathbb{C} \setminus \{f(\mathbb{D}) \cup h(\mathbb{D}^)\}$ has Lebesgue measure zero.*

Lemma 5.5. *Let $\rho \in \mathcal{N}_+$ and $(f_t)_{t \geq 0}$ is the associated Loewner chain. Then*

$$\mathcal{D}_{\mathbb{D}}(\arg[f_t(z)/z]) = \frac{1}{\pi} \int_{\mathbb{D}} \left| \frac{f'_t(z)}{f_t(z)} - \frac{1}{z} \right|^2 dA(z) \leq 2t. \quad (5.4)$$

Proof. Since D_t and \mathbb{D}^* are disjoint, and since $f'_t(0) = e^{-t}$, Lemma 5.4 applied to the pair $(f_t, \operatorname{Id}_{\mathbb{D}^*})$ shows that

$$\int_{\mathbb{D}} \left| \frac{f'_t(z)}{f_t(z)} - \frac{1}{z} \right|^2 dA(z) \leq -2\pi \log |f'_t(0)| = 2\pi t$$

as claimed. \square

Corollary 5.6. *Suppose $\rho \in \mathcal{N}_+^\infty$. Then for all $t \geq 0$, $\mathcal{D}_{\mathbb{D}}(\arg f'_t) \leq 32S_{[0,t]}(\rho) + 4t$.*

Proof. Fix $t \geq 0$. Without loss of generality, we assume that ρ_s is the uniform measure for all $s > t$. Combining Proposition 5.3, Lemma 5.5, and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \mathcal{D}_{\mathbb{D}}(\arg f'_t) &= \mathcal{D}_{\mathbb{D}}(\vartheta[f_t] + \arg[f_t(z)/z]) \leq 2\mathcal{D}_{D_t}(\vartheta[g_t]) + 2\mathcal{D}_{\mathbb{D}}(\arg[f_t(z)/z]) \\ &\leq 2\mathcal{D}_{\mathbb{D}}(\varphi) + 4t = 32S_{[0,t]}(\rho) + 4t \end{aligned}$$

as claimed. \square

5.2 Foliation by Weil–Petersson quasicircles

This section proves that the interfaces generated by a finite energy measure are Weil–Petersson quasicircles that together form a foliation (Corollary 5.12). We show first that each leaf ∂D_t is a Weil–Petersson quasicircle. For this, we need the following quantitative upper bound for $\mathcal{D}_{\mathbb{D}}(\arg f'_t)$ that depends only on $S_+(\rho)$ and t .

Lemma 5.7. *If $S_+(\rho) < \infty$, then for all $t \geq 0$, $\mathcal{D}_{\mathbb{D}}(\arg f'_t) \leq 32S_{[0,t]}(\rho) + 4t$.*

Proof. Fix $t > 0$. We will approximate ρ by a sequence of measures $\rho^{(k)} \in \mathcal{N}_+^\infty$ which converges to ρ weakly, and such that $S_{[0,t]}(\rho^{(k)}) \leq S_{[0,t]}(\rho)$. The corresponding sequence of conformal maps $f_t^{(k)}$ then converges uniformly on compacts to f_t by Lemma 2.2, which, by Corollary 5.6, implies

$$\mathcal{D}_{\mathbb{D}}(\arg f'_t) \leq \liminf_{k \rightarrow \infty} \mathcal{D}_{\mathbb{D}}(\arg (f_t^{(k)})') \leq 32 S_{[0,t]}(\rho^{(k)}) + 4t \leq 32 S_{[0,t]}(\rho) + 4t.$$

We construct the approximation in two steps. We first let $\rho^{(n)}$ be the “time-averaged” measure, which is piecewise constant on dyadic time intervals $[jt/2^n, (j+1)t/2^n)$ and is defined by

$$\rho_s^{(n)} := \sigma^j := \frac{2^n}{t} \int_{jt/2^n}^{(j+1)t/2^n} \rho_r \, dr \in \mathcal{M}_1(S^1), \quad \forall s \in \left[\frac{jt}{2^n}, \frac{(j+1)t}{2^n} \right).$$

To see that $S_{[0,t]}(\rho^{(n)}) \leq S_{[0,t]}(\rho)$, the key observation is that the map $\sigma \mapsto L(\sigma)$ from $\mathcal{M}_1(S^1)$ to $[0, \infty]$ is convex [16] (see also [5, Thm. 3.4]). We can therefore apply Jensen’s inequality:

$$\begin{aligned} S_{[0,t]}(\rho^{(n)}) &= \frac{t}{2^n} \sum_{j=0}^{2^n-1} L \left(\frac{2^n}{t} \int_{jt/2^n}^{(j+1)t/2^n} \rho_r \, dr \right) \\ &\leq \sum_{j=0}^{2^n-1} \int_{jt/2^n}^{(j+1)t/2^n} L(\rho_t) \, dt = S_{[0,t]}(\rho). \end{aligned}$$

It is clear that $\rho^{(n)}$ restricted to $S^1 \times [0, t]$ converges weakly to ρ : integrating against a continuous function u on $S^1 \times [0, t]$, which is then uniformly continuous, $\int u \, d\rho^{(n)}$ converges to $\int u \, d\rho$.

Since $\rho^{(n)}$ might not be strictly positive and smooth, the second step is to approximate each $\sigma = \sigma^j$ of the 2^n measures in $\mathcal{M}_1(S^1)$ on dyadics by

$$d\sigma_r(\theta) := \frac{d\theta}{2\pi} \int_{\xi \in S^1} P_{\mathbb{D}}(re^{i\theta}, e^{i\xi}) d\sigma(\xi)$$

where $r < 1$, which has positive and smooth density with respect to $d\theta$.

Now we prove that $L(\sigma_r) \leq L(\sigma)$. Let $f \in C^0(S^1)$,

$$\begin{aligned} \sigma_r(f) &:= \int_{S^1} f(\theta) d\sigma_r(\theta) = \frac{1}{2\pi} \int_{\theta \in S^1} f(\theta) \int_{\xi \in S^1} P_{\mathbb{D}}(re^{i\theta}, e^{i\xi}) d\sigma(\xi) d\theta \\ &= \frac{1}{2\pi} \int_{\xi \in S^1} \int_{\theta \in S^1} f(\theta) P_{\mathbb{D}}(re^{i(\theta-\xi)}, 1) d\theta d\sigma(\xi) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{\xi \in S^1} \int_{\eta \in S^1} f(\eta + \xi) P_{\mathbb{D}}(re^{i\eta}, 1) d\eta d\sigma(\xi) \\
&= \frac{1}{2\pi} \int_{\eta \in S^1} P_{\mathbb{D}}(re^{i\eta}, 1) \int_{\xi \in S^1} f(\eta + \xi) d\sigma(\xi) d\eta \\
&= \frac{1}{2\pi} \int_{\eta \in S^1} P_{\mathbb{D}}(r, e^{i\eta}) \eta_* \sigma(f) d\eta,
\end{aligned}$$

where $\eta_* \sigma$ is the pull-back measure σ by the rotation $\xi \mapsto \xi + \eta$. In particular, $L(\sigma) = L(\eta_* \sigma)$. We obtain

$$\sigma_r = \frac{1}{2\pi} \int_{\eta \in S^1} P_{\mathbb{D}}(r, e^{i\eta}) \eta_* \sigma d\eta.$$

Since $\frac{1}{2\pi} \int_{\eta \in S^1} P_{\mathbb{D}}(r, e^{i\eta}) d\eta = 1$, σ_r is a probability measure. Using convexity and Jensen's inequality once again,

$$L(\sigma_r) \leq \frac{1}{2\pi} \int_{\eta \in S^1} P_{\mathbb{D}}(r, e^{i\eta}) L(\eta_* \sigma) d\eta = L(\sigma).$$

Finally, note that since f is continuous, $\eta \mapsto \eta_* \sigma(f)$ is continuous on S^1 and equal to $\sigma(f)$ for $\eta = 0$. Therefore, since $\sigma_r(f)$ is the Poisson integral of $\eta \mapsto \eta_* \sigma(f)$ evaluated at r , it follows that $\lim_{r \rightarrow 1} \sigma_r(f) = \sigma(f)$. Hence σ_r converges to σ weakly and this completes the proof. \square

We would now like to use Lemma 5.7 to conclude that ∂D_t is a Weil–Petersson quasicircle. However, we cannot directly apply Lemma 2.1 since we do not know *a priori* that ∂D_t is a Jordan curve. In fact, it is not hard to construct an example of a simply connected domain for which the boundary is self-touching, while $\mathcal{D}_{\mathbb{D}}(\log f') < \infty$ where f is a conformal map onto the domain. In the present case, however, we can use the fact that ∂D_t arises from Loewner evolution: we will consider the evolution for a small time interval and use estimates on the Schwarzian combined with a result of Ahlfors–Weill. We can then complete the proof using Lemma 2.3.

For a function f holomorphic at z such that $f'(z) \neq 0$, recall that the Schwarzian derivative of f at z is defined by

$$\mathcal{S}f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2.$$

Lemma 5.8 (See [61, Lem. I.2.1, Lem. II.1.3 and Lem. II.1.5]). *There exists $\delta > 0$ such that if $\mathcal{D}_{\mathbb{D}}(\log f') < \delta$, then f is univalent and $f(\mathbb{D})$ is a Jordan domain bounded by a Weil–Petersson quasicircle.*

Proof. By Lemma 2.1 and the assumption that $\mathcal{D}_{\mathbb{D}}(\log f') < \delta < \infty$, it is enough to prove that $f(\mathbb{D})$ is a Jordan domain. We will prove more and show that $f(\mathbb{D})$ is a quasidisk. By a theorem of Ahlfors–Weill (see [46, Cor. 5.24]), to show that $f(\mathbb{D})$ is a quasidisk it suffices to show that for small enough δ ,

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |\mathcal{S}f(z)| < 2. \tag{5.5}$$

We will estimate the left-hand side of (5.5) in terms of $\mathcal{D}_{\mathbb{D}}(\log f')$. The required estimate is a combination of two bounds. First we claim that if f is holomorphic on \mathbb{D} and $f' \neq 0$, then,

$$\int_{\mathbb{D}} |\mathcal{S}f(z)|^2 (1 - |z|^2)^2 dA(z) \leq \pi \mathcal{D}_{\mathbb{D}}(\log f') + \frac{\pi}{8} (\mathcal{D}_{\mathbb{D}}(\log f'))^2.$$

Indeed, this follows from the proof of [61, Lem. II.1.5] where the bound depends on a constant from [61, Lem. II.1.3]. On the other hand, it follows directly from [61, Lem. I.2.1] that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |\mathcal{S}f(z)| \leq \sqrt{\frac{12}{\pi}} \left(\int_{\mathbb{D}} |\mathcal{S}f(z)|^2 (1 - |z|^2)^2 dA(z) \right)^{1/2}.$$

Combining these bounds we see that (5.5) indeed holds provided δ is chosen sufficiently small. \square

Proposition 5.9. *If $S_+(\rho) < \infty$, then for all $t \geq 0$, ∂D_t is a Weil–Petersson quasicircle.*

Proof. Let $\delta > 0$ be the small constant from Lemma 5.8. Pick $t_1 > 0$ such that $32S_{[0, t_1]}(\rho) + 4t_1 < \delta$. For $t \leq t_1$, Lemma 5.7 and Lemma 5.8 show that ∂D_t is a Weil–Petersson quasicircle.

Now we consider the general case $t \geq 0$. Lemma 5.7 shows that $\mathcal{D}_{\mathbb{D}}(\arg f'_t)$ is finite, so it suffices to prove that D_t is a Jordan domain to conclude that ∂D_t is a Weil–Petersson quasicircle by Lemma 2.1. For this, let $0 = t_0 < t_1 < t_2 < \dots$ be a sequence tending to ∞ such that $32S_{[t_j, t_{j+1}]}(\rho) + 4(t_{j+1} - t_j) < \delta$. We show by induction that f_t is a homeomorphism from \mathbb{D} onto \overline{D}_t for all $t \leq t_j$. We have already proved this in the case $j = 1$. Assume it is true for all $t \leq t_j$. Using Lemma 2.3 and the choices of δ and $|t_{j+1} - t_j|$ we obtain that $g_{t_j}(\partial D_t)$ is a Weil–Petersson quasicircle for each $t_j \leq t \leq t_{j+1}$, in particular a Jordan curve. Since by assumption f_{t_j} is a homeomorphism of \mathbb{D} , $\partial D_t = f_{t_j} \circ g_{t_j}(\partial D_t)$ is also a Jordan curve and this completes the induction. \square

We now show that $t \mapsto \gamma_t$ is continuous, where γ_t is viewed as a parametrized curve $S^1 \rightarrow \gamma_t$ by $\theta \mapsto f_t(e^{i\theta})$. We will need the following lemma.

Lemma 5.10. *Suppose D is a simply connected domain containing 0 and let $f : \mathbb{D} \rightarrow D$ be a conformal map with $f(0) = 0$ and assume that $\mathcal{D}_{\mathbb{D}}(\log f') < \infty$. There exists a constant $C < \infty$ depending only on $\mathcal{D}_{\mathbb{D}}(\log f')$ such that*

$$|f'(re^{i\theta})| \leq C|f'(0)| \exp \sqrt{C \log(1-r)^{-1}}. \quad (5.6)$$

Remark. Note that it is not assumed that D is a Jordan domain. The estimate (5.6) easily implies that f is continuous on $\overline{\mathbb{D}}$.

Proof. We may assume $f'(0) = 1$. Any ϕ that is holomorphic in \mathbb{D} and such that $\mathcal{D}_{\mathbb{D}}(\phi) < \infty$ has non-tangential limits a.e. on S^1 , and writing ϕ for that function as well we have the following weak-type estimate: there exist universal constants C_1, C_2 such that $|\{\theta \in S^1 : |\phi| > \lambda\}| \leq C_1 e^{-C_2 \lambda^2 / (|\phi(0)|^2 + \mathcal{D}_{\mathbb{D}}(\phi))}$. See [21, Cor. 3.3.2] for a proof. We apply this estimate with $\phi = \log f'$ which is normalized so that $\log f'(0) = 0$. Hence the upper bound in the weak-type estimate depends only on $\mathcal{D}_{\mathbb{D}}(\log f')$ and implies there

exist C_1, C_2 depending only on $\mathcal{D}_{\mathbb{D}}(\log f')$ (but which, in what follows, are allowed to change from line to line) such that

$$\int_0^{2\pi} \exp\left(C_1 |\log |f'(e^{i\theta})||^2\right) d\theta \leq C_2.$$

Since $\log |f'(z)|$ is harmonic, $\exp(C_1 |\log |f'(z)||^2)$ is subharmonic and it follows that for $0 \leq r < 1$

$$\int_0^{2\pi} \exp\left(C_1 |\log |f'(re^{i\theta})||^2\right) d\theta \leq C_2.$$

Therefore, if $r_n = 1 - 2^{-n}$ and $z_{k,n} = r_n e^{i2\pi k/2^n}$, using Koebe's distortion theorem (see, e.g., [20, Ch. 2.3]), there is a universal constant $C_3 < \infty$ such that, taking C_2 larger if necessary,

$$\frac{1}{2^n} \sum_{k=1}^{2^n} \exp\left(C_1 (|\log |f'(z_{k,n})||^2)\right) \leq \int_0^{2\pi} \exp\left(C_1 (|\log |f'(r_n e^{i\theta})|| + C_3)^2\right) d\theta \leq C_2.$$

Whence,

$$|f'(z_{k,n})| \leq \exp \sqrt{C_1^{-1} \log(C_2 2^n)}.$$

Using the distortion theorem again we deduce

$$|f'(re^{i\theta})| \leq C \exp \sqrt{C \log(1-r)^{-1}},$$

where C depends only on $\mathcal{D}_{\mathbb{D}}(\log f')$, as claimed. \square

If $(f_t)_{t \geq 0}$ is the Loewner chain generated by a finite energy measure ρ then by Lemma 5.7 we have $\mathcal{D}_{\mathbb{D}}(\log |f'_t|) \leq 32S_+(\rho) + 4T$ for all $t \leq T$. Therefore, by Lemma 5.10, if $\sigma(x) := C \exp \sqrt{C \log(x)}$ where C depends only on $S_+(\rho)$ and T , and

$$|f'_t(re^{i\theta})| \leq |f'_t(0)| \sigma(1/(1-r)) \leq \sigma(1/(1-r)). \quad (5.7)$$

In the rest of the section we use the conformal parametrization of the leaves, namely $\gamma_t(\theta) := f_t(e^{i\theta})$.

