Universal Liouville action as a renormalized volume
and its gradient flow

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Abstract

The universal Liouville action (also known as the Loewner energy for Jordan curves) is a Kähler potential on the Weil–Petersson universal Teichmüller space which is identified with the family of Weil–Petersson quasicircles via conformal welding. This action is invariant under Möbius transformations of the Riemann sphere. Our main result shows that it equals the renormalized volume of the non-compact subset of the hyperbolic 3-space bounded by the two Epstein–Poincaré surfaces associated with the quasicircle in analogy to the theory for convex co-compact hyperbolic 3-manifolds. We also study the gradient descent flow of the universal Liouville action with respect to the Weil–Petersson metric and show that the flow always converges to the origin (the circle). This provides a bound of the Weil–Petersson distance to the origin by the universal Liouville action.

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For a Jordan curve $\gamma \subset \hat{\mathbb{C}}$, we let $\Omega$ and $\Omega^*$ be the two connected components of $\hat{\mathbb{C}} \setminus \gamma$, $\rho_\Omega$ and $\rho_{\Omega^*}$ be the Poincaré (hyperbolic) metric (of constant $-1$ Gauss curvature) in $\Omega$ and $\Omega^*$ respectively. The Riemann sphere $\hat{\mathbb{C}}$ is the conformal boundary of the hyperbolic 3-space $\mathbb{H}^3$. In [11] C. Epstein gave a natural way to associate to each conformal metric on $\hat{\mathbb{C}}$ a surface in $\mathbb{H}^3$. We will recall the basics on Epstein surfaces in Section 3. Let $E_{\rho_\Omega} : \Omega \to \mathbb{H}^3$ be the Epstein–Poincaré map, namely, the Epstein map associated with the metric $\rho_\Omega$, similarly for $E_{\rho_{\Omega^*}} : \Omega^* \to \mathbb{H}^3$. The maps $E_{\rho_\Omega}, E_{\rho_{\Omega^*}}$ are smooth, extend continuously to the identity map on $\gamma$, and are immersions almost everywhere. We call their images as the Epstein–Poincaré surfaces $\Sigma_\Omega$ and $\Sigma_{\Omega^*}$. In particular, we note that the Epstein–Poincaré surfaces are non-compact and not necessarily embedded and have infinite area. We show the following results.

**Proposition 1.1** (See Proposition 4.1). If $\gamma$ is not a circle, then the two Epstein–Poincaré surfaces $\Sigma_\Omega$ and $\Sigma_{\Omega^*}$ are disjoint except at $\gamma$.

It follows directly from the definition of Epstein–Poincaré map that if $\gamma$ is a circle, then both $\Sigma_\Omega$ and $\Sigma_{\Omega^*}$ are the totally geodesic plane bounded by $\gamma$ with opposite orientation (see Example 3.1).

**Proposition 1.2** (See Corollary 3.13). When $\gamma$ is asymptotically conformal (see Theorem 3.10), there is a neighborhood of $\gamma$ in $\hat{\mathbb{C}}$ on which the Epstein–Poincaré maps $E_{\rho_\Omega}$ and $E_{\rho_{\Omega^*}}$ are immersions and embeddings which fix $\gamma$.

Quasicircles are in natural correspondence with points in the universal Teichmüller space $T(1)$, where we identify a quasicircle with its conformal welding homeomorphism. We are interested in a special class of quasicircles, i.e. Weil–Petersson quasicircles, which corresponds to the Weil–Petersson universal Teichmüller space $T_0(1)$. This space has been studied extensively for it being the connected component of the unique homogeneous
Kähler metric on $T(1)$ (i.e. the Weil–Petersson metric) \cite{32}, and have a big number of equivalent descriptions from very different perspectives, see, e.g., \cite{3, 8, 13, 28, 35, 36}.

Weil–Petersson quasicircles are asymptotically conformal, so Propositions 1.1 and 1.2 allow us to define the signed volume between $\Sigma$ and $\Sigma^*$. A priori, this volume takes value in $(-\infty, \infty]$ (see Section 4.2 for more details). However, we show the following result.

**Theorem 1.3.** If $\gamma$ is a Weil–Petersson quasicircle, then the signed volume between the two Epstein–Poincaré surfaces, denoted as $V(\gamma)$, is finite.

See Proposition 4.5 for the proof for smooth Jordan curves. The result for general Weil–Petersson quasicircles is obtained via an approximation argument, see Corollary 5.16.

Since $T_0(1)$ has a unique homogeneous Kähler structure, its Kähler potential is of critical importance. Takhtajan and Teo defined the universal Liouville action $S$ on $T_0(1)$ and showed it to be such a Kähler potential \cite{32}. In this work, we will consider the universal Liouville action as defined for Jordan curves (see Section 2.3), and denote it as $\tilde{S}$ for clarity. The functional $\tilde{S}(\gamma)$ can actually be defined for arbitrary Jordan curve, but it is finite if and only if $\gamma$ is a Weil–Petersson quasicircle. Moreover, $\tilde{S}$ is invariant under Möbius transformations of $\tilde{C}$ (i.e. under the $\text{PSL}_2(\mathbb{C})$ action). As the $\text{PSL}_2(\mathbb{C})$ action extends to orientation preserving isometries of $\mathbb{H}^3$, it is very natural to search for a characterization of the class of Weil–Petersson quasicircles and an expression of $\tilde{S}$ in terms of geometric quantities in $\mathbb{H}^3$.

A pioneering work of C. Bishop \cite{3} shows that the class of Weil–Petersson quasicircles can be characterized as Jordan curves bounding minimal surfaces in $\mathbb{H}^3$ with finite total curvature. We obtain the following similar characterization in terms of Epstein–Poincaré surfaces. See also Section 7 where we compare Epstein–Poincaré surfaces to minimal surfaces and the convex core, answering a question of Bishop \cite{2}.

In fact, the Epstein maps come with a well-defined unit normal $\vec{n}$ pointing away from $\Omega$ and from $\Omega^*$ respectively. The mean curvature $H := \text{Tr}(B)/2$ is defined using the shape operator $B(v) := -\nabla_v \vec{n}$.

**Theorem 1.4** (See Corollary 3.9). We have for all Jordan curves,

$$\int_{\Sigma_\Omega} H \text{d}a = \int_{\Sigma_\Omega} |\det B| \text{d}a = \int_{\mathbb{D}} |\mathcal{S}(f)(z)|^2 \frac{(1 - |z|^2)^2}{4} \text{d}^2z$$

where $f : \mathbb{D} \to \Omega$ is any conformal map, $\mathcal{S}(f) = f'''/f' - (3/2)(f''/f')^2$ is the Schwarzian derivative of $f$, $\text{d}a$ is the area form induced from $\mathbb{H}^3$, and $\text{d}^2z$ is the Euclidean area form.

In particular, $\Sigma_\Omega$ has finite total mean curvature (and finite total curvature) if and only if $\gamma$ is a Weil–Petersson quasicircle.

However, no exact identity between the Kähler potential and geometric quantity in $\mathbb{H}^3$ was known. The main result of this work is to provide such an identity.

**Definition 1.5.** Let $\gamma$ be a Weil–Petersson quasicircle. We define the renormalized volume (or W-volume) associated with $\gamma$ as

$$V_R(\gamma) := V(\gamma) - \frac{1}{2} \int_{\Sigma_\Omega \cup \Sigma^*} H \text{d}a \in (-\infty, \infty).$$
The definition is reminiscent to the renormalized volume\(^1\) for quasi-Fuchsian manifolds [17, 31]. But we emphasize again that \(\Sigma_\Omega\) and \(\Sigma_\Omega^*\) are non-compact so the analysis has additional technicality.

**Theorem 1.6** (See Corollary 5.10 and Theorem 5.11). If \(\gamma\) is a \(C^{5,\alpha}\) Jordan curve with \(\alpha > 0\), we have

\[
\tilde{S}(\gamma) = 4V_R(\gamma). \tag{1.1}
\]

If \(\gamma\) is a Weil–Petersson quasicircle, then we have \(\tilde{S}(\gamma) \geq 4V_R(\gamma)\).

Let us comment briefly on the proof of this theorem. It is easy to check that when \(\gamma\) is a circle, both sides of (1.1) are zero. We show under regularity assumptions that the first variation of both sides are equal. The variation of \(\tilde{S}\) was proved in [32], which we recall in Theorem 2.1 (and improve in Proposition 2.4). The first variation of \(V_R\) is more laborious since the Epstein–Poincaré surfaces are not compact and are immersed only almost everywhere. After administering appropriate truncation (where we make use of the regularity assumption), we re-derive the Schlaffi formula which expresses the variation of \(V_R\) in terms of the mean curvature \(H\), the metric \(I\) and the second fundamental form \(II\) on Epstein surfaces (Theorem 5.1), then translate the variation formula into quantities defined directly on \(\Omega, \Omega^* \subset \hat{C}\) (Theorem 5.7 and Corollary 5.9).

For a general Weil–Petersson quasicircle \(\gamma\), we use an approximation by equipotentials (they are analytic curves and the universal Liouville action increases to that of \(\gamma\)). We believe the identity (1.1) also holds for a general Weil–Petersson quasicircle. However, our approximation argument only implies the inequality due to the lack of tightness for the volume between the Epstein–Poincaré surfaces, see Section 5.3.

The second topic of this work concerns the gradient descent flow of \(S\) with respect to the Weil–Petersson metric. We proceed similarly as in Bridgeman-Brock-Bromberg [5]. For \([\mu] \in T(1)\) we have a natural isomorphism \(T[\mu]T(1) \simeq \Omega^{-1,1}(\mathbb{D}^*)\).

**Theorem 1.7** (See Theorem 6.1). The negative gradient of \(S\) with respect to the Weil–Petersson metric is the vector field

\[
V[\mu] := -4\frac{\mathcal{F}(g_\mu)}{\rho_{\mathbb{D}^*}} \in \Omega^{-1,1}(\mathbb{D}^*).
\]

Moreover, the gradient descent flow of \(S\) starting from any point in \(T_0(1)\) converges to the origin \([0]\) which corresponds to the round circle.

Using the gradient flow, we also obtain bounds of the Weil–Petersson distance on \(T_0(1)\) in terms of the universal Liouville action.

**Theorem 1.8** (See Theorem 6.3). There exist universal positive constants \(c\) and \(K\) such that for all \([\mu] \in T_0(1)\), we have \(c(\text{dist}_{WP}([\mu], [0]) - Ke) \leq S([\mu])\).

---

\(^1\) Renormalized volume of a convex co-compact hyperbolic 3-manifold is referred to the difference between the volume and half of the boundary area defined through a foliation near the ends. Our formula is similar to the definition of the \(W\)-volume. However, in the convex co-compact case, they only differ by a multiple of Euler characteristics of the boundary [17, Lem. 4.5].
Finally, let us make a few remarks on the motivation behind this work and additional comments on the relation with the literature.