Proposition 5.11. *Suppose $S_+(\rho) < \infty$. Then the function $t \mapsto (\gamma_t : S^1 \rightarrow \mathbb{C}) : [0, \infty) \rightarrow (C^0, \|\cdot\|_{\infty})$ is continuous.*

Proof. Throughout C denotes a constant whose value is allowed to change from line to line. Fix any $T < \infty$. Then if $0 \leq s \leq t \leq T$, for any $0 \leq r < 1$,

$$|\gamma_s(\theta) - \gamma_t(\theta)| \leq |\gamma_s(\theta) - f_s(re^{i\theta})| + |\gamma_t(\theta) - f_t(re^{i\theta})| + |f_s(re^{i\theta}) - f_t(re^{i\theta})|. \quad (5.8)$$

By integrating (5.7) we have

$$|\gamma_s(\theta) - f_s(re^{i\theta})| + |\gamma_t(\theta) - f_t(re^{i\theta})| \leq C(1-r)\sigma(1/(1-r)), \quad (5.9)$$

where σ is a subpower function depending only on $S_+(\rho)$ and T . Using the Loewner equation and once again (5.6), we have for $z = re^{i\theta}$,

$$|f_s(re^{i\theta}) - f_t(re^{i\theta})| = \left| \int_s^t z f'_u(z) H_u(z) du \right| \leq C\sigma(1/(1-r)) \int_s^t |H_u(z)| du.$$

Since for a.e. t , $d\rho_t(\theta) = \nu_t(\theta)^2 d\theta$, we can estimate using the Cauchy-Schwarz inequality

$$|\nu_t(\theta_1)^2 - \nu_t(\theta_2)^2| \leq 2\|\nu_t\|_\infty \|\nu_t'\|_{L^2} |\theta_1 - \theta_2|^{1/2} \leq M_t |\theta_1 - \theta_2|^{1/2},$$

where

$$M_t := 2(1/\sqrt{2\pi} + \sqrt{2\pi}\|\nu_t'\|_{L^2})\|\nu_t'\|_{L^2}.$$

Indeed, since ν_t is continuous and $\int \nu_t^2 = 1$, we can assume that $\nu_t(0) < 1/\sqrt{2\pi}$. Then $|\nu_t(\theta) - \nu_t(0)| \leq \sqrt{\theta}\|\nu_t'\|_{L^2} \leq \sqrt{2\pi}\|\nu_t'\|_{L^2}$, so $\|\nu_t\|_\infty \leq 1/\sqrt{2\pi} + \sqrt{2\pi}\|\nu_t'\|_{L^2}$. We claim that for a.e. t , $\sup_{z \in \mathbb{D}} |H_t(z)| \leq CM_t$. Since $\operatorname{Re} H_t(z) = 2\pi P_{\mathbb{D}}[\rho_t](z)$, we have $|H_t'(re^{i\theta})| \leq CM_t(1-r)^{-1/2}$. (See, e.g., [19, Thm. 5.8 and 5.1].) So by integration, the claim follows. Consequently $\int_s^t |H_u(z)| du \leq C \int_s^t M_u du$. Hence

$$|f_s(re^{i\theta}) - f_t(re^{i\theta})| \leq C\sigma(1/(1-r)) \int_s^t M_u du. \quad (5.10)$$

Now we choose r so that $1-r = \int_s^t M_u du \wedge 1$ and plug in (5.9) and (5.10) into (5.8) to conclude that

$$\sup_{\theta \in [0, 2\pi)} |\gamma_s(\theta) - \gamma_t(\theta)| = o(1)$$

as $|t-s| \rightarrow 0$. Since $T < \infty$ was arbitrary, this completes the proof. \square

Remark. Under the stronger assumption that $L(\rho_t)$ is uniformly bounded, the proof of Lemma 5.10 shows that the mapping $t \mapsto (\gamma_t : S^1 \rightarrow \mathbb{C}) : [0, \infty) \rightarrow (C^0, \|\cdot\|_\infty)$ is weakly Lipschitz continuous, that is, it admits a modulus of continuity of the form $|\cdot| \sigma(1/|\cdot|)$, where σ is a subpower function.

We are now ready to prove the main result of this section.

Corollary 5.12. *If $S_+(\rho) < \infty$, then ρ generates a foliation of $\overline{\mathbb{D}} \setminus \{0\}$ by Weil–Petersson quasicircles.*

Proof. Proposition 5.9 shows γ_t is a Weil–Petersson quasicircle for each t and $t \mapsto \gamma_t$ is continuous in the supremum norm for the conformal parametrization by Proposition 5.11. Lemma 4.2 shows that $\tau(z) < \infty$ for all $z \in \overline{\mathbb{D}} \setminus \{0\}$ and this completes the proof. \square

The next result is not used in the rest of the paper, but as it is an interesting and immediate consequence of Lemma 5.10, we choose to state it here.

Corollary 5.13. *Suppose D is a simply connected domain containing 0 and let $f : \mathbb{D} \rightarrow D$ be a conformal map with $f(0) = 0$ and assume that $\mathcal{D}_{\mathbb{D}}(\log f') < \infty$. Then the conformal parametrization of ∂D is weakly Lipschitz continuous on S^1 with subpower function depending only on $\mathcal{D}_{\mathbb{D}}(\log f')$.*

Remark. The condition $\mathcal{D}_{\mathbb{D}}(\log f') < \infty$ allows f' to be unbounded and the conformal parametrization is not Lipschitz in general in this setting, so up to the exact form of the subpower function this modulus of continuity is sharp.

Proof. We have already noted that f is continuous on $\overline{\mathbb{D}}$. By Lemma 5.10, we have for $0 < r < 1$,

$$\begin{aligned} |f(e^{i\theta_1}) - f(e^{i\theta_2})| &\leq |f(e^{i\theta_1}) - f(re^{i\theta_1})| + |f(e^{i\theta_2}) - f(re^{i\theta_2})| + |f(re^{i\theta_1}) - f(re^{i\theta_2})| \\ &\leq 2 \int_r^1 \sigma(1/(1-u)) du + \sigma(1/(1-r)) |\theta_1 - \theta_2| \\ &\leq ((1-r) + |\theta_1 - \theta_2|) \tilde{\sigma}(1/(1-r)), \end{aligned}$$

where $\tilde{\sigma}$ is a subpower function that depends only on $\mathcal{D}_{\mathbb{D}}(\log f')$. Now choose $r = 1 - |\theta_1 - \theta_2|$ and we obtain the desired estimate. \square

6 Disk energy duality: proof of Theorem 1.7

The proof of Theorem 1.7 is completed at the end of the section. We assume that $\rho \in \mathcal{N}_+$ generates a foliation of $\overline{\mathbb{D}} \setminus \{0\}$ throughout. The proof is carried out in two steps: in Section 6.1 we assume $S_+(\rho) < \infty$ and derive energy duality. Then in Section 6.2 we assume $\mathcal{D}_{\mathbb{D}}(\varphi) < \infty$ and prove that this implies $S_+(\rho) < \infty$. An overview of the argument presented in this section was provided in Section 1.5.

6.1 $S_+(\rho) < \infty$ implies $\mathcal{D}_{\mathbb{D}}(\varphi) = 16S_+(\rho)$

For $\rho \in \mathcal{N}_+$, we define (for a.e. t)

$$\alpha_t(z) = \text{Im}(zH'_t(z)), \quad z \in \mathbb{D},$$

where H_t is the Herglotz integral of ρ_t . In this section, we assume $S_+(\rho) < \infty$, and write as before $d\rho_t = \nu_t^2(\theta)d\theta$. In this case, we have

$$H_t(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \nu_t^2(\theta) d\theta.$$

Lemma 6.1. *If $L(\rho_t) < \infty$, then $H'_t \in \mathcal{H}^2$.*

Proof. We have $(\nu_t^2)' = 2\nu_t'\nu_t \in L^2(S^1, d\theta)$ since ν_t is bounded and $\nu_t' \in L^2(S^1, d\theta)$ by assumption. Writing the complex derivative in polar coordinates shows that $\text{Im } zH'_t(z) = -\partial_\theta \text{Re } H_t(z)$. Let

$$P_r(\theta - s) := P_{\mathbb{D}}(re^{i(\theta-s)}) = (1-r^2)/|1-re^{i(\theta-s)}|^2.$$

Then using integration by parts

$$\begin{aligned} \partial_\theta \text{Re } H_t(z) &= \int_0^{2\pi} \partial_\theta P_r(\theta - s) \nu_t(s)^2 ds = \int_0^{2\pi} (-\partial_s P_r(\theta - s)) \nu_t(s)^2 ds \\ &= \int_0^{2\pi} P_r(\theta - s) [\nu_t(s)^2]' ds = 2\pi P_{\mathbb{D}}[(\nu_t^2)'](z). \end{aligned}$$

Therefore, since $(\nu_t^2)' \in L^2$, we get that $\text{Im } H'_t \in \mathfrak{h}^2$. By [19, Thm.4.1] this in turn implies $\text{Re } H'_t \in \mathfrak{h}^2$ and we conclude that $H'_t \in \mathcal{H}^2$. \square

Lemma 6.2. *If $H'_t \in \mathcal{H}^1$ then for a.e. $\theta \in S^1$,*

$$\operatorname{Im}(e^{i\theta} H'_t(e^{i\theta})) = -4\pi\nu_t(\theta)\nu'_t(\theta), \quad (6.1)$$

where the left-hand side is understood in terms of radial limits and we have $\alpha_t = -4\pi P_{\mathbb{D}}[\nu_t\nu'_t]$. In particular, this holds if $L(\rho_t) < \infty$.

Proof. If $H'_t \in \mathcal{H}^1$, then by [19, Thm. 5.2] (see in particular the last paragraph of the proof), H_t is continuous on $\overline{\mathbb{D}}$ and the boundary function $H_t(e^{i\theta})$ is absolutely continuous on S^1 . Since $2\pi\nu_t^2(\theta) = \operatorname{Re} H_t(e^{i\theta})$, $4\pi\nu_t(\theta)\nu'_t(\theta) = \partial_\theta \operatorname{Re} H_t(e^{i\theta})$ exists a.e. on S^1 . Moreover, H'_t has radial limits a.e. on S^1 and by [19, Thm. 3.11], $\partial_\theta H_t(e^{i\theta}) = ie^{i\theta} \lim_{r \rightarrow 1} H'_t(re^{i\theta})$ a.e. on S^1 . This gives the identity (6.1). Since $H'_t \in \mathcal{H}^1$, $\alpha_t(z) = \operatorname{Im} z H'_t(z)$ is the Poisson integral of its boundary values and this gives the second assertion. The final statement follows directly using Lemma 6.1. \square

The following lemma holds for all $\rho \in \mathcal{N}_+$.

Lemma 6.3. *For all $z \in D_T$, $\vartheta[g_T](z) = \int_0^T \alpha_t(g_t(z)) dt$.*

Proof. Since $t \mapsto \vartheta[g_t](z)$ is absolutely continuous on $[0, T]$, we can use the Loewner equation (2.5) to see that for a.e. t ,

$$\begin{aligned} \partial_t \vartheta[g_t](z) &= \operatorname{Im} \left(\frac{\partial_t g'_t(z)}{g'_t(z)} - \frac{\partial_t g_t(z)}{g_t(z)} \right) \\ &= \operatorname{Im} \left(\frac{g'_t(z) H_t(g_t(z)) + g_t(z) H'_t(g_t(z)) g'_t(z)}{g'_t(z)} - H_t(g_t(z)) \right) \\ &= \operatorname{Im}(g_t(z) H'_t(g_t(z))) = \alpha_t(g_t(z)). \end{aligned}$$

Since $g_0(z) = z$ we get the claim after integration. \square

We define

$$\beta(z) = \int_0^{\tau(z)} \alpha_t(g_t(z)) dt, \quad z \in \mathbb{D}. \quad (6.2)$$

It is not obvious that this quantity is finite a.e. However, part of the conclusion of the next result is that $\beta \in \mathcal{E}_0(\mathbb{D})$ and we shall later prove that β is the unique $\mathcal{E}_0(\mathbb{D})$ extension of the winding function φ .

Proposition 6.4. *Suppose $S_+(\rho) < \infty$. Let $u(\theta, t) := -2\nu'_t(\theta)/\nu_t(\theta)$ if $\nu_t(\theta) \neq 0$, and $u(\theta, t) := 0$ otherwise. Then $u \in L^2(2\rho)$ and $\varkappa[u] = \beta$. In particular, $\beta \in \mathcal{E}_0(\mathbb{D})$ and $\mathcal{D}_{\mathbb{D}}(\beta) = 16 S_+(\rho)$.*

Proof. We verify directly that $u \in L^2(2\rho)$ and

$$\|u\|_{L^2(2\rho)}^2 = 2 \int_0^\infty \int_{S^1} 1_{\nu_t \neq 0} \left[\frac{2\nu'_t(\theta)}{\nu_t(\theta)} \right]^2 \nu_t^2(\theta) d\theta dt = 16 S_+(\rho) < \infty. \quad (6.3)$$

Corollary 5.12 shows that ρ generates a foliation, therefore the disintegration isometry of Section 3.2 applies. Using Proposition 3.3 and Lemma 6.2,

$$\varkappa[u](z) = 2\pi \int_0^{\tau(z)} P_{\mathbb{D}}[u_t \nu_t^2](g_t(z)) dt = -4\pi \int_0^{\tau(z)} P_{\mathbb{D}}[\nu_t \nu'_t](g_t(z)) dt$$

$$= \int_0^{\tau(z)} \alpha_t(g_t(z)) dt = \beta(z).$$

Moreover, by Theorem 3.5 and (6.3), we obtain $16 S_+(\rho) = \|u\|_{L^2(2\rho)}^2 = \mathcal{D}_{\mathbb{D}}(\mathcal{K}[u]) = \mathcal{D}_{\mathbb{D}}(\beta)$ as claimed. \square

Corollary 6.5. *For $T > 0$, let $\rho_t^T = \rho_t$ for $t \leq T$ and let ρ_t^T be the uniform measure for $t > T$. Let β^T be the associated function as in (6.2). Then we have $\vartheta[g_T] = \beta^T$ on D_T . In particular,*

$$\mathcal{D}_{D_T}(\vartheta[g_T]) \leq \mathcal{D}_{\mathbb{D}}(\beta^T) = 16 S_+(\rho^T) = 16 S_{[0,T]}(\rho).$$

Proof. We only need to note that for $t \geq T$, $\alpha_t \equiv 0$. Therefore, for $z \in D_T$,

$$\beta^T(z) = \int_0^{\min\{\tau(z), T\}} \alpha_t(g_t(z)) dt = \int_0^T \alpha_t(g_t(z)) dt,$$

since $\tau(z) > T$. Lemma 6.3 implies $\vartheta[g_T] = \beta^T$ on D_T . \square

We now show that β is the unique extension of the winding function (2.9) φ in $\mathcal{E}_0(\mathbb{D})$.

Lemma 6.6. *If $S_+(\rho) < \infty$, then for all $t \geq 0$,*

$$\beta|_{\partial D_t} = \varphi|_{\partial D_t} \quad \text{arclength-a.e.},$$

where β is as in (6.2) and its trace is taken in the sense of Jonsson–Wallin (3.9).

In other words, β is the unique extension of φ in $\mathcal{E}_0(\mathbb{D})$ by Proposition 3.6 (and from now on we will not distinguish β and φ).

Proof. We will identify functions in $\mathcal{E}_0(\mathbb{D})$ with their extension to $W^{1,2}(\mathbb{C})$ by 0 in \mathbb{D}^* . By Corollary 6.5, $\beta^t = \beta$ in $\mathbb{C} \setminus D_t$ and the Jonsson–Wallin traces satisfy

$$\beta|_{\gamma_t} = \beta^t|_{\gamma_t} = \vartheta[g_t]|_{\gamma_t} = \varphi|_{\gamma_t} \quad \text{arclength-a.e.}$$

Here, the first equality is a property of the Jonsson–Wallin trace, see [62, Lem. A.2]. The second equality follows from Corollary 6.5, where we interpret $\vartheta[g_t]|_{\gamma_t}$ as the non-tangential limit from inside D_t using [62, Lem. A.5]. The last equality is the definition of φ . \square

Corollary 6.7. *If $S_+(\rho) < \infty$, then $\varphi \in \mathcal{E}_0(\mathbb{D})$. For ρ -a.e. (θ, t) ,*

$$\iota[\varphi](\theta, t) = -2\nu'_t(\theta)/\nu_t(\theta)$$

and $\mathcal{D}_{\mathbb{D}}(\varphi) = 16S_+(\rho)$.

Proof. The proof is immediate by combining Proposition 6.4 and Lemma 6.6. \square

6.2 $\mathcal{D}_{\mathbb{D}}(\varphi) < \infty$ implies $S_+(\rho) < \infty$

This section proves the following result.

Proposition 6.8. *Suppose $\rho \in \mathcal{N}_+$ generates a foliation and assume that the winding function $\varphi : \mathcal{T} \mapsto \mathbb{R}$ can be extended to a function in $\mathcal{E}_0(\mathbb{D})$ (also denoted φ). Then $S_+(\rho) < \infty$.*

Assuming this proposition, we can complete the proof of Theorem 1.7.

Proof of Theorem 1.7. Let $\rho \in \mathcal{N}_+$. If $S_+(\rho) < \infty$, Corollary 6.7 shows that $\varphi \in \mathcal{E}_0(\mathbb{D})$ and $\mathcal{D}_{\mathbb{D}}(\varphi) = 16S_+(\rho)$. Conversely, Proposition 6.8 shows that if $\varphi \in \mathcal{E}_0(\mathbb{D})$, then $S_+(\rho) < \infty$. Therefore the identity also holds. \square

To explain our present goal towards the proof of Proposition 6.8, note that we do not know *a priori* that ρ_t is absolutely continuous with respect to Lebesgue measure, nor that its density (if it exists) is differentiable almost everywhere. We need to show that this is the case.

Under the assumptions of Proposition 6.8, we let as before φ_s^0 be the zero trace part of the function $\varphi \circ f_s$ and recall that $\alpha_t(z) = \text{Im}(zH_t'(z))$ for $z \in \mathbb{D}$. Since ρ generates a foliation and since $\varphi \in \mathcal{E}_0(\mathbb{D})$ by assumption, using Theorem 3.5 we may consider

$$u := \iota[\varphi] \in L^2(2\rho).$$

Lemma 6.9. *For a.e. $t \geq 0$, we have*

$$\alpha_t(z) = 2\pi P_{\mathbb{D}}[u_t \rho_t](z), \quad u_t(\cdot) := u(\cdot, t). \quad (6.4)$$

Proof. Proposition 3.3 shows that a.e. in \mathbb{D} ,

$$\varphi(w) = \varkappa[u](w) = 2\pi \int_0^{\tau(w)} P_{\mathbb{D}}[u_t \rho_t](g_t(w)) dt. \quad (6.5)$$

Lemma 2.10 shows that the unique orthogonal decomposition of φ with respect to D_s is given by the zero-trace part $\varphi_s^0 \circ g_s \in \mathcal{E}_0(D_s)$ and the harmonic part $\varphi - \varphi_s^0 \circ g_s$ which equals $\vartheta[g_s]$ in D_s . On the other hand, Corollary 3.4 shows that this decomposition is also given by $\varkappa[u \mathbf{1}_{S^1 \times [0, s]}]$ and $\varkappa[u \mathbf{1}_{S^1 \times [s, \infty)}]$. Hence, for all $w \in D_s$ (so that $\tau(w) > s$),

$$\vartheta[g_s](w) = 2\pi \int_0^s P_{\mathbb{D}}[u_t \rho_t](g_t(w)) dt.$$

Taking a derivative in s in the above expression we obtain from Lemma 6.3 that for a fixed $w \in \mathbb{D}$ and a.e. $t < \tau(w)$, $2\pi P_{\mathbb{D}}[u_t \rho_t](g_t(w)) = \alpha_t(g_t(w))$. Indeed, by choosing a countable and dense family J of points in \mathbb{D} , we have for a.e. $t \geq 0$ and all $w \in D_t \cap J$, $2\pi P_{\mathbb{D}}[u_t \rho_t](g_t(w)) = \alpha_t(g_t(w))$. Since both α_t and $P_{\mathbb{D}}[u_t \rho_t]$ are continuous (actually harmonic) in \mathbb{D} , we therefore obtain a.e. t , $\alpha_t = 2\pi P_{\mathbb{D}}[u_t \rho_t]$ for all $w \in \mathbb{D}$. \square

We are now ready to prove the main result of this section.

Proof of Proposition 6.8. Since $u = \iota[\varphi] \in L^2(2\rho)$, for a.e. $t \in \mathbb{R}_+$, $u_t \in L^2(S^1, 2\rho_t)$. By the Cauchy-Schwarz inequality and the fact that $\rho_t \in \mathcal{M}_1(S^1)$ is a probability measure, we also know that $u_t \in L^1(S^1, 2\rho_t)$. Hence $\alpha_t = 2\pi P_{\mathbb{D}}[u_t \rho_t] \in \mathfrak{h}^1$ by Lemma 6.9. This implies $H'_t \in \mathcal{H}^p$ for any $p < 1$. (See [19, Thm. 4.2], and use that an analytic function is in \mathcal{H}^p if and only if its real and imaginary parts are in \mathfrak{h}^p .) Using [19, Thm. 5.12] this in turn implies $H_t \in \mathcal{H}^p$ for every $p < \infty$. (More precisely, if $f' \in \mathcal{H}^p$ for some $p < 1$, then $f \in \mathcal{H}^q$ with $q = p/(1-p)$.) Therefore, the radial limits of H_t exist a.e. and define a function in $L^p(S^1, d\theta)$ for all $p < \infty$. It follows that the positive function $\operatorname{Re} H_t/2\pi$ is the Poisson integral of a function in $L^p(S^1, d\theta)$ for all $p < \infty$, and we denote this function by $\nu_t^2(\theta)$. (See, e.g., Corollary 2 to [19, Thm. 3.1].) In other words we have shown that the measure ρ_t is absolutely continuous and $d\rho_t(\theta) = \nu_t^2(\theta)d\theta$, where $\nu_t^2 \in L^p(S^1, d\theta)$ for every $p < \infty$.

Since $u_t^2 \nu_t^2 \in L^1(S^1, d\theta)$ and $\nu_t^2 \in L^p(S^1, d\theta)$ for all $p < \infty$, Hölder's inequality with $p = 2/(2-\varepsilon)$, $q = 2/\varepsilon$ implies $u_t \nu_t^2 \in L^{2-\varepsilon}(S^1, d\theta)$ for any $\varepsilon \in (0, 2)$. This in turn implies that the Herglotz integral of $u_t \nu_t^2$ is in $\mathcal{H}^{2-\varepsilon}$ if $\varepsilon \in (0, 1)$. Lemma 6.9 then implies that $H'_t \in \mathcal{H}^{2-\varepsilon}$. It follows that H_t is continuous on $\overline{\mathbb{D}}$ and absolutely continuous on S^1 (see [19, Thm. 3.11]) and consequently so is the density $\theta \mapsto \nu_t(\theta)$ and $\nu_t'(\theta)$ is well-defined for Lebesgue-a.e. (θ, t) .

Moreover, by taking radial limits in (6.4), we obtain using Lemma 6.2 and Fubini's theorem that for Lebesgue-a.e. $(\theta, t) \in S^1 \times \mathbb{R}_+$,

$$u_t(\theta) \nu_t^2(\theta) = \frac{1}{2\pi} \alpha_t(e^{i\theta}) = -2\nu_t'(\theta) \nu_t(\theta).$$

It follows that Lebesgue-a.e., when $\nu_t(\theta) \neq 0$, $u_t^2(\theta) \nu_t^2(\theta) = 4\nu_t'(\theta)^2$, and both sides are equal to 0 otherwise. We conclude the proof by integrating over $S^1 \times \mathbb{R}_+$, and we obtain $S_+(\rho) < \infty$ since $u = \iota[\varphi] \in L^2(2\rho)$. \square

7 Whole-plane energy duality: proof of Theorem 1.2

In this section we deduce whole-plane energy duality, Theorem 1.2, from disk energy duality, Theorem 1.7.

7.1 Whole-plane Loewner evolution

We now describe the whole-plane Loewner chain. We define similarly as before a space of driving measures

$$\mathcal{N} := \{\rho \in \mathcal{M}(S^1 \times \mathbb{R}) : \rho(S^1 \times I) = |I| \text{ for all intervals } I\}.$$

The whole-plane Loewner chain driven by $\rho \in \mathcal{N}$, or equivalently by its measurable family of disintegration measures $\mathbb{R} \rightarrow \mathcal{M}_1(S^1) : t \mapsto \rho_t$, is the unique family of conformal maps $(f_t : \mathbb{D} \rightarrow D_t)_{t \in \mathbb{R}}$ such that

- (i) For all $s < t$, $0 \in D_t \subset D_s$.
- (ii) For all $t \in \mathbb{R}$, $f_t(0) = 0$ and $f_t'(0) = e^{-t}$ (in other words, D_t has conformal radius e^{-t} seen from 0).

- (iii) For all $s \in \mathbb{R}$, $(f_t^{(s)} := f_s^{-1} \circ f_t : \mathbb{D} \rightarrow D_t^{(s)})_{t \geq s}$ is the Loewner chain driven by $(\rho_t)_{t \geq s}$, which satisfies (2.4) with the initial condition $f_s^{(s)}(z) = z$, as discussed in Section 2.2.

Remark. If $\rho_t \in \mathcal{M}_1(S^1)$ is the uniform measure for all $t \leq 0$, then $f_t(z) = e^{-t}z$ for $t \leq 0$ and $(f_t)_{t \geq 0}$ is the Loewner chain driven by $(\rho_t)_{t \geq 0} \in \mathcal{N}_+$ as in Section 2.2. Indeed, we check directly that $(f_t)_{t \in \mathbb{R}}$ satisfy the three conditions above. In other words, the Loewner chain in \mathbb{D} is a special case of the whole-plane Loewner chain.

Note that the range $D_t^{(s)}$ of $f_t^{(s)}$ has conformal radius e^{s-t} seen from 0 and the family $(f_t^{(s)})_{t \geq s}$ is uniquely defined for all $s \in \mathbb{R}$ and satisfies for $t \geq s$ and $z \in \mathbb{D}$,

$$\partial_t f_t^{(s)}(z) = -z f_t^{(s)\prime}(z) H_t(z), \quad H_t(z) = \int_{S^1} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\rho_t(\theta),$$

with the initial condition $f_s^{(s)}(z) = z$ (see Section 2.2). The condition (iii) is then equivalent to for all $t \in \mathbb{R}$ and $z \in \mathbb{D}$,

$$\partial_t f_t(z) = -z f_t'(z) H_t(z) \tag{7.1}$$

as described in the introduction. For the purposes of our proofs later on it is convenient to give a slightly more explicit construction of the family $(f_t)_{t \in \mathbb{R}}$ and explain why it is uniquely determined by $t \rightarrow \rho_t$, even though this statement is well-known. For this, note that once we determine f_n , (7.1) gives f_t for all $t \geq n$ by

$$f_t = f_n \circ f_t^{(n)}. \tag{7.2}$$

Existence: Consider for $-\infty < s \leq t$, the conformal map

$$F_t^{(s)}(z) := e^{-s} f_t^{(s)}(z)$$

which maps \mathbb{D} onto $e^{-s} D_t^{(s)}$. Since $F_t^{(s)\prime}(0) = e^{-t}$, $(F_t^{(s)})_{s \in (-\infty, t]}$ is a normal family for any t . We extract a sequence (s_k) converging to $-\infty$, such that for all $n \in \mathbb{Z}$, $F_n^{(s_k)}$ converges uniformly on compacts to a univalent function that we call F_n . We construct (f_t) by taking $f_n := F_n$ and generate the other f_t using (7.2). We need to verify compatibility, that is, that we have that $F_{n+1} = F_n \circ f_{n+1}^{(n)}$. To see this, notice that for $s < t_1 < t_2$,

$$(F_{t_1}^{(s)})^{-1} \circ F_{t_2}^{(s)} = (f_{t_1}^{(s)})^{-1} \circ f_{t_2}^{(s)} = f_{t_2}^{(t_1)}$$

is independent of s . The last equality follows from the fact that as a function of t_2 on $[t_1, \infty)$, both terms satisfy the same differential equation with same initial condition. Hence, we have $F_n^{-1} \circ F_{n+1} = f_{n+1}^{(n)}$ as claimed.

Uniqueness: If there are two such families $(f_t : \mathbb{D} \rightarrow D_t)_{t \in \mathbb{R}}$ and $(\tilde{f}_t : \mathbb{D} \rightarrow \tilde{D}_t)_{t \in \mathbb{R}}$. Let $\psi_t := \tilde{f}_t \circ f_t^{-1} : D_t \rightarrow \tilde{D}_t$. Since (f_t) and (\tilde{f}_t) are driven by the same process of measures, for all $s \leq t$,

$$f_s^{-1} \circ f_t = \tilde{f}_s^{-1} \circ \tilde{f}_t.$$

And for $z = f_t(w) \in D_t$,

$$\psi_s(z) = \tilde{f}_s \circ f_s^{-1} \circ f_t(w) = \tilde{f}_s \circ \tilde{f}_s^{-1} \circ \tilde{f}_t(w) = \tilde{f}_t \circ f_t^{-1}(z) = \psi_t(z).$$

That is, $\psi_s|_{D_t} = \psi_t$. Hence, ψ_t extends to a conformal map $\cup_{t \in \mathbb{R}} D_t \rightarrow \cup_{t \in \mathbb{R}} \tilde{D}_t = \mathbb{C}$. This shows that ψ_t is the identity map and $D_t = \tilde{D}_t$, since $\psi_t(0) = 0$ and $\psi'_t(0) = 1$ and completes the proof of the uniqueness.