S. Rohde and the last author introduced the \textit{Loewner energy} for Jordan curves [25, 34] which is originally motivated from the large deviation theory of random fractal curves Schramm-Loewner evolutions (SLE) [34, 37]. In a certain sense, the Loewner energy is the \textit{action functional} which characterizes the law of SLE. It turns out quite surprisingly that the Loewner energy equals exactly $S/\pi$ as proved in [35]. Since we will not make use of Loewner theory but only the fact of $S$ is a Kähler potential on $T_0(1)$, we adopt the terminology of \textit{universal Liouville action} here. SLEs play a central role in the emerging field of two-dimensional random conformal geometry. In particular, they provide a mathematical description of the interfaces in statistical mechanics models [18, 27, 29] and also a new way of thinking about 2D conformal field theory (CFT) [1, 9, 14, 21]. On the other hand, $\mathbb{H}^3$ is the Riemannian analog of AdS$_3$ space. Our main result Theorem 1.6 can be interpreted as the \textit{holographic principle} for the Loewner energy that is reminiscent of the conjectural AdS$_3$/CFT$_2$ correspondence pioneered by Maldacena [19] (see also, e.g., [20, 38]). The authors are not aware of a (even conjectural) holographic principle for SLE nor for random conformal geometry in general, this work may be a first step towards this direction. We also mention [15] gives a holographic expression for determinants of discrete Dirac operator on periodic bipartite isoradial graphs.

Renormalized volume as a Liouville action has been previously studied for convex co-compact group actions in $\mathbb{H}^3$ (see work by Takhtajan–Teo [31] and Krasnov–Schlenker [17]), or equivalently, for conformally compact hyperbolic metrics. A set of applications of this study are bounds for the hyperbolic volume of mapping tori of pseudo-Anosov maps in term of their Weil–Petersson translation length (by Brock [7]) or their entropy (by Kojima–McShane [16]). This uses a bound (by Schlenker [26]) for renormalized volume in terms of Weil–Petersson distance by studying the gradient of the Liouville action, similar to our bound in Theorem 6.3. Moreover, we show in Theorem 6.1 that every flowline of the gradient converges to the absolute minimum, in analogy to the result done by the first three authors [6] for the relatively acylindrical case. This builds on work by the first two authors and Brock [5], where they used the gradient flow to find the minimum of renormalized volume for a boundary incompressible hyperbolic 3-manifold.

The paper is organized as follows: In Section 2 we collect all the basics about universal Teichmüller space, its Kähler geometry, characterizations of the Weil–Petersson universal Teichmüller space, and the universal Liouville action. In Section 3 we recall the definition of Epstein surfaces and the correspondence between geometric quantities on the surface and those on the conformal boundary. We also prove the immersion and embeddedness of the Epstein–Poincaré surfaces associated with an asymptotically conformal curve. In Section 4 we study the relation between the two Epstein–Poincaré surfaces associated with the same curve. We show that they are disjoint (except for a circle), and that if the curve is regular enough, the signed volume between the Epstein–Poincaré surfaces is finite. In Section 5, we prove the variational formula for the renormalized volume and prove the main theorem Theorem 1.6. Section 6 is independent from Sections 3, 4 and 5 and deals with the gradient flow of the universal Liouville action. Similarly, Section
7 describes the relative position of Epstein–Poincaré surfaces with respect to minimal surfaces and convex core.

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2 Universal Weil–Petersson Teichmüller space

2.1 Universal Teichmüller space

We first briefly recall a few equivalent descriptions of the universal Teichmüller space \( T(1) \). Let \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \), \( \mathbb{D} = \{z \, : \, |z| < 1\} \), \( \mathbb{D}^* = \hat{\mathbb{C}} - \mathbb{D} \) and \( S^1 = \partial \mathbb{D} \). The group of orientation preserving conformal automorphism of \( \hat{\mathbb{C}} \) is

\[
\text{Möb}(\hat{\mathbb{C}}) = \text{PSL}_2(\mathbb{C}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\} / A \sim -A
\]

which acts on \( \hat{\mathbb{C}} \) by Möbius transformations \( z \mapsto \frac{az + b}{cz + d} \). The subgroup preserving \( S^1 \) is

\[
\text{Möb}(S^1) = \text{PSU}_{1,1} = \left\{ A = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \right\} / A \sim -A
\]

which is isomorphic to \( \text{PSL}_2(\mathbb{R}) \). There are a number of equivalent descriptions that we will use.

Quasisymmetric maps: We write \( \text{QS}(S^1) \) for the group of sense preserving quasisymmetric homeomorphisms of \( S^1 \). The universal Teichmüller space is

\[
T(1) := \text{Möb}(S^1) \backslash \text{QS}(S^1) \simeq \{ \varphi \in \text{QS}(S^1), \varphi \text{ fixes } -1, -i \text{ and } 1 \}.
\]

\( T(1) \) is endowed with a group operation given by the composition and the origin is the identity map \( \text{Id}_{S^1} \).

Beltrami Differentials: Given a Beltrami differential

\[
\mu \in L^{\infty}_1(\mathbb{D}^*) = \{ \mu \in L^{\infty}(\mathbb{D}^*), ||\mu||_{\infty} < 1 \},
\]

we extend it to \( \hat{\mathbb{C}} \) by reflection, i.e. define for \( z \in \mathbb{D} \),

\[
\mu(z) = \mu \left( \frac{1}{\overline{z}} \right) \frac{z^2}{\overline{z}^2}.
\]
Let $w_\mu : \hat{C} \to \hat{C}$ be the solution to the Beltrami equation $\partial \bar{z} w_\mu = \mu \partial_z w_\mu$ fixing $-1, -i$ and $1$. Then $w_\mu$ preserves $S^1$ and $w_{\mu|S^1} \in QS(S^1)$. Since every quasisymmetric circle homeomorphism can be extended to a quasiconformal self-map of $\overline{D}$, we have

$$T(1) = L^\infty_1(\mathbb{D}^*)/\sim$$

where $\mu \sim \nu$ if and only if $w_{\mu|S^1} = w_{\nu|S^1}$. We denote by $\Phi : L^\infty(\mathbb{D}^*) \to T(1)$ the projection $\mu \mapsto [\mu]$. Here the origin corresponds to $[0]$.

**Univalent maps:** If instead we extend $\mu$ by 0 on $\mathbb{D}$ and let $w^\mu$ be the unique solution to $w_\mu^\mu = \mu w_\mu^\mu$ fixing $-1, -i$ and 1, then $w^\mu$ is conformal on $\mathbb{D}$. The map $[\mu] \mapsto w^\mu|_{\mathbb{D}}$ identifies $T(1)$ with

\[ \{ f : \mathbb{D} \to \hat{C}, \text{ univalent fixing } -1, -i \text{ and } 1, \text{ extendable to q.c. map of } \hat{C} \}, \tag{2.1} \]

since $\mu \sim \nu$ if and only if $w^\mu = w^\nu$ on $\mathbb{D}$. The origin corresponds to $\text{Id}_\mathbb{D}$.

**Quasicircles:** By Riemann mapping theorem, the previous identification also gives

$$T(1) \simeq \{ \gamma \text{ quasicircle passing through } -1, -i, \text{ and } 1 \} \tag{2.2}$$

by the map $[\mu] \mapsto \gamma_\mu := w^\mu(S^1)$. The origin corresponds to $\gamma_1 = S^1$. We can recover the quasisymmetric circle homeomorphism from $\gamma_\mu$ via conformal welding. Let $\Omega$ (resp. $\Omega^*$) denote the connected component of $\hat{C} \setminus \gamma_\mu$ where $-1, -i, 1$ are in the counterclockwise direction of $\partial \Omega$ (resp. clockwise direction of $\partial \Omega^*$). Let $f_\mu = w^\mu|_{\mathbb{D}} : \mathbb{D} \to \Omega$ and $g_\mu : \mathbb{D}^* \to \Omega^*$ be the conformal maps fixing $-1, -i, 1$. Then,

$$w_{\mu|S^1} = g_{\mu}^{-1} \circ f_{\mu|S^1}$$

since $g_{\mu} = w^\mu \circ w_{\mu}^{-1}|_{\mathbb{D}^*}$. We call $g_{\mu}^{-1} \circ f_{\mu|S^1}$ the *welding homeomorphism* of the quasicircle $\gamma_\mu$ passing through $-1, -i, 1$.

### 2.2 Kähler Structure and Weil–Petersson Teichmüller space

We first define the following spaces,

$$A_\infty(\mathbb{D}^*) = \{ \phi : \mathbb{D}^* \to \mathbb{C} \text{ holomorphic}, \sup_{\mathbb{D}^*} |\phi| \rho_{\mathbb{D}^*}^{-1} < \infty \},$$

$$A_2(\mathbb{D}^*) = \{ \phi : \mathbb{D}^* \to \mathbb{C} \text{ holomorphic}, \int_{\mathbb{D}^*} |\phi|^2 \rho_{\mathbb{D}^*}^{-1} \, d^2z < \infty \} \subset A_\infty(\mathbb{D}^*),$$

where $\rho_{\mathbb{D}^*}(z) = 4/(1 - |z|^2)^2$ is the hyperbolic density function and $d^2z = dx \wedge dy$ if $z = x + iy$. The inclusion is shown in [32, Lem.1.2.1]. We define the similar spaces $A_\infty(\mathbb{D})$ and $A_2(\mathbb{D})$ (and also $A_\infty(\Omega)$ and $A_2(\Omega)$). We will also use the spaces of harmonic Beltrami differentials defined as

$$\Omega^{-1,1}(\mathbb{D}^*) = \{ \nu \in L^\infty(\mathbb{D}^*), \nu = \rho_{\mathbb{D}^*}^{-1} \overline{\phi}, \phi \in A_\infty(\mathbb{D}^*) \};$$

$$H^{-1,1}(\mathbb{D}^*) = \{ \nu \in L^\infty(\mathbb{D}^*), \nu = \rho_{\mathbb{D}^*}^{-1} \overline{\phi}, \phi \in A_2(\mathbb{D}^*) \} \subset \Omega^{-1,1}(\mathbb{D}^*).$$

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The universal Teichmüller space $T(1)$ has a canonical complex structure such that $\Phi : L^\infty(\mathbb{D}^*) \to T(1)$ is a holomorphic surjection. The holomorphic tangent space at the origin is

$$T_0 T(1) = L^\infty(\mathbb{D}^*) / \ker(D_0 \Phi) \simeq \Omega^{-1,1}(\mathbb{D}^*)$$

where

$$\ker(D_0 \Phi) = \mathfrak{N}(\mathbb{D}^*) := \{ \nu \in L^\infty(\mathbb{D}^*) : \int_{\mathbb{D}^*} \nu \phi = 0, \forall \phi \text{ holomorphic and } \int_{\mathbb{D}^*} |\phi| d^2z < \infty \}$$

is the space of infinitesimally trivial Beltrami differentials.

The space $L^\infty(\mathbb{D}^*)$ has a natural group structure given by the associated quasiconformal maps. We define $\lambda = \nu \ast \mu^{-1}$ if $w_\lambda = w_\nu \circ w_\mu^{-1}$. Thus

$$\lambda = \left( \frac{\nu - \mu}{1 - \overline{\nu} \frac{\partial_w w_\mu}{\partial_z w_\mu}} \right) \circ w_\mu^{-1}.$$

We define $R_\mu$ to be right multiplication by $\mu$ on $L^\infty(\mathbb{D}^*)$. This descends to give a map $R_\mu : T(1) \to T(1)$. Furthermore, the complex structure on $T(1)$ is right-invariant. Therefore, $D_0 R_\mu : T_0 T(1) \to T_0 T(1)$ is a complex linear isomorphism between holomorphic tangent spaces, and we obtain the identification of $T_0 T(1) \simeq \Omega^{-1,1}(\mathbb{D}^*)$.