We remark that if $\rho \in \mathcal{N}$, then $\cup_{t \in \mathbb{R}} D_t = \mathbb{C}$. Indeed, D_t has conformal radius e^{-t} , therefore contains the centered ball of radius $e^{-t}/4$ by Koebe's 1/4 theorem. We define for all $z \in \mathbb{C}$,

$$\tau(z) := \sup\{t \in \mathbb{R} : z \in D_t\} \in (-\infty, \infty].$$

Similar to the definition of foliations of $\bar{\mathbb{D}} \setminus \{0\}$, we say that $\rho \in \mathcal{N}$ generates a foliation $(\gamma_t := \partial D_t)_{t \in \mathbb{R}}$ of $\mathbb{C} \setminus \{0\}$ if

1. For all $t \in \mathbb{R}$, γ_t is a chord-arc Jordan curve.
2. It is possible to parametrize each curve $\gamma_t, t \in \mathbb{R}$, by S^1 so that the mapping $t \mapsto \gamma_t$ is continuous in the supremum norm.
3. For all $z \in \mathbb{C} \setminus \{0\}$, $\tau(z) < \infty$.

We have the whole-plane version of Lemma 2.4:

Lemma 7.1. *Assume that ρ generates a foliation $(\gamma_t)_{t \in \mathbb{R}}$ of $\mathbb{C} \setminus \{0\}$. For all $z \neq 0$, we have $z \in \gamma_{\tau(z)}$. In particular, $\cup_{t \geq 0} \gamma_t = \mathbb{C} \setminus \{0\}$.*

We associate similarly to a foliation of $\mathbb{C} \setminus \{0\}$ its *winding function* defined by $\varphi(0) = 0$,

$$\varphi(z) := \vartheta[g_t](z), \quad \text{for arclength-a.e. } z \in \gamma_t \text{ and } \forall t \in \mathbb{R},$$

where $g_t = f_t^{-1}$. We may also define $\varphi(z)$ by $\vartheta[g_{\tau(z)}](z)$. We say that φ has an extension to $W_{\text{loc}}^{1,2}$ if $\varphi|_{\gamma_t}$ coincides a.e. with the Jonsson–Wallin trace of its extension on γ_t , for all $t \in \mathbb{R}$.

7.2 Proof of Theorem 1.1 and Theorem 1.2

Recall the definition of the Loewner–Kufarev energy in the whole-plane setting: for $\rho \in \mathcal{N}$,

$$S(\rho) = \int_{\mathbb{R}} L(\rho_t) dt \quad \text{and} \quad S_{[a,b]}(\rho) = \int_a^b L(\rho_t) dt.$$

We use the notation from Section 7.1 and write for $s \leq t$,

$$g_t^{(s)} := (f_t^{(s)})^{-1} = g_t \circ f_s, \quad g_t = f_t^{-1}.$$

Then $(g_t^{(s)})_{t \geq s}$ is the uniformizing Loewner chain driven by $\rho^{(s)} := (\rho_t)_{t \geq s}$, mapping $D_t^{(s)} = g_s(D_t)$ onto \mathbb{D} . We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Conditions (i) and (iii) we have that

$$\cap_{t \in \mathbb{R}} D_t = \cap_{t \in \mathbb{R}_+} D_t = f_0(\cap_{t \geq 0} D_t^{(0)}) = \{0\},$$

where we used Lemma 4.2 and $S_+(\rho) < \infty$. Therefore $\tau(z) < \infty$ for all $z \neq 0$.

Now we show that ∂D_t is a Weil–Petersson quasicircle. Note that on D_t ,

$$g_t = (g_t \circ f_s) \circ g_s = g_t^{(s)} \circ g_s, \quad \forall s \leq t.$$

Since $S_{[s,\infty)}(\rho^{(s)}) < \infty$, Proposition 5.9 shows that $g_s(D_t) = g_s(f_t(\mathbb{D})) = f_t^{(s)}(\mathbb{D}) = D_t^{(s)}$ is bounded by a Weil–Petersson quasicircle. Moreover, from the proof of Lemma 4.2, we know that for all $t \in \mathbb{R}$ there is $s_0 < t$, such that $D_t^{(s_0)} \subset (1/2)\mathbb{D}$. As $f_{s_0} : \mathbb{D} \rightarrow D_{s_0}$ is conformal, $\gamma_t = f_{s_0}(\partial D_t^{(s_0)})$ is also a Weil–Petersson quasicircle (e.g. by Lemma 2.1). We also obtain the continuity of $t \mapsto \gamma_t$ from the continuity of $t \mapsto \partial D_t^{(s_0)}$ by Corollary 5.12. \square

We are now ready to prove whole-plane energy duality.

Proof of Theorem 1.2. Assume first $S(\rho) < \infty$, we show $\mathcal{D}_{\mathbb{C}}(\varphi) < \infty$. For this, note that the winding function $\varphi^{(s)}$ of the foliation in $\overline{\mathbb{D}} \setminus \{0\}$ driven by $\rho^{(s)} = (\rho_t)_{t \geq s}$ is given by

$$\varphi^{(s)}(w) = \vartheta[g_{\tau(w)}^{(s)}](w) = \vartheta[g_{\tau(z)}](z) - \vartheta[g_s](z) = \varphi(z) - \vartheta[g_s](z)$$

where $z = f_s(w) \in D_s$ and we used the chain rule (2.8). We have

$$\begin{aligned} \mathcal{D}_{D_s}(\varphi) &= \mathcal{D}_{D_s}(\varphi^{(s)} \circ g_s) + \mathcal{D}_{D_s}(\vartheta[g_s]) = \mathcal{D}_{\mathbb{D}}(\varphi^{(s)}) + \mathcal{D}_{D_s}(\vartheta[g_s]) \\ &= 16 S_{[s,\infty)}(\rho) + \mathcal{D}_{\mathbb{D}}(\vartheta[f_s]). \end{aligned} \quad (7.3)$$

The first equality follows from orthogonality of $\vartheta[g_s]$ and $\varphi^{(s)} \circ g_s$ in $\mathcal{E}_0(\mathbb{D})$, since $\vartheta[g_s]$ is harmonic in D_s and $\varphi^{(s)} \circ g_s \in \mathcal{E}_0(D_s)$. The second equality follows from the conformal invariance of the Dirichlet energy, and the third from Theorem 1.7. We obtain immediately the lower bound

$$\mathcal{D}_{\mathbb{C}}(\varphi) \geq 16 S(\rho).$$

For the opposite inequality, since $F_s^{(r)} := e^{-r} f_s^{(r)}$ (see Section 7.1) converges uniformly on compact subsets of \mathbb{D} to f_s as $r \rightarrow -\infty$, we have that $\vartheta[F_s^{(r)}](z) = \vartheta[f_s^{(r)}](z)$ converges uniformly on compact sets to $\vartheta[f_s](z)$. The lower semicontinuity of the Dirichlet energy then shows that for all compact $K \subset \mathbb{D}$,

$$\mathcal{D}_K(\vartheta[f_s]) \leq \liminf_{r \rightarrow -\infty} \mathcal{D}_K(\vartheta[f_s^{(r)}]).$$

On the other hand, Corollary 6.5 shows that

$$\mathcal{D}_K(\vartheta[f_s^{(r)}]) = \mathcal{D}_{f_s^{(r)}(K)}(\vartheta[g_s^{(r)}]) \leq 16 S_{[r,s]}(\rho),$$

letting $r \rightarrow -\infty$ yields

$$\mathcal{D}_{\mathbb{D}}(\vartheta[f_s]) \leq 16 \int_{-\infty}^s L(\rho_t) dt.$$

Combining this with (7.3) shows that $\mathcal{D}_{D_s}(\varphi) \leq 16 S(\rho)$. Letting $s \rightarrow -\infty$ we obtain the upper bound and hence $\mathcal{D}_{\mathbb{C}}(\varphi) = 16 S(\rho)$.

For the converse, if ρ generates a foliation of $\mathbb{C} \setminus \{0\}$ with $\mathcal{D}_{\mathbb{C}}(\varphi) < \infty$, Proposition 6.8 and (7.3) imply that

$$16 S_{[s,\infty)}(\rho^{(s)}) = \mathcal{D}_{\mathbb{D}}(\varphi^{(s)}) \leq \mathcal{D}_{D_s}(\varphi) \leq \mathcal{D}_{\mathbb{D}}(\varphi).$$

Letting $s \rightarrow -\infty$ we obtain $S(\rho) < \infty$ and this completes the proof. \square

8 Applications of energy duality

In this section we derive several consequences of Theorem 1.2.

8.1 Reversibility of the Loewner–Kufarev energy

Let $(\gamma_t)_{t \in \mathbb{R}}$ be a foliation generated by $\rho \in \mathcal{N}$ and let $(D_t)_{t \in \mathbb{R}}$ be the corresponding family of domains. We will consider the evolution of its time-reversal, that is the Loewner chain corresponding to the family $(\tilde{D}_{s(t)} := j(\hat{\mathbb{C}} \setminus D_t))_{t \in \mathbb{R}}$, where $j(z) := 1/z$ and $e^{-s(t)}$ is the conformal radius of $j(\hat{\mathbb{C}} \setminus D_t)$. Let \tilde{f}_s be the conformal map from \mathbb{D} onto \tilde{D}_s with $\tilde{f}_s(0) = 0$ and $\tilde{f}'_s(0) = e^{-s}$.

Lemma 8.1. *The function $t \mapsto s(t)$ is a decreasing homeomorphism of \mathbb{R} .*

Proof. To show that $t \mapsto s$ is decreasing, let $t_1 < t_2$. Since $D_{t_2} \subsetneq D_{t_1}$, we have $\tilde{D}_{s(t_1)} \subsetneq \tilde{D}_{s(t_2)}$. Schwarz' lemma then shows that $s(t_2) < s(t_1)$. We claim that $s(t) \rightarrow \infty$ as $t \rightarrow -\infty$. In fact, Koebe's 1/4-theorem shows that D_t contains the centered disk of radius $e^{-t}/4$, therefore $\tilde{D}_{s(t)}$ is contained in the centered disk of radius $4e^t$. Schwarz' lemma shows $s(t) \geq -t - \log 4$ which goes to ∞ as $t \rightarrow -\infty$. Since the diameter of D_t tends to 0 as $t \rightarrow \infty$ by Lemma 4.2, $\tilde{D}_{s(t)}$ has conformal radius tending to ∞ . Therefore $s(t) \rightarrow -\infty$ as $t \rightarrow \infty$. It remains to verify that s is continuous. To see this, note that the continuity of $t \mapsto \gamma_t$ in the supremum norm shows that as $t \rightarrow t_0 \in \mathbb{R}$, $\tilde{D}_{s(t)}$ converges in the Carathéodory kernel sense to $\tilde{D}_{s(t_0)}$. Therefore, $\tilde{f}'_{s(t)}(0)$ tends to $\tilde{f}'_{s(t_0)}(0)$, and equivalently, $s(t)$ tends to $s(t_0)$. \square

Lemma 8.1 implies that $(\tilde{f}_s)_{s \in \mathbb{R}}$ is a whole-plane Loewner chain as defined in Section 7. Note that in Section 7 we took the measure ρ as starting point for the definition whereas we have constructed the monotone family of domains here. However, using Pommerenke's theorem as discussed in Section 2.2 the existence of a measure $\tilde{\rho}$ generating the family follows easily. Energy duality, Theorem 1.2, now implies the following reversibility of the Loewner–Kufarev energy.

Theorem 8.2 (Energy reversibility). *Let $\tilde{\rho} \in \mathcal{N}$ be the measure generating $(\tilde{D}_s)_{s \in \mathbb{R}}$. Then $S(\rho) = S(\tilde{\rho})$.*

For the proof of Theorem 8.2 we will use the following lemma which allows us to relate the winding function associated with ρ to that with $\tilde{\rho}$. A weaker lemma sufficient for our purposes appeared in an earlier version of the paper and in [66]. The proof of the present stronger version was suggested by an anonymous referee.

Lemma 8.3. *Let γ be a Jordan curve separating 0 from ∞ . Suppose $z \in \mathcal{T}_\gamma$. Then the winding functions φ_γ and $\varphi_{1/\gamma}$ are well-defined at z and $1/z$, respectively, and*

$$\varphi_\gamma(z) = \varphi_{1/\gamma}(1/z).$$

Geometrically, $\varphi_\gamma(z)$ measures the angle between γ and the circle centered at 0 passing through z modulo 2π . The map $z \mapsto 1/z$ is conformal and maps centered circles to centered circles, so the angle is preserved. The difficulty lies in proving the equality

without “modulo 2π ” with our choice for the branch of the argument. Before giving the proof, we first complete the proof of Theorem 8.2 assuming the lemma.

Proof of Theorem 8.2 assuming Lemma 8.3. Assume that $S(\rho) < \infty$. Theorem 1.1 implies that every leaf of the associated foliation is a Weil–Petersson quasicircle, hence rectifiable and differentiable almost everywhere. Lemma 8.3 implies that the winding function $\tilde{\varphi}$ associated with the foliation generated by $\tilde{\rho}$ satisfies

$$\tilde{\varphi}(1/z) = \varphi(z)$$

for all $z \in \mathcal{T}_{\gamma_t}$, $t \in \mathbb{R}$. Since the Dirichlet energy is invariant under conformal mappings, we obtain $\mathcal{D}_{\mathbb{C}}(\tilde{\varphi}) = \mathcal{D}_{\mathbb{C}}(\varphi)$ which implies $S(\rho) = S(\tilde{\rho})$ by Theorem 1.2.

Finally, if $S(\rho) = \infty$, then $S(\tilde{\rho}) = \infty$. Indeed, if this does not hold we would have $S(\rho) < \infty$, since ρ corresponds to the time-reversed foliation of that associated with $\tilde{\rho}$. This contradiction completes the proof. \square

It remains to prove Lemma 8.3. For this we will need some more notation and another lemma. Let γ be a Jordan curve separating 0 from ∞ . Recall that the winding function $\varphi_{\gamma}(z)$ at $z \in \mathcal{T}_{\gamma}$ is by definition the non-tangential limit at z of $\vartheta[g]$ where $g : D \rightarrow \mathbb{D}$ is a conformal map fixing 0 (the fact that the limit exists follows from the remark after Lemma 2.5). We first give an equivalent expression for the winding function at a given $z \in \mathcal{T}_{\gamma}$ in terms of the turning of a smooth curve approaching z normally.

We introduce the following notation. For a simply connected subset $V \subset \mathbb{C}$ containing 0, and $z \notin V$, let $w \mapsto \arg_V(z/(z-w))$ be the unique real-valued continuous function on V such that $\arg_V(z/(z-0)) = 0$ and $\exp(i \arg_V(z/(z-w))) = z|z-w|/|z|(z-w)$. If $v : [0, 1] \rightarrow \mathbb{C}$ is a smooth path joining 0 to $z \in \mathbb{C}$, $z \neq 0$, we define the *turning* of v around z as

$$\Theta_v(z) = \lim_{s \rightarrow 1^-} \arg_{v[0,1]} \left(\frac{z}{z-v(s)} \right) = \int_v d \arg \frac{z}{z-w}.$$

The limit exists since v is smooth and has a well-defined tangent at z and $z \neq 0$.