To define a Kähler metric on $T(1)$, one needs to endow $T(1)$ with a Hilbert manifold structure. It is known since [4] that on the subspace $\mathcal{M} = \text{Möb}(S^1) \setminus \text{Diff}(S^1)$ there is a unique Kähler metric up to a scalar multiple. However, $\mathcal{M}$ is not complete under the Kähler metric. Takhtajan and Teo extend the Hilbert manifold structure on $T(1)$ by defining the Hermitian metric on the distribution $\mathfrak{D}([\mu]) = D_0 R_\mu(H^{-1,1}(\mathbb{D}^*)) \subset T_0 T(1)$ induced from $H^{-1,1}(\mathbb{D}^*)$:

$$\langle \mu, \hat{\nu} \rangle := \int_{\mathbb{D}^*} \overline{\mu} \rho_{D^*} d^2z, \quad \forall \mu, \hat{\nu} \in H^{-1,1}(\mathbb{D}^*).$$

They prove that this distribution is integrable and define $T_0(1)$ to be the connected component containing $[0]$ which is called the Weil–Petersson Teichmüller space. The Hermitian metric defined above is called the Weil–Petersson metric. (One may draw the similarity with the Weil–Petersson metric on Teichmüller spaces of a Fuchsian group $\Gamma$ where the integral is over $\mathbb{D}^*/\Gamma$.) In terms of the four equivalent definitions of $T(1)$, the subspace $T_0(1)$ is characterized as follows:

- **Quasisymmetric maps:** Y. Shen [28] showed $\varphi \in T_0(1)$ if and only if $\varphi$ is absolutely continuous with respect to the arclength measure, and $\log |\varphi'| \in H^{1/2}(S^1)$, the fractional Sobolev space of functions $u$ such that

$$\|u\|^2_{H^{1/2}} := \iint_{S^1 \times S^1} \left| \frac{u(\zeta) - u(\xi)}{\zeta - \xi} \right|^2 d\zeta d\xi < \infty. \quad (2.3)$$

- **Beltrami Differentials:** It is shown in [32] that $[\mu] \in T_0(1)$ if and only if it has a representative $\mu \in L^\infty(\mathbb{D}^*)$ such that

$$\int_{\mathbb{D}^*} |\mu(z)|^2 \rho_{D^*}(z) d^2z < \infty.$$
• **Univalent maps:** It is shown in [32, Thm. II.1.12] (see also [8]) that a univalent function \( f : \mathbb{D} \to \hat{\mathbb{C}} \) fixing \(-1, -i, 1\) and extendable to a quasiconformal map of \( \hat{\mathbb{C}} \), corresponds to an element of \( T_0(1) \) via the identification (2.1) if and only if the Schwarzian derivative

\[
\mathcal{S}(f) := \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2
\]

satisfies

\[
\int_{\mathbb{D}} |\mathcal{S}(f)|^2 \rho_{\mathbb{D}}^{-1} d^2z < \infty. \tag{2.4}
\]

In other words, the Bers’ embedding \( \beta([\mu]) := \mathcal{S}(f) \in A_2(\mathbb{D}) \).

Furthermore, let \( \tilde{f} = A \circ f \) where \( A \) is a Möbius map sending \( \Omega = f(\mathbb{D}) \) to a bounded domain (as a priori, \( \tilde{\Omega} \) may contain \( \infty \)). Then \( f \in T_0(1) \) if and only if

\[
\int_{\mathbb{D}} |\mathcal{N}(\tilde{f})|^2 d^2z < \infty \tag{2.5}
\]

where \( \mathcal{N}(\tilde{f}) = \tilde{f}''/\tilde{f}' \) is the pre-Schwarzian of \( \tilde{f} \). We note that the expression in (2.4) is invariant under the transformation \( f \to A \circ f \circ B \), for all \( A \in \text{PSL}_2(\mathbb{C}) \) and \( B \in \text{PSU}_{1,1} \) but the expression in (2.5) is not invariant under such transformations.

• **Quasicircles:** A quasicircle passing through \(-1, -i, 1\) which corresponds via (2.2) to an element of \( T_0(1) \) is called a Weil–Petersson quasicircle. It is easy to see that if \( \gamma \) and \( \tilde{\gamma} \) are two quasicircles passing through \(-1, -i, 1\) and \( \tilde{\gamma} = A(\gamma) \) for some \( A \in \text{PSL}_2(\mathbb{C}) \), then \( \tilde{\gamma} \) is Weil–Petersson if and only if \( \gamma \) is Weil–Petersson. Therefore, we may extend the definition to say that a Jordan curve \( \gamma \) is Weil–Petersson if and only if it is \( \text{PSL}_2(\mathbb{C}) \)-equivalent to a Weil–Petersson quasicircle passing through \(-1, -i, 1\).

### 2.3 Universal Liouville action

Takhtajan and Teo introduced the **universal Liouville action** \( S \) on \( T_0(1) \) and showed it to be a Kähler potential on \( T_0(1) \). See [32, Thm. II.4.1]. We will consider it as a functional on the space of Weil–Petersson quasicircles.

Indeed, let \( \gamma \) be a Jordan curve which does not pass through \( \infty \). Let \( D \) and \( D^* \) be respectively the bounded and unbounded connected component of \( \hat{\mathbb{C}} \setminus \gamma \), \( f : \mathbb{D} \to D \) and \( g : \mathbb{D}^* \to D^* \) be any conformal maps such that \( g(\infty) = \infty \) (note that \( D \) might not be \( \Omega \), it can also be \( \Omega^* \), and \( f \) and \( g \) are different from the canonical maps \( f_{\mu} \) and \( g_{\mu} \)). Define

\[
\tilde{S}(\gamma) := \int_{\mathbb{D}} |\mathcal{N}(f)|^2 d^2z + \int_{\mathbb{D}^*} |\mathcal{N}(g)|^2 d^2z + 4\pi \log |f'(0)/g'(\infty)| \tag{2.6}
\]

and is \( \text{PSL}_2(\mathbb{C}) \)-invariant (it can be seen via the identity with \( \pi \) times the Loewner energy of \( \gamma \) [35]) and finite if and only if \( \gamma \) is a Weil–Petersson quasicircle. The universal Liouville action \( S([\mu]) \) for \( [\mu] \in T_0(1) \) is defined as \( \tilde{S}(A(\gamma_{\mu})) \) where \( \gamma_{\mu} \) is the Weil–Petersson quasicircle passing through \(-1, -i, 1\) corresponding to \([\mu]\) and \( A \in \text{PSL}_2(\mathbb{C}) \) is any Möbius transformation such that \( A(\gamma_{\mu}) \) is bounded. The universal Liouville action \( S \) satisfies the following properties:
The first variation formula of $S$ from [32] will be a key ingredient in our proofs. We now state it for $S$. Let $\gamma$ be the Weil–Petersson quasicircle passing through $-1, -i, 1$ corresponding to an element $[\mu]$ of $T_0(1)$. Let $\Omega$ and $\Omega'$ be the connected components of $\hat{C} \setminus \gamma$ as in Section 2.1. Let $f_\mu : \mathbb{D} \to \Omega$ and $g_\mu : \mathbb{D}^* \to \Omega'$ be the conformal maps fixing $-1, -i, 1$. Let $\dot{\nu} \in H^{-1,1}(\mathbb{D}^*) \simeq T_{[\mu]}T_0(1)$, $t \in (-\|\dot{\nu}\|_{\infty}, \|\dot{\nu}\|_{\infty}^{-1})$, $w_t : \hat{C} \to \hat{C}$ be the solution fixing $-1, -i, 1$ to the Beltrami equation

$$\partial_{\bar{z}} w_t(z) = \begin{cases} 0 & z \in \Omega, \\ t(g_\mu)_* \dot{\nu}(z) \partial_z w_t(z) & z \in \Omega'. \end{cases}$$

where

$$(g_\mu)_* \dot{\nu}(z) = \dot{\nu} \circ g_\mu^{-1} \left( \frac{(g_\mu^{-1})'}{(g_\mu^*)'} \right).$$

We let $\gamma_t = w_t(\gamma)$ which is a small deformation of $\gamma$.

**Theorem 2.1 ([32, Cor. II.3.9]).** The universal Liouville action satisfies the following first variation formula. Let $\dot{\nu} \in H^{-1,1}(\mathbb{D}^*) \simeq T_{[\mu]}T_0(1)$,

$$(dS)_{[\mu]}(\dot{\nu}) = \left. \frac{d}{dt} \right|_{t=0} S(\gamma_t) = 4 \text{Re} \int_{\mathbb{D}^*} \nu \mathcal{J}(g_\mu) d^2 z = -4 \text{Re} \int_{\Omega'} ((g_\mu)_* \dot{\nu}) \mathcal{J}(g_\mu^{-1}) d^2 z.$$ 

**Remark 2.2.** We note that compared to the formula in [32], we take the derivative of $S$ in the real tangent space (which is canonically isomorphic to the holomorphic tangent space) while [32] takes derivative in the holomorphic tangent space and both derivatives are related by

$$(dS)_{[\mu]}(\dot{\nu}) = 2 \text{Re} \partial_\nu S([\mu]).$$

The last equality in Theorem 2.1 follows from a change of variable and the chain rule for Schwarzian derivatives which shows

$$\mathcal{J}(g^{-1}) = -\mathcal{J}(g) \circ g^{-1}(g^{-1})^2.$$ 

**Remark 2.3.** We choose $\dot{\nu}$ to be harmonic Beltrami differential as $H^{-1,1}(\mathbb{D}^*)$ is isomorphic to $T_{[\mu]}T_0(1)$, in particular, supplementary to the infinitesimally trivial Beltrami differentials $\mathcal{H}(\mathbb{D}^*)$. Clearly, the variational formula also holds for $\dot{\nu} \in H^{-1,1}(\mathbb{D}^*) \oplus \mathbb{R}(\mathbb{D}^*)$ if $\int |\mathcal{J}(g)| d^2 z < \infty$, which is the case, e.g., whenever the curve $\gamma$ is smooth.

Combining Theorem 2.1 and Remark 2.3 we obtain the following slightly modified version of the variational formula for $S$. (We will not need the two-sided deformation, but it is more natural when considering the Liouville action as defined for quasicircles and there is almost no cost to add this.) We write $\mathcal{H}(\Omega)$ (resp. $\mathcal{H}(\Omega^*)$) for the space of infinitesimally trivial Beltrami differentials on $\Omega$ (resp. $\Omega^*$).

**Proposition 2.4.** Let $\gamma \subset \mathbb{C}$ be a smooth Jordan curve. Let $\Omega$ and $\Omega^*$ be the connected components of $\hat{C} \setminus \gamma$. Let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ be any conformal maps. Let
\( \nu_1 \in H^{-1,1}(\Omega) \oplus \mathcal{N}(\Omega) \) and \( \nu_2 \in H^{-1,1}(\Omega^*) \oplus \mathcal{N}(\Omega^*) \) and \( w_t \) be any solution to the Beltrami equation

\[
\frac{\partial}{\partial z} w_t = \begin{cases} t\nu_1, & z \in \Omega, \\ t\nu_2, & z \in \Omega^*. \end{cases}
\]

Then we have

\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{S}(w_t(\gamma)) = -4 \text{Re} \left( \int_{\Omega} \nu_1 \mathcal{J}(f^{-1}) d^2 z + \int_{\Omega^*} \nu_2 \mathcal{J}(g^{-1}) d^2 z \right).
\]

The normalization of \( \gamma, w_t, f \) and \( g \) are not needed as the formula is invariant under other choices.

**Proof.** We only need to justify how the variation formula for one-sided quasiconformal deformation implies the two-sided deformation.

We consider first the two-variable family of quasiconformal maps \( w_{s,t} \) whose Beltrami coefficients are \( s\nu_1 \) in \( \Omega \) and \( t\nu_2 \) in \( \Omega^* \) for \( t, s \in \mathbb{R} \) small enough. We have by the composition rule of quasiconformal maps

\[ w_{s,t} = u_s^t \circ w_{0,t} \]

where

\[
\frac{\partial}{\partial z} w_{0,t} = \begin{cases} 0, & z \in \Omega, \\ t\nu_2, & z \in \Omega^*. \end{cases}
\]

From the one-sided variation we get

\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{S}(w_{0,t}(\gamma)) = -4 \text{Re} \int_{\Omega^*} \nu_2(z) \mathcal{J}(g^{-1})(z) d^2 z
\]

and

\[
\frac{d}{ds} \bigg|_{s=0} \mathcal{S}(w_{s,t}(\gamma)) = -4 \text{Re} \int_{w_{0,t}(\Omega)} (w_{0,t})_* \nu_1(z) \mathcal{J}(f^{-1} \circ w_{0,t}^{-1})(z) d^2 z
\]

\[= -4 \text{Re} \int_{\Omega} \nu_1(z) \mathcal{J}(f^{-1})(z) d^2 z + 4 \text{Re} \int_{\Omega} \nu_1(z) \mathcal{J}(w_{0,t})(z) d^2 z. \tag{2.7} \]

Lemma 1.2.9 in [32] shows that there exists \( C \) such that

\[ \| \mathcal{J}(w_{0,t}) \|_2 = \left( \int_{\Omega} \frac{|\mathcal{J}(w_{0,t})|^2}{\rho^2} d^2 z \right)^{1/2} \leq C |t| \| P(\nu_2) \|_2 = C |t| \left( \int_{\Omega} |P(\nu_2)|^2 \rho_{\Omega} d^2 z \right)^{1/2} \]

where \( P : H^{-1,1}(\Omega) \oplus \mathcal{N}(\Omega) \to H^{-1,1}(\Omega) \) is the projection parallel to \( \mathcal{N}(\Omega) \). The second term in \( (2.7) \) converges to \( 0 \) as \( t \to 0 \) by Cauchy-Schwarz inequality. Therefore we can apply the chain rule and get

\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{S}(w_t(\gamma)) = \frac{d}{dt} \bigg|_{t=0} \mathcal{S}(w_{t,t}(\gamma)) = -4 \text{Re} \left( \int_{\Omega} \nu_1 \mathcal{J}(f^{-1}) d^2 z + \int_{\Omega^*} \nu_2 \mathcal{J}(g^{-1}) d^2 z \right)
\]

as claimed. \( \square \)
3 Epstein–Poincaré surfaces

3.1 Epstein hypersurfaces associated with conformal metrics

In [12] Epstein developed a formula for envelopes of horosphere in terms of conformal metrics in $S^n = \partial_{\infty} \mathbb{H}^{n+1}$. Here, the hyperbolic space $\mathbb{H}^{n+1}$ is represented as the interior of the unit ball $\mathbb{B}^{n+1}$ with the metric:

$$ds^2 = \frac{4(dx_1^2 + \cdots + dx_{n+1}^2)}{(1 - |x|^2)^2},$$

and $S^n$ is represented by the unit sphere in $\mathbb{R}^{n+1}$. Let $\rho_{S^n}$ denote the metric on $S^n$ induced by the Euclidean metric, namely, the round metric.

Given a domain $\Omega \subseteq S^n$ and a smooth function $\varphi : \Omega \to \mathbb{R}$, we can associated the conformal metric $\rho = e^\varphi \rho_{S^n}$ to the parametrized surface

$$\text{Ep}_\rho : z \in \Omega \mapsto \frac{|\nabla \varphi|^2(e^{2\varphi} - 1)}{|\nabla \varphi|^2(e^{2\varphi} + 1)^2}z + \frac{2\nabla \varphi}{|\nabla \varphi|^2(e^{2\varphi} + 1)^2} \in \mathbb{H}^{n+1},$$

(3.1)

where $\nabla$ denotes the gradient with respect to $\rho_{S^n}$. As proved in [12, Section 2], the map $\text{Ep}_\rho$ solves the envelop equation of the family of horospheres $\{H(z, \varphi)\}_{z \in \Omega}$, where

$$H(z, \varphi) := \left\{ \frac{e^{\varphi(z)}}{e^{\varphi(z)} + 1} z + \frac{1}{e^{\varphi(z)} + 1} Y \bigg| Y \in S^n \setminus \{z\} \right\}$$

(3.2)

is a horosphere centered at $z$ and determined by the value of $\varphi(z)$. Solving the envelop equation means for all $z \in \Omega$,

$$\text{Ep}_\rho(z) \in H(z, \varphi) \quad \text{and} \quad D_z \text{Ep}_\rho(T_z S^n) \subseteq T_{\text{Ep}_\rho(z)} H(z, \varphi).$$

(3.3)

We can expand the Epstein map $\text{Ep}_\rho$ to the Epstein Gauss map $\widetilde{\text{Ep}}_\rho : \Omega \to T^1 \mathbb{H}^{n+1}$ by defining $\widetilde{\text{Ep}}_\rho(z)$ as the outer normal vector to $H(z, \varphi)$ at $\text{Ep}_\rho(z)$. The geodesic flow in the direction $-\widetilde{\text{Ep}}_\rho(z)$ converges to $z$. We have that $\widetilde{\text{Ep}}_\rho$ is always an embedding. In contrast, even though we have called $\text{Ep}$ a parametrized surface, the map $\text{Ep}_\rho$ need not to be an immersion. For instance, $\varphi \equiv 0$ implies that for any $z \in S^n$ we have that $\text{Ep}_\rho(z) = 0$ while $\widetilde{\text{Ep}}_\rho(z) = (0, -z)$. Regardless, we will see in Section 5 that because we can parametrize the normal bundle by $\widetilde{\text{Ep}}_\rho$ we will be able to treat the $\text{Ep}_\rho$ as a parametrized surface.

Geometrically, we can use visual metrics to describe $H(z, \varphi)$. Given $x \in \mathbb{H}^{n+1}$, we define the visual metric of $x$, denoted by $\nu_x$, as the metric in $S^n = \partial_{\infty} \mathbb{H}^{n+1}$ defined by the pullback $h^*(\rho_{S^n})$, where $h$ is any isometry of $\mathbb{H}^{n+1}$ so that $h(x) = 0$. The metric $\nu_x$ is well-defined, as the stabilizer of $0$ in Isom$(\mathbb{H}^{n+1})$ acts by isometries in $S^n$. Moreover, as Isom$(\mathbb{H}^{n+1})$ acts conformally in $S^n$, the metric $\nu_x$ is conformal to $\rho_{S^n}$, meaning that we can write $\nu_x = e^{\varphi(x)} \rho_{S^n}$ for some smooth function $\varphi : S^n \to \mathbb{R}$ that depends only on $x$.

Then it is a simple exercise to verify that $H(z, \varphi)$ coincides with the locus

$$\{x \in \mathbb{H}^{n+1} \big| \nu_x(z) = e^{\varphi(z)} \rho_{S^n} \},$$

(3.4)
and the inside of \( H(z, \varphi) \) is the locus
\[
\{ x \in \mathbb{H}^{n+1} \mid \nu_x(z) > e^{\varphi(z)} \rho_S \}.
\] (3.5)

By definition, it is easy to verify that if \( x \in \mathbb{H}^{n+1}, \ h \in \text{Isom}_+(\mathbb{H}^{n+1}) \) we have that \( h^*(\nu_h(x)) = \nu_x \). Hence it follows
\[
\text{Ep}_\rho = h \circ \text{Ep}_{h^*\rho}
\] (3.6)

where \( h^*\rho \) is the pull-back metric of \( \rho \) under \( h \).

### 3.2 Explicit expression of Epstein maps in the upper-space model

Here and in the sequel, we restrict ourselves to the case \( n = 3 \). For the computation purpose, it is convenient to use the upper-space model of the hyperbolic 3-space. Namely,
\[
\mathbb{H}^3 = \{(y, \xi) \in \mathbb{C} \times \mathbb{R}_+ \}
\]

with the hyperbolic metric
\[
ds^2 = \frac{|dy|^2 + d\xi^2}{\xi^2}.
\]

The results presented in this section were obtained in [17] and [12]. We collect them here for the readers’ convenience, also because our choice of convention of Epstein map, which coincides with the horosphere envelop interpretation of the Epstein map as described in Section 3.1, is slightly different than [17]. The difference of convention results mainly in constant factors at various places. We choose to include the simple derivations or examples to verify the constant factors.

Let \( e^{\varphi}|dz|^2 \) be a smooth conformal metric on an open set \( U \subset \mathbb{C} \). The Epstein map
\[
\text{Ep}_\varphi := \text{Ep}_{e^{\varphi}|dz|^2} : z \in U \mapsto (y, \xi) \in \mathbb{C} \times \mathbb{R}_+ = \mathbb{H}^3
\]
is given explicitly by
\[
\xi = \frac{2e^{-\varphi/2}}{1 + |\varphi_e| e^{-\varphi}}, \quad y = z + \frac{2\varphi_e e^{-\varphi}}{1 + |\varphi_e| e^{-\varphi}} = z + \xi \cdot \psi,
\] (3.7)

where
\[
\psi := \varphi_e e^{-\varphi/2}, \quad \varphi_e = \partial_e \varphi.
\]

The Epstein Gauss map is \( \text{Ep}_\varphi^* : U \subset \mathbb{C} \rightarrow T_1\mathbb{H}^3 \) such that the base point is \( \text{Ep}_\varphi \) and the vector component is \( \xi \frac{\eta}{\xi} \) where
\[
\frac{\eta}{\xi} = \left( \frac{2\varphi_e e^{-\varphi/2}}{1 + |\varphi_e| e^{-\varphi}} \frac{1 - |\varphi_e|^2 e^{-\varphi}}{1 + |\varphi_e|^2 e^{-\varphi}} \right) = \left( \frac{2\psi}{1 + \psi^2} \frac{1 - |\psi|^2}{1 + |\psi|^2} \right)
\] (3.8)
is a Euclidean normal vector. It is straightforward to check that the geodesic flow \( \alpha(t) \in T_1\mathbb{H}^3 \) starting from \( -\text{Ep}_\varphi(z) = (\text{Ep}_\varphi(z), -\frac{\eta}{\xi}) \) satisfies
\[
\alpha(t) = -\text{Ep}_\varphi + 2t(z),
\]
and the base point of \( \alpha(t) \) tends to \( z \) as \( t \rightarrow \infty \).
Example 3.1.  
• If \( \varphi \equiv 2t \), then for all \( z \),
  \[
  \text{Ep}_\varphi(z) = (z, 2e^{-t}) \quad \eta = (0, 1).
  \]

• If \( e^\varphi = \frac{4}{1+|z|^2} \), then for all \( z \in \mathbb{C} \), \( (y, \xi) = (0, 1) \).