Lemma 8.4. *Let γ be a Jordan curve separating 0 and ∞ , and let D be the bounded connected component of $\mathbb{C} \setminus \gamma$ containing 0. Suppose $z \in \mathcal{T}_{\gamma}$. Let $v : [0, 1] \rightarrow \overline{D}$ be any smooth path joining 0 to z in D (with $v(1) = z \in \overline{D}$). Then $|\Theta_v(z) - \varphi_{\gamma}(z)| \leq \pi/2$. In particular, if v approaches z normally, then $\Theta_v(z) = \varphi_{\gamma}(z)$.*

Proof. By (2.7) we have

$$\varphi_{\gamma}(z) = \int_0^1 d \arg \frac{g(z) - g(v(s))}{z - v(s)} = \lim_{s \rightarrow 1^-} \left[\arg \frac{g(z) - g(v(s))}{z - v(s)} - \arg \frac{g(z)}{z} \right]$$

where the branch of \arg is chosen continuously along v (and the difference vanishes when $s = 0$). This may be rewritten as

$$\begin{aligned} \varphi_{\gamma}(z) &= \lim_{s \rightarrow 1^-} \left[\arg_D \left(\frac{z}{z - v(s)} \right) - \arg_{\mathbb{D}} \left(\frac{g(z)}{g(z) - g(v(s))} \right) \right] \\ &= \Theta_v(z) - \lim_{s \rightarrow 1^-} \arg_{\mathbb{D}} \left(\frac{g(z)}{g(z) - g(v(s))} \right). \end{aligned}$$

Since g takes values in \mathbb{D} and $g(z) \in S^1$, we have that $w \mapsto \frac{g(z)}{g(z)-w}$ maps \mathbb{D} to $1/(1+\mathbb{D}) = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 1/2\}$. Therefore,

$$\arg_{\mathbb{D}} \left(\frac{g(z)}{g(z) - g(v(s))} \right) \in (-\pi/2, \pi/2).$$

We obtain $|\Theta_v(z) - \varphi_\gamma(z)| \leq \pi/2$.

If v approaches z normally, then from the geometric interpretation of φ_γ (see Figure 1.1), we have $\Theta_v(z) = \varphi_\gamma(z)$ modulo 2π . The last paragraph shows that $\Theta_v(z) = \varphi_\gamma(z)$. \square

Proof of Lemma 8.3. Write $\tilde{\gamma} = 1/\gamma$. Let D and \tilde{D} be the bounded connected component of $\mathbb{C} \setminus \gamma$ and of $\mathbb{C} \setminus \tilde{\gamma}$ respectively. Given $z \in \mathcal{T}_\gamma$, let v be a smooth path connecting 0 to z in D , approaching z normally as in Lemma 8.4. Similarly, let \tilde{v} be a smooth path connecting 0 to $1/z$ in \tilde{D} , approaching $1/z$ normally. We assume furthermore that the paths v and $1/\tilde{v}$ can be concatenated to form a smooth path from 0 to ∞ . That is,

$$u(s) := \begin{cases} v(s), & s \in [0, 1) \\ z, & s = 1 \\ 1/\tilde{v}(1/s), & s \in (1, \infty] \end{cases}$$

is also smooth in a neighborhood of $s = 1$.

Now we define the function

$$F_u(s) = \begin{cases} \frac{u(1)}{u(1) - u(s)} \left| \frac{u(1) - u(s)}{u(1)} \right|, & s \in [0, 1) \\ \frac{u(1)}{u'(1)} \left| \frac{u'(1)}{u(1)} \right|, & s = 1 \\ \frac{u(s)}{u(s) - u(1)} \left| \frac{u(s) - u(1)}{u(s)} \right|, & s \in (1, \infty) \\ 1, & s = \infty. \end{cases}$$

We note that $F_u(0) = 1 = F_u(\infty) = \lim_{s \rightarrow \infty} F_u(s)$. Hence, F_u defines a continuous map $[0, \infty]_{/0 \sim \infty} \simeq S^1 \rightarrow S^1$. We note that for $s > 1$,

$$F_u(s) = \frac{u(s)}{u(s) - u(1)} \left| \frac{u(s) - u(1)}{u(s)} \right| = \frac{1/z}{1/z - \tilde{v}(1/s)} \left| \frac{1/z - \tilde{v}(1/s)}{1/z} \right| \xrightarrow{s \rightarrow 1^+} \exp(i\Theta_{\tilde{v}}(1/z)).$$

By definition

$$\lim_{s \rightarrow 1^-} F_u(s) = \exp(i\Theta_v(z)).$$

Now we scale and reparametrize the path u : consider for $r \in (0, 1]$,

$$u_r(s) := r^{-1}u(rs)$$

and define $F_{u_r} : [0, \infty]_{/0 \sim \infty} \simeq S^1 \rightarrow S^1$ analogously to F_u (with z replaced by $u_r(1) = r^{-1}u(r)$). It is not hard to see that F_{u_r} converges uniformly as $r \rightarrow 0^+$ to the constant function 1, so $r \mapsto F_{u_r}$ defines a homotopy from the constant function 1 to F_u . Thus, there exists a continuous argument along $s \mapsto F_u(s)$ such that $\arg F_u(0) = \arg F_u(\infty) = 0$, which implies

$$\Theta_v(z) = \arg F_u(1^-) = \arg F_u(1^+) = \Theta_{\tilde{v}}(1/z).$$

By Lemma 8.4 we conclude that $\varphi_\gamma(z) = \varphi_{1/\gamma}(1/z)$. \square

8.2 Characterization of Weil–Petersson quasicircles

Let γ be a Weil–Petersson quasicircle separating 0 from ∞ . We will associate with γ a particular measure $\rho = \rho^\gamma$ (and foliation) with $S(\rho) < \infty$ and in this way prove Theorem 1.4. We assume for notational simplicity that the bounded component of $\mathbb{C} \setminus \gamma$ has conformal radius 1⁴. Let f (resp. h) be the conformal map from \mathbb{D} (resp. \mathbb{D}^*) to the bounded component D_0 (resp. unbounded component D_0^*) of $\mathbb{C} \setminus \gamma$ such that f (resp. h) fixes 0 (resp. ∞) and has derivative $f'(0) = 1$ (resp. $h'(\infty) > 0$). Consider the foliation $(\gamma_t)_{t \in \mathbb{R}}$ that consists of γ together with the family of equipotentials on both sides of γ . By equipotential which we mean image of a circle rS^1 under f (resp. under h), and we include all equipotentials corresponding to $r < 1$ (resp. $r > 1$). The parametrization of $(\gamma_t)_{t \in \mathbb{R}}$ by t is chosen so that the connected component D_t of $\hat{\mathbb{C}} \setminus \gamma_t$ containing 0 has conformal radius e^{-t} .

Let $\rho^\gamma \in \mathcal{N}$ be the measure associated with $(\gamma_t)_{t \in \mathbb{R}}$ and let φ be the corresponding winding function. Along with the Loewner chain $(f_t : \mathbb{D} \rightarrow D_t)_{t \in \mathbb{R}}$, we consider also the conformal maps $h_t : \mathbb{D}^* \rightarrow D_t^*$, such that $h_t(\infty) = \infty$ and $h'_t(\infty) > 0$. In particular, $f = f_0$ and $h = h_0$. We also set $g_t := f_t^{-1}$ and $k_t := h_t^{-1}$. For a conformal map k fixing ∞ , we define similarly

$$\vartheta[k](z) = \arg \frac{zk'(z)}{k(z)},$$

where the argument is chosen so that $\vartheta[k](z) \rightarrow 0$ as $z \rightarrow \infty$.

Lemma 8.5. *For a.e. $t \geq 0$, ρ_t^γ is the uniform measure. Moreover, $\varphi|_{\mathbb{C} \setminus \gamma}$ is harmonic, and we have*

$$\varphi|_{D_0} = \vartheta[g_0], \quad \text{and} \quad \varphi|_{D_0^*} = \vartheta[k_0].$$

Proof. By our definition of equipotential we have $\gamma_t = f(e^{-t}S^1)$ for $t \geq 0$. The flow $(g_t \circ f)_{t \geq 0}$, driven by $(\rho_t^\gamma)_{t \geq 0}$ equals $(z \mapsto e^t z)$, which shows that $(\rho_t^\gamma)_{t \geq 0}$ is a.e. the uniform measure on S^1 . The identity $\varphi = \vartheta[g_0]$ on D_0 and hence, the harmonicity of $\varphi|_{D_0}$ follows as in Lemma 2.9.

Now we consider D_0^* . Using the notation of Theorem 8.2, the winding function in D_0^* is given by $\varphi(z) = \tilde{\varphi} \circ j(z)$ where $\tilde{\varphi}$ is the winding function associated with the foliation under the inversion map $j : z \mapsto 1/z$ by Lemma 8.3. Since the family of equipotentials is preserved under the inversion, it follows that $\tilde{\varphi}$ is harmonic in $j(D_0^*)$ and

$$\tilde{\varphi} = \vartheta[j \circ k_0 \circ j] = \vartheta[k_0] \circ j,$$

using the chain rule (2.8) and $\vartheta[j] \equiv 0$. Therefore $\varphi|_{D_0^*} = \vartheta[k_0]$. □

Recall that the Loewner energy of the curve γ is

$$L^L(\gamma) = \mathcal{D}_{\mathbb{D}}(\arg f') + \mathcal{D}_{\mathbb{D}^*}(\arg h') + 4 \log |f'(0)/h'(\infty)|. \quad (8.1)$$

(We have normalized so that $f'(0) = 1$ but we will keep it in the notation since (8.1) holds more generally.)

⁴This assumption is only made for convenience, so that the curve γ corresponds to time-index 0 in the foliation. All results in this section hold for a general Weil–Petersson quasicircle γ separating 0 from ∞ .

Theorem 8.6. *Let γ be a Weil–Peterson quasicircle separating 0 from ∞ . Let ρ^γ be the measure associated to γ . Then*

$$16 S(\rho^\gamma) = I^L(\gamma) - 2 \log |f'(0)/h'(\infty)| < \infty.$$

Moreover, if $\rho \in \mathcal{N}$ generates γ as a leaf, then $S(\rho^\gamma) \leq S(\rho)$.

Proof. By Lemma 8.5, the winding function associated to ρ^γ satisfies

$$\begin{aligned} \mathcal{D}_{\mathbb{C}}(\varphi) &= \frac{1}{\pi} \int_{D_0} |\nabla \vartheta[g_0]|^2 dA(z) + \frac{1}{\pi} \int_{D_0^*} |\nabla \vartheta[k_0]|^2 dA(z) \\ &= \frac{1}{\pi} \int_{\mathbb{D}} |\nabla \vartheta[f]|^2 dA(z) + \frac{1}{\pi} \int_{\mathbb{D}^*} |\nabla \vartheta[h]|^2 dA(z) \\ &= \frac{1}{\pi} \int_{\mathbb{D}} \left| \frac{f''}{f'} - \frac{f'}{f} + \frac{1}{z} \right|^2 dA(z) + \frac{1}{\pi} \int_{\mathbb{D}^*} \left| \frac{h''}{h'} - \frac{h'}{h} + \frac{1}{z} \right|^2 dA(z) \\ &= \frac{1}{\pi} \int_{\mathbb{D}} \left| \frac{f''}{f'} \right|^2 dA(z) + \frac{1}{\pi} \int_{\mathbb{D}^*} \left| \frac{h''}{h'} \right|^2 dA(z) - \frac{1}{\pi} \int_{\mathbb{D}} \left| \frac{f'}{f} - \frac{1}{z} \right|^2 dA(z) \\ &\quad - \frac{1}{\pi} \int_{\mathbb{D}^*} \left| \frac{h'}{h} - \frac{1}{z} \right|^2 dA(z) \\ &= \frac{1}{\pi} \int_{\mathbb{D}} \left| \frac{f''}{f'} \right|^2 dA(z) + \frac{1}{\pi} \int_{\mathbb{D}^*} \left| \frac{h''}{h'} \right|^2 dA(z) + 2 \log |f'(0)/h'(\infty)| \\ &= I^L(\gamma) - 2 \log |f'(0)/h'(\infty)|, \end{aligned}$$

where we used (8.1) and Lemma 8.7 proved just below. Theorem 1.2 then implies the identity. The claim $S(\rho^\gamma) \leq S(\rho)$ follows from the fact that the foliation formed by the equipotentials in D is generated by the zero energy measure for $t \geq 0$, thus minimizes the energy among all foliations in D , and the reversibility Theorem 8.2. \square

Lemma 8.7. *We have the identity*

$$\begin{aligned} \operatorname{Re} \left[\int_{\mathbb{D}} \frac{f''}{f'} \left(\overline{\frac{f'}{f} - \frac{1}{z}} \right) dA(z) + \int_{\mathbb{D}^*} \frac{h''}{h'} \left(\overline{\frac{h'}{h} - \frac{1}{z}} \right) dA(z) \right] \\ = \int_{\mathbb{D}} \left| \frac{f'}{f} - \frac{1}{z} \right|^2 dA(z) + \int_{\mathbb{D}^*} \left| \frac{h'}{h} - \frac{1}{z} \right|^2 dA(z) = 2\pi \log \left| \frac{h'(\infty)}{f'(0)} \right|. \end{aligned}$$

Proof. Consider $\tilde{f} := j \circ h \circ j$ and $\tilde{h} := j \circ f \circ j$, the conformal maps associated to $j(\gamma)$. We have $\tilde{f}'(0) = h'(\infty)^{-1}$ and

$$\frac{\tilde{f}''(z)}{\tilde{f}'(z)} = -\frac{1}{z^2} \left(\frac{h''(1/z)}{h'(1/z)} - \frac{2h'(1/z)}{h(1/z)} + 2z \right)$$

and similarly for \tilde{h} . We compute

$$\begin{aligned} I^L(j(\gamma)) &= \frac{1}{\pi} \int_{\mathbb{D}} \left| \frac{\tilde{f}''}{\tilde{f}'} \right|^2 dA(z) + \frac{1}{\pi} \int_{\mathbb{D}^*} \left| \frac{\tilde{h}''}{\tilde{h}'} \right|^2 dA(z) + 4 \log \left| \frac{\tilde{f}'(0)}{\tilde{h}'(\infty)} \right| \\ &= \frac{1}{\pi} \int_{\mathbb{D}^*} \left| \frac{h''}{h'} - \frac{2h'}{h} + \frac{2}{z} \right|^2 dA(z) + \frac{1}{\pi} \int_{\mathbb{D}} \left| \frac{f''}{f'} - \frac{2f'}{f} + \frac{2}{z} \right|^2 dA(z) + 4 \log \left| \frac{f'(0)}{h'(\infty)} \right| \end{aligned}$$

$$\begin{aligned}
&= I^L(\gamma) - \int_{\mathbb{D}^*} 4 \operatorname{Re} \left[\frac{h''}{h'} \left(\frac{\overline{h'} - 1}{z} \right) \right] dA(z) - \int_{\mathbb{D}} 4 \operatorname{Re} \left[\frac{f''}{f'} \left(\frac{\overline{f'} - 1}{z} \right) \right] dA(z) \\
&\quad + 4 \int_{\mathbb{D}^*} \left| \frac{h'}{h} - \frac{1}{z} \right|^2 dA(z) + 4 \int_{\mathbb{D}} \left| \frac{f'}{f} - \frac{1}{z} \right|^2 dA(z).
\end{aligned}$$

Since the Loewner energy of a Jordan curve is Möbius invariant, we have $I^L(\gamma) = I^L(j(\gamma))$ and we obtain the first equality. The second equality follows from Lemma 5.4 since γ has Lebesgue measure zero. \square

Combining Theorem 1.1 and Theorem 8.6, we obtain a new characterization of Weil–Petersson quasicircles.