• If \( \varphi = \log 4 - 2 \log(1 - |z|^2) \), i.e., \( e^\varphi|dz|^2 \) is the hyperbolic metric in \( \mathbb{D} \), then for \( z = re^{i\theta} \in \mathbb{D} \),
  \[
  \text{Ep}_\varphi(re^{i\theta}) = \left( \frac{2r}{1 + r^2} e^{\theta}, \frac{1 - r^2}{1 + r^2} \right) = \eta.
  \]

We see (and one of the advantage of choosing this convention is) that \( \text{Ep}_\varphi \) maps \( \mathbb{D} \) onto the totally geodesic plane in \( \mathbb{H}^3 \) bounded by \( \partial \mathbb{D} \).

Fix \( \varphi \in C^\infty(U, \mathbb{R}) \). Let \( \Sigma_t \) denote the Epstein surface associated with the metric
\( e^\varphi + 2|dz|^2 \), \( \Sigma = \Sigma_0 \). Let \( I \) and \( I_t \) denote the first fundamental form on \( \Sigma \) and on \( \Sigma_t \). Let
\( B(v) := -\nabla_v \bar{n} \) be the shape operator on \( \Sigma \), wherever \( \text{Ep}_\varphi \) is an immersion. We let
\[
I^* := I + 2I I + I I I = I((id + B) \cdot, (id + B) \cdot); \\
II^* := I - I I I = I((id + B) \cdot, (id - B) \cdot); \\
B^* := (I^*)^{-1} = \frac{id - B}{id + B}; \\
III^* := I^*(B^*, B^*) = I((id - B) \cdot, (id - B) \cdot) = I - 2II + III
\]
where \( id \) is the identity operator. We define similarly \( H^* = \text{tr}(B^*)/2 \).

Theorem 3.3 (See [17, Lem. 5.7, Thm.5.8, Cor.5.11]). We have
\[
I_t = \frac{1}{4}(e^{2t}I^* + 2II^* + e^{-2t}III^*).
\]

In particular,
\[
4e^{-2t}I_t \xrightarrow{t \to \infty} I^*.
\]

Moreover, we have
\[
I^* = e^\varphi|dz|^2, \quad II^* = \vartheta dz^2 + \bar{\vartheta} d\bar{z}^2 + 2\varphi_{zz} dz d\bar{z},
\]
where \( \vartheta = \varphi_{zz} - \frac{1}{2}(\varphi_z)^2 \). The eigenvalues of \( B^* \) are given by
\[
k_{\pm}^* = \frac{1 - k_{\pm}}{1 + k_{\pm}} = 2e^{-\varphi} \left( \varphi_{zz} \pm \sqrt{\vartheta \bar{\vartheta}} \right) = -K^* \pm 2e^{-\varphi} \sqrt{\vartheta \bar{\vartheta}}.
\]
In the last equality we used the identity
\[
\varphi_{\bar{z}z} = \frac{1}{4} \Delta \varphi = -\frac{1}{2} e^{\varphi} K^*
\] (3.9)
where \( K^* \) denotes the Gauss curvature of the metric \( \Gamma^* \).

**Example 3.4.** When \( \varphi \equiv 0 \), we have \( I_t(y, \xi) = \xi^{-2} |dy|^2 = (e^{2t}/4) |dy|^2 \).

**Corollary 3.5.** Let \( da \) denote the area form induced by \( I \), \( da^* = e^\varphi d^2 z \) the area form induced by \( \Gamma^* \), we have \( da^* = (1 + k_+(1 + k_-)) da \),

\[
H da = \left( \frac{1 - (K^*)^2}{4} + |\vartheta|^2 e^{-2\varphi} \right) da^*,
\]
\[
H^* = \frac{k^*_+ + k^*_-}{2} = -K^*.
\]
and
\[
k_+ k_- da = \frac{(1 - k^*_+)(1 - k^*_-)}{4} da^* = \left[ \frac{(1 + K^*)^2}{4} - e^{-2\varphi} |\vartheta|^2 \right] da^*.
\]

**Proof.** It follows directly from Definition 3.2 that
\[
k_\pm = \frac{1 - k^*_\pm}{1 + k^*_\pm} \quad \text{and} \quad da^* = (1 + k_+(1 + k_-)) da.
\]
We obtain from Theorem 3.3 and (3.9) that
\[
H da = \frac{k_+ + k_-}{2(1 + k_+(1 + k_-))} da^* = \frac{1}{4} (1 - k^*_+ k^*_-) e^\varphi d^2 z
\]
\[
= \frac{1}{4} (1 - 4 e^{-2\varphi} (\varphi_{\bar{z}z} - |\vartheta|^2)) e^\varphi d^2 z
\]
\[
= \frac{1}{4} (1 - 4 e^{-2\varphi} (\frac{1}{2} e^\varphi K^*))^2 - |\vartheta|^2) e^\varphi d^2 z
\]
\[
= \left( 1 - (K^*)^2 + |\vartheta|^2 e^{-2\varphi} \right) da^*.
\]
Similarly,
\[
H^* = \frac{k^*_+ + k^*_-}{2} = 2 e^{-\varphi} \varphi_{\bar{z}z} = -K^*
\]
and
\[
k_k - da = \frac{k_+ k_-}{(1 + k_+(1 + k_-))} da^* = \frac{1 - k^*_+ k^*_-}{4} da^*
\]
\[
= \frac{1}{4} (1 + K^* - 2 e^{-\varphi} \sqrt{|\vartheta|}(1 + K^* + 2 e^{-\varphi} \sqrt{|\vartheta|}) da^*
\]
\[
= \left[ \frac{(1 + K^*)^2}{4} - e^{-2\varphi} |\vartheta|^2 \right] da^*
\]
as claimed. \( \square \)
3.3 Epstein–Poincaré map on a simply connected domain

We apply the results to the special case of Epstein–Poincaré surfaces, namely, the Epstein surfaces associated with the Poincaré (or hyperbolic, namely, $K^* = -1$) metric $\rho_\Omega|dz|^2 = e^\varphi|dz|^2$ on a simply connected domain $\Omega \subseteq \mathbb{C}$. Let $f : \mathbb{D} \to \Omega$ be a conformal map. Then using the same notation as Theorem 3.3, we have

$$\vartheta = \mathcal{S}(f^{-1}), \quad k^*_\pm = 1 \pm 2\|\mathcal{S}(f^{-1})\|$$

where

$$\|\mathcal{S}(f^{-1})\| := |\mathcal{S}(f^{-1})|e^{-\varphi}.$$  

**Remark 3.6.** The fact that $\vartheta$ equals the Schwarzian derivative of a uniformization map follows from a direct computation, this holds true if and only if the metric on $\Omega$ has constant curvature. Moreover, from the Nehari bound, we have

$$\|\mathcal{S}(f^{-1})\|(z) \leq \frac{3}{2}, \quad \forall z \in \Omega.$$

From the computation above and Corollary 3.5 we obtain the following result.

**Theorem 3.7** (Epstein–Poincaré surface [12, Prop. 7.4]). If $\|\mathcal{S}(f^{-1})\|(z) \neq 1$, then $\text{Ep}_\Omega$ is an immersion near $z$ and the principal curvatures of $\Sigma_\Omega$ at $\text{Ep}_\Omega(z)$ are given by

$$k_\pm = -\|\mathcal{S}(f^{-1})\| \|\mathcal{S}(f^{-1})\| \pm 1.$$  \hspace{1cm} (3.10)

In particular, we have

$$Hda = |\vartheta|^2 e^{-2\varphi}da^* = \|\mathcal{S}(f^{-1})\|^2 \rho_\Omega d^2z.$$  \hspace{1cm} (3.11)

We have the total curvature

$$\int_{\Sigma_\Omega} |\det B| da := \int_{\Sigma_\Omega} |k_-k_+|da = \int_{\Omega} e^{-2\varphi}|\vartheta|^2da^* = \int_{\Omega} \|\mathcal{S}(f^{-1})\|^2 \rho_\Omega|dz|^2 = \int_{\Sigma_\Omega} Hda.$$

We note that $\int_{\Sigma_\Omega}$ is understood as the integral on the non-singular locus

$$\{\text{Ep}_\Omega(z) : \|\mathcal{S}(f^{-1})\|(z) \neq 1\}$$

which has full measure.

**Example 3.8.** If $\Omega = \mathbb{D}$, (3.10) shows $k_\pm \equiv 0$ which is consistent with the fact that $\Sigma$ is a totally geodesic plane. See Example 3.1.

We obtain immediately the following consequence which is reminiscent to the results of Bishop obtained in [3].

**Corollary 3.9.** A Jordan curve $\gamma$ is a Weil–Petersson quasicircle if and only if

$$0 \leq \int_{\Sigma_\Omega} Hda = \int_{\Sigma_\Omega} |\det B| da < \infty.$$
Proof. This follows from the characterization (2.4) of Weil–Petersson quasicircle, the identity
\[ \int_{D} \| \mathcal{J}(f) \|^{2}(\zeta)\rho d^{2}\zeta = \int_{\Omega} \| \mathcal{J}(f^{-1}) \|^{2}(z)\rho d^{2}z, \]
and Theorem 3.7.

The Epstein surface is uniquely determined and naturally associated with a simply connected domain $\Omega$. There are two connected components of $\gamma$ in $\hat{C}$, we will study the relation between the two Epstein–Poincaré surfaces later in Section 4 and it will be crucial to define the renormalized volume. However, let us first record some properties of a single Epstein–Poincaré surface. We will use a few classical results from geometric function theory.

**Theorem 3.10.** Let $f$ be a conformal map from $\mathbb{D}$ onto a domain bounded by a Jordan curve $\gamma$.

- See [23, Prop. 1.2]. For all $\zeta \in \mathbb{D}$, we have
  \[ \left| \frac{1 - |\zeta|^{2}}{2} f''(\zeta) - \frac{1}{f'(\zeta)} \right| \leq 2. \]  
  \[ (3.12) \]
- See [23, Cor. 1.4]. For all $\zeta \in \mathbb{D}$, we have
  \[ \frac{1}{4}(1 - |\zeta|^{2}) |f'(\zeta)| \leq \text{dist}(f(\zeta), \gamma) \leq (1 - |\zeta|^{2}) |f'(\zeta)|. \]  
  \[ (3.13) \]
- See [23, Thm. 11.1]. The following are equivalent:
  
  (AC1) $\gamma$ is asymptotically conformal;
  (AC2) $\lim_{|\zeta| \to 1} \frac{f''(\zeta)}{f'(\zeta)}(1 - |\zeta|^{2}) = 0$;
  (AC3) $\lim_{|\zeta| \to 1} \| \mathcal{J}(f) \|(\zeta) = 0$;
  (AC4) $f(\zeta) - f(x) \overline{(\zeta - x)f'(\zeta)} \to 1$ as $\zeta \to x$, $x \in \mathbb{D}$ and $|z - x| \leq a$ (for all $a > 0$).

**Example 3.11.** Weil–Petersson quasicircles are asymptotically conformal. See, e.g., Corollary II.1.4 of [32].

**Lemma 3.12.** The Epstein–Poincaré map $E_{p_{\Omega}} : \Omega \to \Sigma_{\Omega}$ has the following explicit expression. For $z = f(\zeta)$, $\zeta \in \mathbb{D}$, we have

\[ \psi(z) = \varphi_{z} e^{-\psi / 2} = \frac{|f'(\zeta)|}{f'(\zeta)} \left( -\frac{f''(\zeta)}{f'(\zeta)} \frac{(1 - |\zeta|^{2})}{2} + \frac{1}{|\zeta|^{2}} \right), \]

\[ e^{-\psi(z)/2} = \frac{1}{2} |e^{-\psi(z)/2}| \left( 1 - |\zeta|^{2} \right), \]

\[ |\varphi(z)| = \frac{2e^{-\psi / 2}}{1 + |\varphi|^{2}} = \frac{|f'(\zeta)|(1 - |\zeta|^{2})}{1 + |f''(\zeta)/f'(\zeta)|^{2} + |\zeta|^{2}}, \]

\[ y(z) = z + \xi \cdot \psi = f(\zeta) + \left( -\frac{f''(\zeta)}{f'(\zeta)} \frac{(1 - |\zeta|^{2})}{2} + \frac{1}{|\zeta|^{2}} \right) f'(\zeta)(1 - |\zeta|^{2}) \overline{(\zeta - x)f'(\zeta)} \]
Furthermore, it extends continuously to $\partial \mathbb{D}$, namely, $(y, \xi) \xrightarrow{\zeta \to e^{i\theta}} (f(e^{i\theta}), 0)$. In particular, $E_{p_{\Omega}}$ extends continuously to $\gamma$ as the identity map.

**Proof.** We express $\rho_{\Omega}$ as

$$\rho_{\Omega} = e^{\varphi(z)} |dz|^2 = \frac{4|dz|^2}{|f'(f^{-1}(z))|^2(1 - |f^{-1}(z)|^2)^2}. \quad (3.14)$$

In $\zeta$ variable,

$$e^{-\varphi(f(\zeta))} = \left( \frac{|f'(\zeta)(1 - |\zeta|^2)}{2} \right)^2, \quad (3.15)$$

it gives

$$\varphi(f(\zeta)) = \log 4 - \log |f'(\zeta)|^2 - 2 \log(1 - |\zeta|^2).$$

Taking derivative in $\zeta$ we obtain

$$\varphi_z f'(\zeta) = -\frac{f''(\zeta)}{f'(\zeta)} \frac{2\zeta}{1 - |\zeta|^2} + 2 \zeta \frac{1 - |\zeta|^2}{2},$$

hence

$$\varphi_z = -\frac{f''(\zeta)}{f'(\zeta)} \frac{2\zeta}{(f'(\zeta))^2} + \frac{2\zeta}{f'(\zeta)(1 - |\zeta|^2)}.$$  

Combining with equation (3.15) we get

$$\psi(z) = \varphi_{\zeta} e^{-\varphi/2} = \frac{|f'(\zeta)|}{f'(\zeta)} \left( \frac{f''(\zeta)(1 - |\zeta|^2)}{f'(\zeta)} + \zeta \right). \quad (3.16)$$

The expression for $\xi$ and $y$ follows from their definition (3.7). We note that (3.12) implies that $|\psi| \leq 2$ and (3.13) implies that $e^{-\varphi(z)/2} \to 0$ as $\zeta \to e^{i\theta}$. We obtain the limit $(y(z), \xi(z)) \xrightarrow{\zeta \to e^{i\theta}} (f(e^{i\theta}), 0).$  

Lemma 3.12 and condition (AC2) show that when $\gamma$ is asymptotically conformal, we have as $\zeta \to \partial \mathbb{D}$,

$$y \circ f(\zeta) - f(\zeta) \sim \frac{\zeta f'(\zeta)(1 - |\zeta|^2)}{2} \simeq \frac{\zeta f'(\zeta)}{|\zeta f'(\zeta)|} \text{dist}(f(\zeta), \gamma);$$

$$\xi \circ f(\zeta) \sim \frac{|f'(\zeta)(1 - |\zeta|^2)|}{2} \simeq \text{dist}(f(\zeta), \gamma).$$

Where “$\sim$” means the ratio goes to 1 and “$\simeq$” means the ratio is bounded from above and below. And for all $\zeta \in \mathbb{D}$, Lemma 3.12, inequalities (3.12), and (3.13) show that

$$\text{dist}(f(\zeta), \gamma) \leq \frac{|f'(\zeta)(1 - |\zeta|^2)|}{5} \leq |\xi \circ f(\zeta)| \leq |f'(\zeta)(1 - |\zeta|^2)| \leq 4 \text{dist}(f(\zeta), \gamma). \quad (3.17)$$

**Corollary 3.13.** If $\Omega$ is bounded by an asymptotically conformal curve $\gamma$, $E_{p_{\Omega}} \circ f$ is an immersion and embedding in a neighborhood of $\partial \mathbb{D}$.  

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Proof. Since $\gamma$ is asymptotically conformal, Theorem 3.7 and (AC3) imply that $\text{Ep}_\Omega \circ f$ is an immersion in a neighborhood of $\partial \mathbb{D}$.

Now we also show that $\text{Ep}_\Omega \circ f$ is also an embedding near the boundary. The inequalities in (3.13) show that there exists $\delta_0 > 0$ such that if $1 - |\zeta| < \delta_0$ then

$$\frac{1}{3} \text{dist}(f(\zeta), \gamma) \leq |y(\zeta) - f(\zeta)|, \quad \xi(\zeta) \leq 3 \text{dist}(f(\zeta), \gamma)$$

(3.18)

and the principal curvatures $\lambda_\pm$ of $\Sigma_\Omega$ at $\text{Ep}_\Omega \circ f(\zeta)$ are bounded by 1/2. Let $A_\delta := \{ \zeta \in \mathbb{D} : 1 - |\zeta| < \delta \}$. We now show that $\text{Ep}_\Omega \circ f|_{A_\delta}$ is an embedding for small enough $\delta$.

If it is not the case, then from (3.17) the self-intersection must occur to points with comparable distance to $\gamma$. In other words, we have for every $\delta < \delta_0$, there is $\zeta_1 = \zeta_1(\delta), \zeta_2 = \zeta_2(\delta) \in A_\delta$ such that $y(\zeta_1) = y(\zeta_2)$ and $\xi(\zeta_1) = \xi(\zeta_2)$. Then we know from (3.18) that $|\zeta_1 - \zeta_2| \to 0$ as $\delta \to 0$. Let $\eta_\delta$ be a geodesic loop from $(y(\zeta_1), \xi(\zeta_1))$ to $(y(\zeta_2), \xi(\zeta_2))$ on the Epstein surface, and for small $\delta$, it is contained in the image of $\text{Ep}_\Omega \circ f|_{A_\delta}$. The proof of [12, Thm.3.4] or [10, Prop.4.15] then shows that it is not possible since the principal curvatures on $\text{Ep}_\Omega \circ f|_{A_{\delta_0}}$ have modulus less than 1/2. \hfill \Box

4 Renormalized volume for a Jordan curve

4.1 Disjoint Epstein–Poincaré surfaces

When $\gamma$ is a circle, Example 3.1 shows that both Epstein surfaces coincide with the geodesic plane bounded by $\gamma$.

**Proposition 4.1.** If $\gamma$ is not a circle, then $\Sigma_\Omega$ and $\Sigma_{\Omega^*}$ are disjoint in the interior.

**Proof.** We will use the horosphere envelop description of the Epstein surfaces described in Section 3.1. It suffices to show that for all $\eta \in \Omega$ and $\eta' \in \Omega^*$, the horospheres at $\eta$ and $\eta'$ for the respective hyperbolic metric are disjoint.

By the invariance property (3.6) of Epstein map under Möbius transformation, we may assume that $\eta$ is the south pole $s = (0, 0, -1)$ and $\eta'$ is the north pole $n = (0, 0, 1)$. Moreover, writing the Poincaré metrics $\rho_\Omega = e^s \rho_{S^2}$ and $\rho_{\Omega^*} = e^\psi \rho_{S^2}$, we may also assume that $\psi(n) = 0$ after possibly applying another Möbius transformation fixing $n$ and $s$. Hence, the horospheres $H(n, \psi) = H(n, 0)$ passes through the origin $(0, 0, 0)$ and is contained in the upper half-ball of $\mathbb{B}^3$.

Let $\pi : S^2 \setminus \{n\} \to \mathbb{C}$ be the stereographic projection from $n$ sending $s$ to $0 \in \mathbb{C}$ and the lower half-sphere $S^2_-$ onto $\mathbb{D}$. The condition $\psi(n) = 0$ is equivalent to $\lim_{z \to \infty} g'(z) = 1$ where $g$ is a conformal map from $\mathbb{D}^* = \pi(S^2_+)$ to $\pi(\Omega^*)$ fixing $\infty$.

On the other hand, we have

$$\pi_* \rho_{S^2_+} = \frac{4}{(1 + |z|^2)^2} |dz|^2, \quad \pi_* \rho_{S^2_-} = \frac{4}{(1 - |z|^2)^2} |dz|^2$$

where $z = \pi(\theta)$, and these expressions coincide when $z = \pi(s) = 0$ which shows

$$\rho_{S^2_+} = \rho_{S^2_-}.$$
Now let $f$ be a conformal map from $\mathbb{D}$ onto $\pi(\Omega)$ fixing 0. We have

$$\pi_\ast \rho_{s^2, s} = \pi_\ast \rho_{s^2, s} = \rho_{\mathbb{D}, 0} = |f'(0)|^2 \rho_{\pi(\Omega), 0} = |f'(0)|^2 \pi_\ast \rho_{\Omega, s},$$

where the second and the last equalities follow from the definition of push-forward, and the third equality follows from the property of the hyperbolic metric.

The following Lemma 4.2 shows that $|f'(0)| < 1$. Therefore, $\rho_{\Omega, s} > \rho_{s^2, s}$ and (3.4) and (3.5) show that $H(s, \varphi)$ is strictly contained in the inside of $H(s, 0)$. In particular, $H(s, \varphi)$ is strictly contained in the lower half-ball and is disjoint from $H(n, \psi)$. \hfill \Box

The following lemma is a special case of the Grunsky inequality, see, e.g., [22, Thm. 4.1, (21)] and [32, P. 70-71].

**Lemma 4.2** (Grunsky inequality). Suppose that $f : \mathbb{D} \to \mathbb{C}$ and $g : \mathbb{D}^* \to \hat{\mathbb{C}}$ are univalent functions on $\mathbb{D}$ and $\mathbb{D}^*$ such that $f(0) = 0$ and $g(\infty) = \infty$, and $f(\mathbb{D}) \cap g(\mathbb{D}^*) = \emptyset$. Then we have

$$\int_{\mathbb{D}} \left| \frac{f'(z)}{f(z)} - \frac{1}{z} \right|^2 d^2 z + \int_{\mathbb{D}^*} \left| \frac{g'(z)}{g(z)} - \frac{1}{z} \right|^2 d^2 z \leq 2 \pi \log \left| \frac{g'(\infty)}{f'(0)} \right|.$$

Equality holds if $\mathbb{C} \setminus \{f(\mathbb{D}) \cup g(\mathbb{D}^*)\}$ has zero Lebesgue measure.