Corollary 8.8. *A Jordan curve γ separating 0 from ∞ is a Weil–Petersson quasicircle if and only if γ can be realized as a leaf in the foliation generated by a measure ρ with $S(\rho) < \infty$.*

Corollary 8.9. *The Loewner energies of the leaves generated by ρ are uniformly bounded by $16 S(\rho)$.*

Proof. For all $t \in \mathbb{R}$, Theorem 1.2 and Theorem 8.6 imply

$$16 S(\rho) \geq 16 S(\rho^\gamma) = I^L(\gamma_t) - 2 \log |f'_t(0)| + 2 \log |h'_t(\infty)|,$$

where f_t, h_t are the conformal maps associated to γ_t . On the other hand, Lemma 5.4 shows that

$$2 \log |h'_t(\infty)| - 2 \log |f'_t(0)| \geq 0.$$

Thus, $I^L(\gamma_t) \leq 16 S(\rho)$ as claimed. \square

Remark. Corollary 8.9 provides a way to generate and simulate Weil–Petersson quasicircles of bounded Loewner energy using a measure with controlled Loewner–Kufarev energy. In fact, we obtain infinitely many such quasicircles for any given measure.

Remark. The Loewner energy $I^L(\gamma)$ is invariant under Möbius transformations, and is known to be a Kähler potential for the Weil–Petersson metric defined on a subspace $T_0(1)$ of the universal Teichmüller space, see [61, Thm. II.4.1]. On the other hand, $\log |f'(0)/h'(\infty)|$ is only invariant under scaling and rotation which is consistent with the fact that the Loewner–Kufarev equation has two marked points 0 and ∞ on the Riemann sphere. We also point out that $\log |f'(0)/h'(\infty)|$ is a Kähler potential for the Velling–Kirillov metric on the universal Teichmüller curve, a complex fiber bundle over $T_0(1)$, see [61, Thm. I.5.3].

8.3 Complex identity: Proof of Proposition 1.5

Recall that a Weil–Petersson quasicircle γ separating 0 from ∞ is said to be compatible with $\varphi \in W_{\text{loc}}^{1,2}$ if the winding function of γ coincides with the trace $\varphi|_\gamma$ arlength-a.e. Since γ by assumption is compatible with $\operatorname{Im} \psi$,

$$\vartheta[f](z) = -\vartheta[f^{-1}](f(z)) = -\operatorname{Im} \psi^h(f(z)), \quad \forall z \in \mathbb{D},$$

and Lemma 8.5 also shows

$$\vartheta[h](z) = -\operatorname{Im} \psi^h(h(z)), \quad \forall z \in \mathbb{D}^*,$$

where $\operatorname{Im} \psi^h + \operatorname{Im} \psi^0$ is the orthogonal decomposition of $\operatorname{Im} \psi$ with respect to $\mathbb{C} \setminus \gamma$ as in (2.1). Hence we can write,

$$\begin{aligned} \zeta &= \left(\operatorname{Re} \psi \circ f + \log \left| \frac{zf'(z)}{f(z)} \right| \right) + i(\operatorname{Im} \psi \circ f + \vartheta[f](z)) = u + i \operatorname{Im} \psi^0 \circ f; \\ \xi &= v + i \operatorname{Im} \psi^0 \circ h, \end{aligned}$$

where

$$u := \operatorname{Re} \psi \circ f + \log \left| \frac{zf'(z)}{f(z)} \right|, \quad v := \operatorname{Re} \psi \circ h + \log \left| \frac{zh'(z)}{h(z)} \right|.$$

Notice that $\log |zf'(z)/f(z)|$ is a harmonic conjugate of $\vartheta[f]$, so they have the same Dirichlet energy. Therefore

$$\begin{aligned} \mathcal{D}_{\mathbb{D}}(\zeta) + \mathcal{D}_{\mathbb{D}^*}(\xi) &= \mathcal{D}_{\mathbb{D}}(u) + \mathcal{D}_{\mathbb{D}}(v) + \mathcal{D}_{\mathbb{C}}(\operatorname{Im} \psi^0) \\ &= \mathcal{D}_{\mathbb{C}}(\operatorname{Re} \psi) + \mathcal{D}_{\mathbb{C}}(\operatorname{Im} \psi^h) + \text{“cross-terms”} + \mathcal{D}_{\mathbb{C}}(\operatorname{Im} \psi^0) \\ &= \mathcal{D}_{\mathbb{C}}(\operatorname{Re} \psi) + \mathcal{D}_{\mathbb{C}}(\operatorname{Im} \psi) + \text{“cross-terms”} \end{aligned}$$

where the “cross-terms” come from expanding the Dirichlet integrals of u and v and equal 2 times

$$\int_{\mathbb{D}} \left\langle \nabla \operatorname{Re} \psi \circ f, \nabla \log \left| \frac{f'(z)z}{f(z)} \right| \right\rangle dA(z) + \int_{\mathbb{D}^*} \left\langle \nabla \operatorname{Re} \psi \circ h, \nabla \log \left| \frac{h'(z)z}{h(z)} \right| \right\rangle dA(z). \quad (8.2)$$

It suffices to show that (8.2) vanishes. We will only prove it assuming that γ is smooth. The general case of a Weil–Petersson quasicircle can be deduced using an approximation argument following exactly the same proof as [62, Thm. 3.1].

Using Stokes’ formula, the first term in (8.2) equals

$$\begin{aligned} &\int_{\partial \mathbb{D}} \operatorname{Re} \psi \circ f(z) \partial_n \log \left| \frac{f'(z)z}{f(z)} \right| |dz| \\ &= \int_{\partial \mathbb{D}} \operatorname{Re} \psi \circ f(z) \partial_n \log |f'(z)| |dz| + \operatorname{Re} \left[\int_{\partial \mathbb{D}} \operatorname{Re} \psi \circ f(z) \partial_n \left(\log \frac{z}{f(z)} \right) |dz| \right] \\ &=: I_1 + I_2 \end{aligned}$$

where ∂_n is the normal derivative in the outward pointing direction. Using the formula $\partial_n \log |f'(z)| = k_{\Omega} \circ f(z) |f'(z)| - k_{\mathbb{D}}(z) = k_{\Omega} \circ f(z) |f'(z)| - 1$, where k_{Ω} is the geodesic curvature of $\partial \Omega$ (see, e.g., [64, Appx. A]), we obtain

$$I_1 = \int_{\partial \Omega} \operatorname{Re} \psi(w) k_{\Omega}(w) |dw| - \int_{\partial \mathbb{D}} \operatorname{Re} \psi \circ f(z) |dz|.$$

Using $z|dz| = -idz$, we have

$$I_2 = \operatorname{Re} \int_{\partial \mathbb{D}} \operatorname{Re} \psi \circ f(z) z \left(-\frac{f'(z)}{f(z)} + \frac{1}{z} \right) |dz|$$

$$\begin{aligned}
&= \operatorname{Re} \int_{\partial \mathbb{D}} i \operatorname{Re} \psi \circ f(z) \frac{f'(z)}{f(z)} dz + \int_{\partial \mathbb{D}} \operatorname{Re} \psi \circ f(z) |dz| \\
&= \operatorname{Re} \int_{\partial \Omega} \frac{i \operatorname{Re} \psi(w)}{w} dw + \int_{\partial \mathbb{D}} \operatorname{Re} \psi \circ f(z) |dz|.
\end{aligned}$$

The sum of I_1 , I_2 and those integrals coming from the second term of (8.2) vanishes since $k_\Omega(y) = -k_{\Omega^*}(y)$ and the contour integral in I_2 winds in the opposite direction for Ω and Ω^* . This concludes the proof. \square

9 Conformal distortion formula

The goal of this section is to compute an explicit formula for the change of the Loewner–Kufarev energy under conformal transformation of the foliation and prove Theorem 1.6 which combines Proposition 9.2 and Corollary 9.3.

We consider the following set up. Let $\rho \in \mathcal{N}_+$ be such that

$$S_{[0,1]}(\rho) = \int_0^1 L(\rho_t) dt < \infty.$$

We write as before $\rho_t = \nu_t^2(\theta)d\theta$. Suppose ψ is a conformal transformation that maps K_1 onto another compact hull $\psi(K_1) = \tilde{K}_1 \subset \bar{\mathbb{D}}$, and is defined on a neighborhood U of K_1 in $\bar{\mathbb{D}}$ which is mapped onto a neighborhood \tilde{U} of \tilde{K}_1 in $\bar{\mathbb{D}}$. Note that we always have $S^1 \subset K_1$ by definition. See Figure 9.1. The image of the hulls ($\tilde{K}_t := \psi(K_t)$), is driven by a measure $t \mapsto \tilde{\rho}_t$. Note that $\tilde{\rho}_t$ may not be a probability measure since $\tilde{D}_t := \mathbb{C} \setminus \tilde{K}_t$ in general does not have conformal radius e^{-t} .

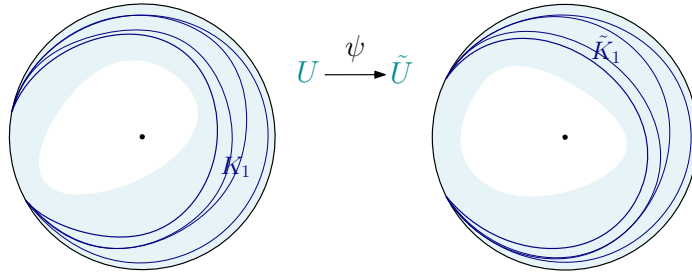


Figure 9.1: Conformal distortion of a foliation. The mapping ψ is conformal in a neighborhood of the hull K generated by ρ , which is mapped to another hull \tilde{K} , generated by a measure $\tilde{\rho}$. Proposition 9.2 computes the difference of the Loewner–Kufarev energies of ρ and $\tilde{\rho}$.

Let $(g_t = f_t^{-1})_{t \leq 1}$ and $(\tilde{g}_t = \tilde{f}_t^{-1})_{t \leq 1}$ be the corresponding uniformizing Loewner chains. Let $\psi_t := \tilde{g}_t \circ \psi \circ f_t$. Then ψ_t is a conformal map of $U_t := g_t(U \setminus K_t)$ onto $\tilde{U}_t := \tilde{g}_t(\tilde{U} \setminus \tilde{K}_t)$. By the Schwarz reflection principle, ψ_t extends to a holomorphic function in a neighborhood of S^1 .

Lemma 9.1. *For a.e. $t \in [0, 1]$, $\tilde{\rho}_t \ll d\theta$. If we write $\tilde{\rho}_t = \tilde{\nu}_t^2(\theta)d\theta$, then*

$$\tilde{\nu}_t^2(\theta_t) = |\psi_t'(e^{i\theta})| \nu_t^2(\theta), \quad (9.1)$$

where $\theta_t \in [0, 2\pi]_{/0 \sim 2\pi}$ satisfies $e^{i\theta_t} = \psi_t(e^{i\theta})$.

Proof. Let \tilde{H}_t be the Herglotz integral of $\tilde{\rho}_t$. Let $0 < r < 1$ and consider for $t \in [0, 1]$ the curve $\gamma_t^r = f_t(rS^1)$ passing through $f_t(re^{i\theta}) =: w$. We can compute the normal velocity of $\psi \circ \gamma_t^r$ at $\psi(w)$ in two ways. First starting from the velocity at w using that ψ is conformal: this gives $r|\psi'(w)||f'_t(re^{i\theta})|\operatorname{Re} H_t(re^{i\theta})$; and second, directly from the Loewner–Kufarev equation driven by $\tilde{\rho}$, which gives

$$|\tilde{f}'_t(\psi_t(re^{i\theta}))|\operatorname{Re} \overline{e^{i\theta}\psi'_t(re^{i\theta})}|\psi'_t(re^{i\theta})|^{-1}\psi_t(re^{i\theta})\tilde{H}_t(\psi_t(re^{i\theta})).$$

Indeed, to see why these formulas hold, first note that a unit normal at w is

$$\mathbf{n}(\gamma_t^r(e^{i\theta})) = -e^{i\theta} \frac{f'_t(re^{i\theta})}{|f'_t(re^{i\theta})|}, \quad \theta \in [0, 2\pi).$$

Using the Loewner equation, we see that the normal velocity with respect to the curve $\gamma_t^r(\theta)$ at time t at the point w is

$$\begin{aligned} -\operatorname{Re} \frac{\partial_t f_t(re^{i\theta}) \overline{e^{i\theta} f'_t(re^{i\theta})}}{|f'_t(re^{i\theta})|} &= \operatorname{Re} \frac{re^{i\theta} f'_t(re^{i\theta}) H_t(re^{i\theta}) \overline{e^{i\theta} f'_t(re^{i\theta})}}{|f'_t(re^{i\theta})|} \\ &= r|f'_t(re^{i\theta})|\operatorname{Re} H_t(re^{i\theta}). \end{aligned}$$

Next, by definition $\psi_t = \tilde{g}_t \circ \psi \circ f_t$, so $\tilde{f}_t \circ \psi_t = \psi \circ f_t$. Since $w = f_t(re^{i\theta})$ we have $\psi'(w)f'_t(re^{i\theta}) = \tilde{f}'_t(\psi_t(re^{i\theta}))\psi'_t(re^{i\theta})$ and the normal velocity of the image curve at $\psi(w)$ is

$$\begin{aligned} -\operatorname{Re} \frac{(\partial_t \tilde{f}_t)(\psi_t(re^{i\theta})) \overline{e^{i\theta}\psi'(w)f'_t(re^{i\theta})}}{|\psi'(w)||f'_t(re^{i\theta})|} \\ &= \operatorname{Re} \frac{\psi_t(re^{i\theta}) \tilde{f}'_t(\psi_t(re^{i\theta})) \tilde{H}_t(\psi_t(re^{i\theta})) \overline{e^{i\theta}\psi'(w)f'_t(re^{i\theta})}}{|\psi'(w)||f'_t(re^{i\theta})|} \\ &= \operatorname{Re} \frac{\psi_t(re^{i\theta}) \tilde{f}'_t(\psi_t(re^{i\theta})) \tilde{H}_t(\psi_t(re^{i\theta})) \overline{e^{i\theta} \tilde{f}'_t(\psi_t(re^{i\theta})) \psi'_t(re^{i\theta})}}{|\tilde{f}'_t(\psi_t(re^{i\theta}))\psi'_t(re^{i\theta})|} \\ &= |\tilde{f}'_t(\psi_t(re^{i\theta}))| \operatorname{Re} \frac{\psi_t(re^{i\theta}) \tilde{H}_t(\psi_t(re^{i\theta})) \overline{e^{i\theta}\psi'_t(re^{i\theta})}}{|\psi'_t(re^{i\theta})|} \end{aligned}$$

We get

$$r|\psi'(w)||f'_t(re^{i\theta})|\operatorname{Re} H_t(re^{i\theta}) = |\tilde{f}'_t(\psi_t(re^{i\theta}))| \operatorname{Re} \left(\frac{e^{i\theta}\psi'_t(re^{i\theta})}{|\psi'_t(re^{i\theta})|} \psi_t(re^{i\theta}) \tilde{H}_t(\psi_t(re^{i\theta})) \right).$$

Note that H_t is continuous on $\overline{\mathbb{D}}$ and ψ_t extends to be holomorphic on a neighborhood of S^1 . Moreover, as $r \rightarrow 1-$, $e^{i\theta}\psi'_t(re^{i\theta})\psi_t(re^{i\theta})/|\psi'_t(re^{i\theta})| \rightarrow 1$. Since $\operatorname{Re} \tilde{H}_t = 2\pi P_{\mathbb{D}}[\tilde{\rho}_t]$, we obtain that $\tilde{\rho}_t \ll d\theta$ and (9.1) by letting $r \rightarrow 1-$ and using the definition $\psi_t = (\tilde{f}_t)^{-1} \circ \psi \circ f_t$ and the chain rule. \square

For the next statement, recall that $\mathcal{S}f$ denotes the Schwarzian derivative of f .