### 4.2 Volume between the Epstein–Poincaré surfaces

Let $\gamma \subset \hat{\mathbb{C}}$ be an asymptotically conformal Jordan curve. We now define the volume between $\Sigma_\Omega$ and $\Sigma_{\Omega^*}$. Without loss of generality, we assume that $\gamma$ does not contain $\infty \in \hat{\mathbb{C}}$ and use the upper space model of $\mathbb{H}^3$. We cautiously note that both Epstein–Poincaré surfaces are non-compact and may not be embedded. For this reason we use an approximation to compute the volume. For $\varepsilon > 0$, let

$$\text{vol}_{\varepsilon} = 1_{\xi \geq \varepsilon} \frac{\text{vol}_{\text{eucl}}}{\xi^3}$$

where $\text{vol}_{\text{eucl}}$ is the Euclidean volume form.

Let $\varphi_\gamma$ be continuous map $\mathbb{H}^3 \to \mathbb{H}^3$, such that $\varphi_\gamma|_\Omega = \text{Ep}_\Omega$, $\varphi_\gamma|_{\Omega^*} = \text{Ep}_{\Omega^*}$, and $\varphi_\gamma|_{\mathbb{H}^3}$ is differentiable. This is possible since $\text{Ep}_\Omega$ and $\text{Ep}_{\Omega^*}$ extend to the identity map on $\gamma$. We define

$$V_2(\gamma)(\varepsilon) := \int_{\mathbb{H}^3} \varphi_\gamma^* \text{vol}_{\varepsilon}.$$

This is the signed volume between the Epstein surfaces bounded by $\gamma$ above level $\varepsilon$.

Since the boundary values of $\varphi_\gamma$ are determined and $\varphi_\gamma(\mathbb{H}^3) \cap \{(y, \xi) : \xi \geq \varepsilon\}$ is compact, we have $V_2(\gamma)(\varepsilon)$ is finite and independent from the choice of $\varphi_\gamma$. Since both Epstein surfaces are disjoint (unless $\gamma$ is a circle) by Proposition 4.1 and embedded near the boundary by Corollary 3.13, without loss of generality, we assume further more that the Jacobian of $\varphi_\gamma$ is positive in a neighborhood $U_\gamma$ of $\gamma$ in $\mathbb{H}^3$. (If $\gamma$ is a circle, then we choose $\varphi_\gamma$ such that the Jacobian is zero.) The limit

$$\lim_{\varepsilon \to 0^+} V_2(\gamma)(\varepsilon) \in (-\infty, \infty] \quad (4.1)$$

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exists since \( \int_{U_\varepsilon} \varphi^*_\gamma \text{vol}_e \) increases as \( \varepsilon \to 0^+ \) and \( \int_{\mathbb{H}^3 \setminus U_\varepsilon} \varphi^*_\gamma \text{vol}_e \) is constant for small enough \( \varepsilon \). The monotonicity and (3.6) also show that the limit is invariant under actions of elements in \( \text{PSL}_2(\mathbb{C}) \) which do not send any point of \( \gamma \) to \( \infty \in \hat{\mathbb{C}} \).

**Definition 4.3.** For an asymptotically conformal Jordan curve \( \gamma \subset \hat{\mathbb{C}} \), we define the **signed volume between the Epstein–Poincaré surfaces** \( V(\gamma) \) to be the limit in (4.1) applied to the curve \( A(\gamma) \), where \( A \) is any element in \( \text{PSL}_2(\mathbb{C}) \) such that \( A(\gamma) \) does not pass through \( \infty \).

The above definition is clearly \( \text{PSL}_2(\mathbb{C}) \)-invariant.

### 4.3 Volume for smooth Jordan curves

In this subsection we will see if the Jordan curve \( \gamma \) is sufficiently smooth, then the map \( \text{Ep}_\Omega \) extends not only continuously to \( \gamma \) but also osculates to the totally geodesic plane bounded by the circle osculating to the curve. This will be useful later to prove that the volume between \( \text{Ep}_\Omega \) and \( \text{Ep}_{\Omega^*} \) is finite, if \( \gamma \) is sufficiently smooth.

For a \( C^2 \) curve \( \gamma(t) = x(t) + iy(t) \) in \( \mathbb{C} \), its curvature is calculated by

\[
k_\gamma(t) = \frac{x'y'' - y'x''}{(x'^2 + (y')^2)^{3/2}} = \frac{1}{|\gamma'|^2} \text{Re} \left( -i\gamma'\gamma'' \right)
\]

If \( \gamma \) is \( C^{2,\alpha} \) for some \( 0 < \alpha < 1 \), Kellogg’s theorem implies that the conformal map \( f : \mathbb{D} \to \Omega \) extends to a \( C^{2,\alpha} \) homeomorphism \( \overline{\mathbb{D}} \to \overline{\Omega} \). Writing \( \gamma(\theta) = f(e^{i\theta}) \) we have that

\[
\begin{align*}
\gamma'(\theta) &= if'(e^{i\theta})e^{i\theta} \\
\gamma''(\theta) &= -f''(e^{i\theta}) - f'(e^{i\theta})e^{i\theta}.
\end{align*}
\]

Then it follows that

\[
k_\gamma(\gamma(\theta)) = \frac{-\text{Re} \left( f''(e^{i\theta})f'(e^{i\theta})e^{i\theta} + |f'(e^{i\theta})|^2 \right)}{|f'(e^{i\theta})|^3} = \frac{-\text{Re}(f''(e^{i\theta})e^{i\theta}) - 1}{|f'(e^{i\theta})|}.
\]

Define then the **osculating circle**, denoted by \( \mathcal{C}_\gamma(\theta) \), as the circle with center \( \gamma(\theta) + \frac{i\gamma'(\theta)}{|\gamma'(\theta)|} \) and radius \( \frac{|\gamma'(\theta)|}{|\gamma''(\theta)|} \), oriented the same as \( \gamma \). The circle \( \mathcal{C}_\gamma(\theta) \) is then tangent to \( \gamma \) at \( \gamma(\theta) \), and agrees with \( \gamma \) at this tangent point up to second order. We define the **osculating plane**, denoted by \( \mathcal{P}_\gamma(\theta) \), as the geodesic plane in \( \mathbb{H}^3 \) so that the boundary of \( \mathcal{P}_\gamma(\theta) \) is \( \mathcal{C}_\gamma(\theta) \).

Now we show that for \( \gamma \) sufficiently smooth we have that \( \text{Ep}_\Omega \) and \( \mathcal{P}_\gamma(\theta) \) agree up to second order. For this, a straightforward computation using the explicit expression of the Epstein–Poincaré map (Lemma 3.12 and (3.8)) gives the following.
Lemma 4.4. Assume that $\gamma$ is $C^{2,\alpha}$. Using the parametrization $(r, \theta) \mapsto f(re^{i\theta})$ of the domain $\Omega$, near $q = f(e^{i\theta}) \in \gamma$, we have

$$\lim_{z \to q} \psi(z) = \frac{|f'(e^{i\theta})|}{f'(e^{i\theta})} e^{i\theta};$$

$$\lim_{z \to q} \nabla f(z) = \lim_{z \to q} \left( \frac{2\psi}{1 + |\psi|^2}, \frac{1 - |\psi|^2}{1 + |\psi|^2} \right) = \left( \frac{|f'(e^{i\theta})|}{f'(e^{i\theta})} e^{i\theta}, 0 \right);$$

$$\partial_r \nabla f(q) = \left( 0, -\Re\left( \frac{f''(e^{i\theta})}{f'(e^{i\theta})} e^{i\theta} \right) - 1 \right);$$

$$\partial_\theta \nabla f(q) = \left( \frac{e^{i\theta} f'(e^{i\theta})}{|f'(e^{i\theta})|^2} \left( \Re\left( \frac{f''(e^{i\theta})}{f'(e^{i\theta})} e^{i\theta} \right) + 1 \right), 0 \right).$$

In particular, this implies that the Epstein–Poincaré surface, viewed as a surface embedded in $\mathbb{R}^3$ near $\gamma$, is umbilic with curvature $-(\Re\left( \frac{f''(e^{i\theta})}{f'(e^{i\theta})} e^{i\theta} \right) + 1)|f'(e^{i\theta})|^{-1} = k_\gamma(q)$ at $q$.

Proposition 4.5. Let $\gamma$ be a $C^{4,\alpha}$ Jordan curve in $\mathbb{C}$. Then $E_{p_\Omega}$ and $\mathcal{P}_\gamma(\theta)$ are tangent at $(\gamma(\theta), 0)$ and agree up to order 2.

Proof. Since $\gamma$ is $C^{4,\alpha}$ then $f : \mathbb{D} \to \mathbb{C}$ extended to $\partial \mathbb{D}$ as a $C^{4,\alpha}$ map. Then the Epstein map extends by identity on $\gamma$ is a $C^{2,\alpha}$ map on $\mathbb{D}$ by Lemma 3.12. We see then from Lemma 4.4 that the continuous extension of $\nabla f$ is the Euclidean unit outward orthogonal vector to $\gamma$. Hence the extension of the Epstein map at $\gamma$ has to agree up to first order with the osculating plane $\mathcal{P}_\gamma$. To verify that they actually agree up to second order, Lemma 4.4 shows that the Epstein–Poincaré surface is umbilic with the same curvature in the Euclidean space as $\mathcal{P}_\gamma$.

Next we show that for sufficiently regular curves $\gamma$ this volume is in fact finite.

Proposition 4.6. Let $\gamma$ be a $C^{5,\alpha}$ Jordan curve in $\mathbb{C}$. Then $V(\gamma)$ is finite.

Proof. Without loss of generality, we assume that $\gamma : S^1 \to \mathbb{C}$ is parametrized by arc-length. Take $\varphi_\gamma$ some continuous map $\mathbb{H}^3 \to \mathbb{H}^3$ as before, meaning that $\varphi_\gamma|_\Omega = E_{p_\Omega}$, $\varphi_\gamma|_{\partial \mathbb{D}} = E_{p_{\partial \mathbb{D}}}$, and $\varphi_\gamma|_{\mathbb{H}^3}$ is differentiable. We take the following $C^{4,\alpha}$ parametrization of a neighbourhood $U$ of $\gamma$ in $\mathbb{H}^3$, denoted $G : S^1_s \times \mathbb{H}^3_{(a,b)} \to \mathbb{H}^3$, by

$$G(s, a, b) = \gamma(s) + ia\gamma'(s) + be_3$$

where $e_3 = (0, 0, 1)$. It is a straightforward calculation to see that the hyperbolic metric in $G$-coordinates is given by

$$\frac{(1 - ak(s))^2}{b^2}ds^2 + \frac{1}{b^2}da^2 + \frac{1}{b^2}db^2,$$

where $k(s)$ is the signed curvature of $\gamma$ given by $\gamma''(s) = ik(s)\gamma'(s)$. Hence the volume form is given by

$$\frac{(1 - ak(s))}{b^3}ds \, da \, db.$$
If we assume that $\gamma$ is $C^{5,\alpha}$, then the Epstein–Poincaré surfaces are $C^{3,\alpha}$ up to the boundary. This means that there are $C^{3,\alpha}$ functions $a_{\Omega}, a_{\Omega^*} : S^1_b \times [0, \varepsilon_0]_b \to \mathbb{R}$ so that the Epstein–Poincaré surfaces in the neighbourhood $U$ of $\gamma$ are given by $G(s, a_{\Omega}(s, b), G(s, a_{\Omega^*}(s, b), b)$. And since by Proposition 4.5 the Epstein–Poincaré surfaces agree up to second order at $\gamma$, then there exists a constant $C > 0$ so that $|a_{\Omega}(s, b) - a_{\Omega^*}(s, b)| \leq Cb^3$.