Proposition 9.2. *We have*

$$L(\tilde{\rho}_t) - L(\rho_t) = \frac{1}{4} \int_{S^1} e^{2i\theta} \mathcal{S}\psi_t(e^{i\theta}) d\rho_t(\theta) + \frac{1}{8} (|\tilde{\rho}_t| - |\rho_t|).$$

Note that by conjugating ψ_t by a Möbius transformation mapping \mathbb{D} to \mathbb{H} , one sees that $e^{2i\theta} \mathcal{S}\psi_t(e^{i\theta}) \in \mathbb{R}$.

Proof. We use the same notation as in Lemma 9.1. Differentiating (9.1) with respect to θ , we obtain using $\partial\theta_t/\partial\theta = |\psi'_t(e^{i\theta})|$ that

$$\tilde{\nu}'_t(\theta_t)\sqrt{|\psi'_t(e^{i\theta})|} = \nu'_t(\theta) + \frac{\partial_\theta |\psi'_t(e^{i\theta})|}{2|\psi'_t(e^{i\theta})|}\nu_t(\theta).$$

Plugging this into the expression for $L(\tilde{\rho}_t)$, we get

$$\begin{aligned} L(\tilde{\rho}_t) &= \frac{1}{2} \int_{S^1} \tilde{\nu}'_t(\theta)^2 d\theta = \frac{1}{2} \int_{S^1} \tilde{\nu}'_t(\psi_t(e^{i\theta}))^2 |\psi'_t(e^{i\theta})| d\theta = \frac{1}{2} \int_{S^1} \left(\nu'_t + \frac{\partial_\theta |\psi'_t|}{2|\psi'_t|} \nu_t \right)^2 d\theta \\ &= L(\rho_t) + \frac{1}{2} \int_{S^1} \frac{\partial_\theta |\psi'_t|}{|\psi'_t|} \nu'_t \nu_t d\theta + \frac{1}{8} \int_{S^1} \left(\frac{\partial_\theta |\psi'_t|}{|\psi'_t|} \right)^2 \nu_t^2 d\theta. \end{aligned}$$

Integrating

$$\partial_\theta \left[\nu_t^2 \frac{\partial_\theta |\psi'_t|}{|\psi'_t|} \right] = 2\nu'_t \nu_t \frac{\partial_\theta |\psi'_t|}{|\psi'_t|} + \nu_t^2 \left[\frac{\partial_\theta^2 |\psi'_t|}{|\psi'_t|} - \left(\frac{\partial_\theta |\psi'_t|}{|\psi'_t|} \right)^2 \right],$$

over S^1 against $d\theta$ gives 0. It follows that $L(\tilde{\rho}_t) - L(\rho_t)$ equals

$$\frac{1}{2} \int_{S^1} \frac{\partial_\theta |\psi'_t|}{|\psi'_t|} \nu'_t \nu_t d\theta + \frac{1}{8} \int_{S^1} \left(\frac{\partial_\theta |\psi'_t|}{|\psi'_t|} \right)^2 \nu_t^2 d\theta = -\frac{1}{4} \int_{S^1} \nu_t^2 \left[\frac{\partial_\theta^2 |\psi'_t|}{|\psi'_t|} - \frac{3}{2} \left(\frac{\partial_\theta |\psi'_t|}{|\psi'_t|} \right)^2 \right] d\theta.$$

Using $|\psi'_t(z)| = \psi'_t(z)z/\psi_t(z)$ and $\partial_\theta = iz\partial_z$, we compute

$$\frac{\partial_\theta |\psi'_t(z)|}{|\psi'_t(z)|} = iz\partial_z \log |\psi'_t(z)| = iz \left(\frac{\psi''_t}{\psi'_t} - \frac{\psi'_t}{\psi_t} + \frac{1}{z} \right)$$

and

$$\begin{aligned} -\frac{1}{2} \left(\frac{\partial_\theta |\psi'_t(z)|}{|\psi'_t(z)|} \right)^2 &= \frac{z^2}{2} \left[\left(\frac{\psi''_t}{\psi'_t} \right)^2 + \left(\frac{\psi'_t}{\psi_t} \right)^2 + \frac{1}{z^2} - 2\frac{\psi''_t}{\psi_t} - 2\frac{\psi'_t}{z\psi_t} + 2\frac{\psi''_t}{z\psi'_t} \right] \\ &= \frac{z^2}{2} \left(\frac{\psi''_t}{\psi'_t} \right)^2 + \frac{z^2}{2} \left(\frac{\psi'_t}{\psi_t} \right)^2 + \frac{1}{2} - z^2 \frac{\psi''_t}{\psi_t} - \frac{z\psi'_t}{\psi_t} + \frac{z\psi''_t}{\psi'_t}. \end{aligned}$$

Moreover,

$$\begin{aligned} \partial_\theta \left(\frac{\partial_\theta |\psi'_t(z)|}{|\psi'_t(z)|} \right) &= iz \left[i \left(\frac{\psi''_t}{\psi'_t} - \frac{\psi'_t}{\psi_t} + \frac{1}{z} \right) + iz \left(\left(\frac{\psi''_t}{\psi'_t} \right)' - \left(\frac{\psi'_t}{\psi_t} \right)' - \frac{1}{z^2} \right) \right] \\ &= -z^2 \left(\frac{\psi''_t}{\psi'_t} \right)' - z \left(\frac{\psi''_t}{\psi'_t} - \frac{\psi'_t}{\psi_t} + \frac{1}{z} \right) + z^2 \left(\frac{\psi''_t}{\psi_t} - \frac{(\psi'_t)^2}{\psi_t^2} \right) + 1. \end{aligned}$$

We obtain

$$\frac{\partial_\theta^2 |\psi'_t|}{|\psi'_t|} - \frac{3}{2} \left(\frac{\partial_\theta |\psi'_t|}{|\psi'_t|} \right)^2 = \partial_\theta \left(\frac{\partial_\theta |\psi'_t|}{|\psi'_t|} \right) - \frac{1}{2} \left(\frac{\partial_\theta |\psi'_t|}{|\psi'_t|} \right)^2$$

$$= -z^2 \mathcal{S}\psi_t - \frac{z^2}{2} \left(\frac{\psi'_t}{\psi_t} \right)^2 + \frac{1}{2} = -z^2 \mathcal{S}\psi_t + \frac{1 - |\psi'_t|^2}{2}.$$

Combining these computations, we get

$$\begin{aligned} L(\tilde{\rho}_t) - L(\rho_t) &= -\frac{1}{4} \int_{S^1} \nu_t^2(\theta) \left[\frac{\partial_\theta^2 |\psi'_t|}{|\psi'_t|} - \frac{3}{2} \left(\frac{\partial_\theta |\psi'_t|}{|\psi'_t|} \right)^2 \right] d\theta \\ &= \frac{1}{4} \int_{S^1} \nu_t^2(\theta) \left[e^{2i\theta} \mathcal{S}\psi_t + \frac{|\psi'_t|^2 - 1}{2} \right] d\theta \\ &= \frac{1}{4} \int_{S^1} \nu_t^2(\theta) e^{2i\theta} \mathcal{S}\psi_t(e^{i\theta}) d\theta + \frac{1}{8} \left(\int_{S^1} \tilde{\nu}_t^2(\theta) d\theta - \int_{S^1} \nu_t^2(\theta) d\theta \right), \end{aligned}$$

where we used (9.1) in the last equality and

$$\int_{S^1} \nu_t^2(\theta) |\psi'_t|^2 d\theta = \int_{S^1} \tilde{\nu}_t^2(\theta) |\psi'_t| d\theta = \int_{S^1} \tilde{\nu}_t^2(\theta) d\theta$$

which completes the proof. \square

Corollary 9.3. *We have*

$$S_{[0,1]}(\tilde{\rho}) - S_{[0,1]}(\rho) = \frac{1}{4} \int_0^1 \int_{S^1} e^{2i\theta} \mathcal{S}\psi_t(e^{i\theta}) \rho_t(\theta) d\theta dt + \frac{1}{8} (\log \tilde{g}'_1(0) - 1). \quad (9.2)$$

Proof. The formula follows by integrating (9.1), using $\int_0^1 |\rho_t| dt = 1$ and $\int_0^1 |\tilde{\rho}_t| dt = \log \tilde{g}'_1(0)$. \square

Remark. The Brownian loop measure is a conformally invariant σ -finite measure on Brownian loops in the plane [34, 35]. The conformal distortion formula of Theorem 1.6 can be interpreted in terms of Brownian loop measures: namely, $S_+(\rho) - S_+(\tilde{\rho})$ also equals the difference between the Brownian loop measure of those loops in \mathbb{D} that intersect both K_1 and $\mathbb{D} \setminus U$ and the measure of those loops in \mathbb{D} that intersect both \tilde{K}_1 and $\mathbb{D} \setminus \tilde{U}$. We will prove this in a forthcoming paper with Lawler. For now we just remark that this interpretation immediately implies that the energy difference depends only on the hulls at time 1 and not on the foliation, a fact that is not immediately apparent from (9.2).

10 Further comments and open problems

We will now indicate further implications and interpretations as well as open problems suggested by our results.

Random conformal geometry and mating-of-trees. We begin by discussing connections to ideas in random conformal geometry, see in particular [18, 39], which served as inspiration for the formulation of our main result, Theorem 1.2. We will also speculate on how to formulate stochastic versions of some of the results obtained in this paper but we do not make rigorous statements here.

Let $(B_t)_{t \in \mathbb{R}}$ be the rotation-invariant two-sided Brownian motion on S^1 and suppose $\kappa \geq 0$. Whole-plane SLE_κ is the random (whole-plane) Loewner chain generated by

the measure $\rho(d\theta dt) = \delta_{B_{\kappa t}}(\theta)dt \in \mathcal{N}$, where $\delta_{B_{\kappa t}}$ is the Dirac mass at $B_{\kappa t}$ and dt is Lebesgue measure on \mathbb{R} . When $\kappa \geq 8$, the hull is generated by a space-filling curve growing in \mathbb{C} from ∞ towards 0, see [48]. The Loewner–Kufarev energy is infinite in the case of SLE, since a Dirac mass is not an absolutely continuous measure. However, an easy generalization of the main result of [5] shows that the Loewner–Kufarev energy is the large deviation rate function of whole-plane SLE_∞ . Roughly speaking, as $\kappa \rightarrow \infty$,

$$\mathbb{P}\{\text{Whole-plane } \text{SLE}_\kappa \text{ domains stay close to } (D_t)_{t \in \mathbb{R}}\} \approx \exp(-\kappa S(\rho)) \quad (10.1)$$

where $(D_t)_{t \in \mathbb{R}}$ is the family of domains of any deterministic whole-plane Loewner chain with driving measure ρ . See [5, Thm. 1.2] for a precise statement in the unit disk setup. On the other hand, the Loewner energy I^L of a Jordan curve can be expressed in terms of Dirichlet energies, see (1.1), and it is believed to be the large deviation rate function of the SLE_{0+} loop: as $\kappa \rightarrow 0+$,

$$\mathbb{P}\{\text{SLE}_\kappa \text{ loop stays close to } \gamma\} \approx \exp(-I^L(\gamma)/\kappa) \quad (10.2)$$

where γ is a given deterministic Jordan curve. See [43, 63] for precise statements in chordal settings. Furthermore, it is well-known that the Dirichlet energy is the large deviation rate function for the Gaussian free field, see [15, Thm. 3.4.12].

SLE processes enjoy a duality property with respect to replacing κ by $16/\kappa$ [17, 37, 68]. Roughly speaking, an SLE_κ curve describes locally the outer boundary of an $\text{SLE}_{16/\kappa}$ hull when $\kappa < 4$. The mating-of-trees theorem [18] further explores this duality and the interplay with an underlying Liouville quantum gravity field. (See also, e.g., [4] and the references therein for some recent progress in this direction.) An impressionistic picture is as follows. A pair of “mated” space-filling trees whose branches are formed by SLE_κ -like curves are constructed as flowlines of a Gaussian field and a coupled whole-plane chordal space-filling $\text{SLE}_{16/\kappa}$ curve from ∞ to ∞ traces the interface between the pair of trees. One can speculate that the union of the pair of trees degenerate to a foliation as $\kappa \rightarrow 0+$ (and $16/\kappa \rightarrow +\infty$) and the mating-of-trees coupling suggests that the large deviation rate functions of the coupled processes should match in this limit. Combining this with the heuristic formulas (10.1) and (10.2) led us to guess Theorem 1.2 where the factor 16 is consistent with the SLE $\kappa \leftrightarrow 16/\kappa$ duality.

However, the setup of [18] uses whole-plane chordal Loewner evolution so the space-filling SLE there runs from ∞ to ∞ , whereas the whole-plane radial SLE runs from ∞ to 0. A coupling of radial SLE with the Gaussian free field is described in [39], but we are not aware of results similar to the mating-of-trees theorem in the current literature for the setting we work in. Our results, in particular Theorem 1.1, Theorem 1.2, and Proposition 1.5 provide analytical evidence for a radial mating-of-trees theorem.

Using the dictionary we outlined in [62, Sec. 1.3 and 3.4], we may speculate that the following statements should hold. For small $\kappa > 0$, run a space filling whole-plane $\text{SLE}_{16/\kappa}$ on an appropriate “quantum sphere” assumed to be independent of the SLE. (A quantum sphere can be described by a Gaussian free field with additional logarithmic singularities and an attached transformation law.) If the SLE process is run up to time t , the unvisited part and visited part of the quantum sphere form two independent “quantum disks” (which are also defined starting from a Gaussian free field) conformally

welded along the frontier of the whole-plane $\text{SLE}_{16/\kappa}$, which itself is an SLE_κ -type loop. The two “quantum disks” are each decorated with an independent radial $\text{SLE}_{16/\kappa}$ curve, and the two $\text{SLE}_{16/\kappa}$ curves are independent conditionally on the position of the tip at time t . Varying t , we expect the separating SLE_κ -type loops to form a (fractal) foliation-like family that sweeps out the twice punctured sphere. This foliation-like process also encodes the whole-plane $\text{SLE}_{16/\kappa}$ evolution. The real part of the complex field in Proposition 1.5 reflects the metric/measure structure of the quantum sphere, whereas the imaginary part encodes the fractal foliation-like process, hence the trajectory of the space filling SLE.

Whole-plane radial SLE Reversibility. Let us next comment on the reversibility of the Loewner–Kufarev energy, Theorem 1.3. An analogous result about the reversibility of the Loewner energy can be explained (and proved) by SLE_{0+} large deviations considerations combined with the fact that chordal SLE is reversible [69] for small κ , see [63]. Chordal SLE is however not reversible for $\kappa > 8$, and it is not known whether whole-plane SLE_κ for $\kappa > 8$ is reversible or not. (For $\kappa \leq 8$, reversibility was established in [39, 70].) Therefore Theorem 1.3 cannot be predicted from the SLE point of view given currently known results, but it does on the other hand suggest that reversibility for whole-plane radial SLE might hold for large κ as well.

Update: In the time since our paper first appeared, proofs of large κ reversibility for whole-plane SLE and a radial mating-of-trees theorem have appeared [3, 6].

Whole-plane chordal Loewner–Kufarev energy. One can ask about a version of our results in chordal settings. The most natural one is the whole-plane chordal version, where the family of curves all pass through ∞ and foliate the plane \mathbb{C} as t ranges from $-\infty$ to ∞ , as in the mating-of-trees theorem. When $\kappa \rightarrow \infty$, the whole-plane chordal SLE_κ Loewner chain converges to the constant identity map for all time t . Therefore, renormalization is needed to obtain both a non-trivial limit and a meaningful large deviation result. (This is one reason to work in the radial setup here as well as in [5].) One way to proceed is to conformally map the two punctures (0 and ∞) in our whole-plane (radial) setup to y and ∞ then let $y \rightarrow \infty$. The third complex degree of freedom (ranging in a non compact space) in the choice of conformal automorphism of the Riemann sphere needs to be chosen carefully to obtain a clean statement.

Foliation loops in Weil–Petersson Teichmüller space. Recall that any quasicircle γ separating 0 from ∞ can be identified with an element of universal Teichmüller space $T(1) \simeq \text{Möb}(S^1) \setminus \text{QS}(S^1)$ via (the equivalence class of) its welding homeomorphism $\phi_\gamma = h^{-1} \circ f|_{S^1}$. The subspace $T_0(1)$ corresponding to Weil–Petersson quasicircles has an infinite dimensional Kähler–Einstein manifold structure when equipped with the Weil–Petersson metric, see [61]. Theorem 1.1 shows that any ρ with $S(\rho) < \infty$ generates a foliation $(\gamma_t)_{t \in \mathbb{R}}$ of Weil–Petersson quasicircles which can be considered as elements of $T_0(1)$ via their welding homeomorphisms. So the Loewner evolution of ρ generates a dynamical process on $T_0(1)$. Theorem 1.4 shows that there exists such a family (obtained by interpolating by equipotentials) passing through any given element of $T_0(1)$. It is not too hard to show that it corresponds to a continuous loop $t \mapsto [\phi_{\gamma_t}]$ in $T_0(1)$ starting and

ending at the origin $[\text{Id}_{\mathbb{S}^1}]$. We believe the class of loops in $T_0(1)$ coming from measures with $S(\rho) < \infty$ may be of interest to study. For instance, a natural question concerns how the length of a loop is related to $S(\rho)$. The example in Section 4.3 shows that one can have $S(\rho) > 0$ while the corresponding loop is trivial, so one can only hope for an upper bound in terms of $S(\rho)$. One can further ask for properties of the minimal energy (equipotential) path to a given element. Another interesting question concerns how transformations on ρ affects a path in $T_0(1)$, and vice versa.

“Foliations” in hyperbolic 3-space. Since Möbius transformations of $\hat{\mathbb{C}}$ extend to isometries of the hyperbolic 3-space \mathbb{H}^3 (whose boundary at ∞ is identified with the Riemann sphere $\hat{\mathbb{C}}$) and being a Weil–Petersson quasicircle is a Möbius invariant property, it is natural to try to relate our foliations by Weil–Petersson quasicircles to objects in \mathbb{H}^3 . In [2], it is shown that every Jordan curve bounds at least one minimal disk in \mathbb{H}^3 , and [8] shows that a Jordan curve is a Weil–Petersson quasicircle if and only if any such minimal disk in \mathbb{H}^3 has finite total curvature. For example, when γ is a circle, the unique minimal surface is the totally geodesic surface, namely the hemisphere bounded by γ . Although the minimal disk for a given boundary curve may not be unique in general, [54, Thm. B] and a bound on the quasiconformal constant of the Weil–Petersson quasicircle together imply uniqueness when the Loewner energy of γ is small enough. Hence, for sufficiently small $S(\rho)$, Theorem 1.1 and Corollary 8.9 imply that the foliation $(\gamma_t)_{t \in \mathbb{R}}$ uniquely determines a family of minimal disks of finite total curvature $(\Sigma_t)_{t \in \mathbb{R}}$ in \mathbb{H}^3 , where Σ_t is bounded by the leaf γ_t . We believe that the family $(\Sigma_t)_{t \in \mathbb{R}}$ forms a smooth foliation of \mathbb{H}^3 in this case. Such families of minimal surfaces seem interesting to study in their own right and, by embedding into a dynamical family, could be useful in the analysis of minimal surfaces in \mathbb{H}^3 bounded by Weil–Petersson quasicircles and in deriving a rigorous AdS₃/CFT₂ holographic principle.

References

- [1] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.
- [2] M. T. Anderson. Complete minimal hypersurfaces in hyperbolic n -manifolds. *Comment. Math. Helv.*, 58(2):264–290, 1983.
- [3] M. Ang. Liouville conformal field theory and the quantum zipper. *Preprint, arxiv:2301.13200*, 2023.
- [4] M. Ang, N. Holden, and X. Sun. Conformal welding of quantum disks. *arXiv preprint:2009.08389*, 2020.
- [5] M. Ang, M. Park, and Y. Wang. Large deviations of radial SLE_∞. *Electron. J. Probab.*, 25(102), 2020.
- [6] M. Ang and P. Yu. Reversibility of whole-plane SLE for $\kappa > 8$. *Preprint, arxiv.org:2309.05176*, 2023.
- [7] J. Becker. Löwnersche differentialgleichung und quasikonform fortsetzbare schlichte funktionen. *J. Reine Angew. Math.*, 255, 1972.

- [8] C. J. Bishop. Weil-Petersson curves, conformal energies, β -numbers, and minimal surfaces. *Preprint <http://www.math.stonybrook.edu/~bishop/papers/wpbeta.pdf>*, 2019.
- [9] M. J. Bowick and S. G. Rajeev. The holomorphic geometry of closed bosonic string theory and $\text{Diff}(S^1)/S^1$. *Nuclear Physics B*, 293:348–384, 1987.
- [10] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.
- [11] L. Carleson and N. Makarov. Aggregation in the plane and Loewner’s equation. *Comm. Math. Phys.*, 216:583–607, 2001.
- [12] G. Cui. Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces. *Sci. China Ser. A*, 43(3):267–279, 2000.
- [13] L. de Branges. A proof of the Bieberbach conjecture. *Acta Math.*, 154(1-2):137–152, 1985.
- [14] C. Dellacherie and P.-A. Meyer. *Probabilities and potential*, volume 29 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam-New York; North-Holland Publishing Co., Amsterdam-New York, 1978.
- [15] J.-D. Deuschel and D. W. Stroock. *Large deviations*, volume 137 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1989.
- [16] M. D. Donsker and S. S. Varadhan. Asymptotic evaluation of certain markov process expectations for large time, i. *Communications on Pure and Applied Mathematics*, 28(1):1–47, 1975.
- [17] J. Dubédat. Duality of Schramm-Loewner evolutions. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):697–724, 2009.
- [18] B. Duplantier, J. Miller, and S. Sheffield. Liouville quantum gravity as a mating of trees. *arXiv preprint: 1409.7055*, 2014.
- [19] P. L. Duren. *Theory of H^p spaces*. Pure and Applied Mathematics, Vol. 38. Academic Press, New York-London, 1970.
- [20] P. L. Duren. *Univalent functions*. Springer-Vlg, New York, 1983.
- [21] O. El-Fallah, K. Kellay, J. Mashreghi, and T. Ransford. *A primer on the Dirichlet space*, volume 203 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2014.
- [22] J. B. Garnett and D. E. Marshall. *Harmonic measure*, volume 2 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008. Reprint of the 2005 original.
- [23] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [24] P. Gumenyuk and I. Prause. Quasiconformal extensions, Loewner chains, and the lambda-lemma. *Anal. Math. Phys.*, 8:621–635, 2018.
- [25] H. Hedenmalm and P. J. Nieminen. The Gaussian free field and Hadamard’s variational formula. *Probab. Theory Related Fields*, 159(1-2):61–73, 2014.
- [26] D. K. Hong and S. G. Rajeev. Universal Teichmüller space and $\text{Diff } S^1/S^1$. *Comm. Math. Phys.*, 135(2):401–411, 1991.
- [27] D. S. Jerison and C. E. Kenig. Hardy spaces, a_∞ , and singular integrals on chord-arc domains. *Math. Scand.*, 50:221–247, Jun. 1982.

- [28] K. Johansson. Strong Szegő theorem on a Jordan curve. In *Toeplitz operators and random matrices—in memory of Harold Widom*, volume 289 of *Oper. Theory Adv. Appl.*, pages 427–461. Birkhäuser/Springer, Cham, [2022] ©2022.
- [29] F. Johansson Viklund, A. Sola, and A. Turner. Scaling limits of anisotropic hastings–levitov clusters. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 48(1):235–257, 2012.
- [30] A. Jonsson and H. Wallin. Function spaces on subsets of \mathbf{R}^n . *Math. Rep.*, 2(1):xiv+221, 1984.
- [31] P. P. Kufareff. On one-parameter families of analytic functions. *Rec. Math. [Mat. Sbornik] N.S.*, 13(55):87–118, 1943.
- [32] N. S. Landkof. *Foundations of modern potential theory*. Springer, Berlin, 1972.
- [33] G. F. Lawler, O. Schramm, and W. Werner. Values of Brownian intersection exponents. I. Half-plane exponents. *Acta Math.*, 187(2):237–273, 2001.
- [34] G. F. Lawler, O. Schramm, and W. Werner. Conformal restriction: the chordal case. *J. Amer. Math. Soc.*, 16(4):917–955, 2003.
- [35] G. F. Lawler and W. Werner. The Brownian loop soup. *Probab. Theory Related Fields*, 128(4):565–588, 2004.
- [36] K. Löwner. Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I. *Math. Ann.*, 89(1-2):103–121, 1923.
- [37] J. Miller and S. Sheffield. Imaginary geometry I: interacting SLEs. *Probab. Theory Related Fields*, 164(3-4):553–705, 2016.
- [38] J. Miller and S. Sheffield. Quantum Loewner evolution. *Duke Mathematical Journal*, 165(17):3241–3378, 2016.
- [39] J. Miller and S. Sheffield. Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees. *Probab. Theory Related Fields*, 169(3-4):729–869, 2017.
- [40] J. Milnor. Fubini foiled: Katok’s paradoxical example in measure theory. *The Mathematical Intelligencer*, 19(2):30–32, 1997.
- [41] S. Nag. Non-perturbative string theory and the diffeomorphism group of the circle. *Topological and Geometrical Methods in Field Theory*, page 267, 1991.
- [42] O. Pekonen. Universal Teichmüller space in geometry and physics. *J. Geom. Phys.*, 15(3):227–251, 1995.
- [43] E. Peltola and Y. Wang. Large deviations of multichordal SLE_{0+} , real rational functions, and zeta-regularized determinants of Laplacians. *J. Eur. Math. Soc.*, published online, 2023.
- [44] C. Pommerenke. Über die Subordination analytischer Funktionen. *Journal für die reine und angewandte Mathematik*, 218:159–173, 1965.
- [45] C. Pommerenke. *Univalent functions*. Vandenhoeck & Ruprecht, Göttingen, 1975. With a chapter on quadratic differentials by Gerd Jensen, *Studia Mathematica/Mathematische Lehrbücher*, Band XXV.
- [46] C. Pommerenke. *Boundary behaviour of conformal maps*, volume 299 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.
- [47] D. Radnell, E. Schippers, and W. Staubach. Quasiconformal Teichmüller theory as an analytical foundation for two-dimensional conformal field theory. In *Lie algebras*,

- vertex operator algebras, and related topics*, volume 695 of *Contemp. Math.*, pages 205–238. Amer. Math. Soc., Providence, RI, 2017.
- [48] S. Rohde and O. Schramm. Basic properties of SLE. *Ann. of Math. (2)*, 161(2):883–924, 2005.
- [49] S. Rohde and Y. Wang. The Loewner energy of loops and regularity of driving functions. *Int. Math. Res. Not. IMRN*, 2021(10):7715–7763, 2021.
- [50] M. Rosenblum and J. Rovnyak. *Topics in Hardy classes and univalent functions*. Birkhäuser, Basel, 1994.
- [51] O. Roth and E. Schippers. The Loewner and Hadamard variations. *Illinois J. Math.*, 52(4):1399–1415, 2008.
- [52] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.
- [53] O. Schramm. Conformally invariant scaling limits: an overview and a collection of problems. In *International Congress of Mathematicians. Vol. I*, pages 513–543. Eur. Math. Soc., Zürich, 2007.
- [54] A. Seppi. Minimal discs in hyperbolic space bounded by a quasicircle at infinity. *Comment. Math. Helv.*, 91(4):807–839, 2016.
- [55] E. Sharon and D. Mumford. 2D-Shape analysis using Conformal Mapping. *International Journal of Computer Vision*, 70(1):55–75, 2006.
- [56] Y. Shen. On Grunsky operator. *Sci. China Ser. A*, 50(12):1805–1817, 2007.
- [57] Y. Shen. Weil-Petersson Teichmüller space. *Amer. J. Math.*, 140(4):1041–1074, 2018.
- [58] A. G. Siskakis. Semigroups of composition operators on spaces of analytic functions, a review. In *Studies on composition operators (Laramie, WY, 1996)*, volume 213 of *Contemp. Math.*, pages 229–252. Amer. Math. Soc., Providence, RI, 1998.
- [59] S. Smirnov. Towards conformal invariance of 2D lattice models. In *International Congress of Mathematicians*, volume 2, pages 1421–1451. Eur. Math. Soc., Zürich, 2006.
- [60] A. Sola. Elementary examples of Loewner chains generated by densities. *Annales Universitatis Mariae Curie-Sklodowska, sectio A Mathematica*, 67(1), 2013.
- [61] L. A. Takhtajan and L.-P. Teo. Weil-Petersson metric on the universal Teichmüller space. *Mem. Amer. Math. Soc.*, 183(861):viii+119, 2006.
- [62] F. Viklund and Y. Wang. Interplay Between Loewner and Dirichlet Energies via Conformal Welding and Flow-Lines. *Geom. Funct. Anal.*, 30(1):289–321, 2020.
- [63] Y. Wang. The energy of a deterministic Loewner chain: reversibility and interpretation via SLE_{0+} . *J. Eur. Math. Soc.*, 21(7):1915–1941, 2019.
- [64] Y. Wang. Equivalent descriptions of the Loewner energy. *Invent. Math.*, 218(2):573–621, 2019.
- [65] Y. Wang. A note on Loewner energy, conformal restriction and Werner’s measure on self-avoiding loops. *Ann. Inst. Fourier (Grenoble)*, 71(4):1791–1805, 2021.
- [66] Y. Wang. Large deviations of Schramm-Loewner evolutions: a survey. *Probab. Surv.*, 19:351–403, 2022.
- [67] E. Witten. Coadjoint orbits of the Virasoro group. *Comm. Math. Phys.*, 114(1):1–53, 1988.
- [68] D. Zhan. Duality of chordal SLE. *Invent. Math.*, 174(2):309–353, 2008.

- [69] D. Zhan. Reversibility of chordal SLE. *Ann. Probab.*, 36(4):1472–1494, 2008.
- [70] D. Zhan. Reversibility of whole-plane SLE. *Probab. Theory Related Fields*, 161(3-4):561–618, 2015.