Hence for small enough neighborhood $U$ of $\gamma$, the integral $\int_U \varphi_s \, \text{vol}$ will be bounded by

$$V_1(\gamma)(\varepsilon_0) = \int_{S^1} \int_0^{\varepsilon_0} \int_{a_{\Omega}(s, b)}^{a_{\Omega^*}(s, b)} \frac{(1 - ak(s))}{b^3} \, da \, db \, ds.$$ 

This integral is well-defined and convergent since

$$\left| \int_{a_{\Omega}(s, b)}^{a_{\Omega^*}(s, b)} \frac{(1 - ak(s))}{b^3} \, da \right| = \frac{1}{b^3} \left| a_{\Omega^*}(s, b) - a_{\Omega}(s, b) \right| \left( \frac{a_{\Omega}(s, b) + a_{\Omega^*}(s, b)}{2} \right) k(s) - 1$$

is bounded by a constant independent of $(s, b)$. Hence $V(\gamma) = \lim_{\varepsilon \to 0+} V_2(\gamma)(\varepsilon)$ is a finite real value.

**Definition 4.7.** Let $\gamma$ be a Weil–Petersson quasicircle in $\mathbb{C}$. Then we define $V_R(\gamma)$, the renormalized volume of $\gamma$, as

$$V_R(\gamma) := V(\gamma) - \frac{1}{2} \int_{\Sigma_{\Omega} \cup \Sigma_{\Omega^*}} H \, da = V(\gamma) - \frac{1}{2} \int_{\Omega} \| \mathcal{J}(f^{-1}) \|^2(z) \rho \, d^2z - \frac{1}{2} \int_{\Omega} \| \mathcal{J}^*(g^{-1}) \|^2(z) \rho \, d^2z. \quad (4.2)$$

**Remark 4.8.** The second identity follows from Theorem 3.7. A priori, $V_R(\gamma) \in (-\infty, \infty]$ as $V(\gamma) \in (-\infty, \infty]$ and the integrals of mean curvature are finite by Corollary 3.9. Proposition 4.6 shows that if $\gamma$ is $C^{5,\alpha}$, then $V_R(\gamma) < \infty$. From the $\text{PSL}_2(\mathbb{C})$-invariance of each summand in ((4.2)) we can easily see that $V_R$ is $\text{PSL}_2(\mathbb{C})$-invariant.

## 5 Universal Liouville action as renormalized volume

Our objective in this section is to prove that the renormalized volume in Definition 4.7 agrees up to a constant with the Loewner energy for $C^{5,\alpha}$ curves.

### 5.1 Variation of the volume

For this subsection we consider a 1-parameter family $(\gamma_t)_{t \in (-1, 1)}$ of $C^{5,\alpha}$ Jordan curves ($\alpha > 0$). We will define a parametrization of the Epstein surfaces that allows us to compute the derivative $\frac{\partial}{\partial t}|_{t=0} V(\gamma_t)$. Since scalar multiplications are isometries of $\mathbb{H}^3$, we can assume without loss of generality that all curves $\gamma_t$ have Euclidean arclength $2\pi$. Furthermore, for any sufficiently small $\varepsilon$ we have that $V(\gamma_t) = V_1(\gamma_t)(\varepsilon) + V_2(\gamma_t)(\varepsilon)$. Moreover, we assume that $V_2(\gamma_t)(\varepsilon) \xrightarrow{\varepsilon \to 0} V(\gamma_t)$ converges uniformly in $t$.

Let $f_t : \overline{\mathbb{H}} \to \Omega_t$, $g_t : \overline{\mathbb{H}}^3 \to \Omega^*_t$ be univalent functions. As the 1-parameter family $\gamma_t$ is $C^{5,\alpha}$, we can take the 1-parameter family of maps $f_t, g_t$ to be $C^{5,\alpha}$ on $\overline{\mathbb{H}}$ and $\overline{\mathbb{H}}^3$.
Figure 1: Illustration of the two Epstein–Poincaré surfaces associated with the two connected component of \( \mathbb{C} \setminus \gamma_t \) and the map \( h_t \).

respectively and in \( t \) parameters. Consider \( \varepsilon \) sufficiently small so that for \( z \in \overline{\mathbb{D}} \) with \( |z| > 1 - \varepsilon \) we have that \( \text{Ep}_{\Omega_t}(f_t(z)) \) belongs to the parametrized neighbourhood \( U_{\gamma_t} \) from Proposition 4.6. Take the horizontal line \( L_{t,z} \) (horocycle centered at \( \infty \in \mathbb{C} \)) obtained by varying the second \( G \)-coordinate in \( U_{\gamma_t} \) starting from \( \text{Ep}_{\Omega_t}(f_t(z)) \), and define \( h_t(z) \in \overline{\mathbb{D}}^c \) to be the point such that \( \text{Ep}_{\Omega^*_{t}}(g_t(h_t(z))) \) is the first point of intersection of the horizontal line with \( \text{Ep}_{\Omega^*_{t}} \). See Figure 1 for an illustration.

Clearly along \( \partial \mathbb{D} \) the map \( h_t \) agrees with \( g_t^{-1} \circ f_t \), and from the regularity of \( f_t, g_t \) and the \( G \)-coordinates of \( U_{\gamma_t} \) we can see that the 1-parameter family of functions \( h_t \) is \( C^{3,\alpha} \) in \( 1 - \varepsilon < |z| \leq 1 \) and \( t \)-parameters. Moreover, we can make \( \varepsilon \) sufficiently small so that \( h_t \) is a diffeomorphism with its image.

For \( r \) sufficiently close to 1, define the cylindrical neighbourhood \( A(r) \) of \( \gamma_0 \) as \( f_0(\{ r \leq |z| \leq 1 \}) \cup g_0(h_0(\{ r \leq |z| \leq 1 \})) \), which we parametrize by \( S^1 \times [r, 1/r] \), sending \( (p, s) \) to \( f_0(sp) \) if \( s \leq 1 \) and sending \( (p, s) \) to \( g_0(h_0(\frac{p}{s})) \) if \( s \geq 1 \). These cylindrical neighbourhoods are nested as \( r \) grows, and their intersection as \( r \to 1^- \) is \( \gamma_0 \). Define as well \( \Omega(r), \Omega^*(r) \) the components of \( \mathbb{C} \setminus A(r) \) in \( \Omega_0 \) and \( \Omega^*_0 \), respectively.

Define a 1-parameter family of homeomorphisms \( F_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) so that for \( z \in \overline{\Omega_0} \) we define \( F_t(z) := f_t(\gamma_t^{-1}(z)) \), for \( z \in g_0(h_0(\{ 1 - \varepsilon < |z| \leq 1 \})) \) we define \( F_t(z) := g_t(h_t^{-1}(g_0^{-1}(z))) \), and we extend \( F_t \) to the rest of \( \Omega^*_0 \) as a \( C^{3,\alpha} \) map in both \( \Omega^*_0 \) and \( t \) parameters. Let us also fix \( F_0 \) to be the identity. It follows then that \( F_t|_{\Omega_0} \) is a conformal map between \( \Omega_0 \) and \( \Omega_t \), and \( F_t|_{\gamma_0} \) parametrizes \( \gamma_t \). Given a parameter \( r \) so that \( 1 - \varepsilon < r < 1 \), we construct the family of piecewise smooth maps \( E_{r,t} : S^2 \to \mathbb{H}^3 \) satisfying the following properties:

(C1) In \( \Omega(r), \Omega^*(r) \) the map \( E_{r,t} \) is defined as the composition of \( F_t \) with the Epstein–Poincaré maps \( \text{Ep}_{\Omega_t}, \text{Ep}_{\Omega^*_{t}} \) of \( \gamma_t \).

(C2) Considering the parametrization of \( A(r) \), for each \( p \in S^1 \), \( E_{r,t}(\{ p \} \times [r, 1/r]) \) is the straight \( \mathbb{R}^3 \) segment joining \( \text{Ep}_{\Omega_t}(f_t(z)) \) and \( \text{Ep}_{\Omega^*_{t}}(g_t(h_t(z))) \).

(C3) The curve \( E_{r,t}(\gamma \times \{ r \}) \) is given by the image of a curve \( \gamma_{r,t} \) in \( \Omega^*(\gamma_t) \) under the Epstein–Poincaré map.

Given that under our conditions the Epstein–Poincaré maps agree up to order 2 at \( \gamma_t \), conditions (C1), (C2) and (C3) can be all satisfied for \( r \) sufficiently close to 1. For such fixed \( r \) the map \( E_{r,t} \) is piecewise smooth, and it is \( C^{3,\alpha} \) while restricted to \( \Omega(r), A(r), \Omega^*(r) \).
on both those parameters and $t$.

We prove the following main theorem in this section.

**Theorem 5.1.** Let $\gamma_t$ be a 1-parameter family of $C^{5,\alpha}$ Jordan curves ($\alpha > 0$). Then the first derivative of the volume $V(\gamma_t)$ is computed by

$$\frac{d}{dt} \bigg|_{t=0} V(\gamma_t) = \int_{\Omega} E_{\Omega}^*(\delta H + \frac{1}{4}(\delta I, I)da) + \int_{\Omega^*} E_{\Omega^*}^*(\delta H + \frac{1}{4}(\delta I, I)da)$$

where $E_{\Omega}, E_{\Omega^*}$ are the Epstein–Poincaré maps of $\Omega, \Omega^*$ (respectively); $I, I, H, da$ are the metric, second fundamental form, mean curvature and area form of the images of $E_{\Omega}, E_{\Omega^*}$; and $\delta$ denotes first order variation.

As in the convex cocompact case (see for instance [24,30]) the main idea is to prove the analogous to the Schl"afli formula for domains with piecewise smooth boundary by using Stokes theorem. While in our case the region bounded by $E_{\Omega}$ and $E_{\Omega^*}$ is non-compact and hence a new difficulty has been introduced, we have already taken the first step to deal with this by approximating with the (compact) region bounded by $E_{r,t}$. Even with this approximation, the map $E_{r,t}$ could fail to be a piecewise immersion, as the Epstein–Poincaré maps define branched surfaces in general. This happens when the principal curvatures at infinity are equal to $\pm 1$ at a given point, and hence the term $\delta H + \frac{1}{4}(\delta I, I)da$ is not well-defined. Regardless, we will see (Lemma 5.2) that we have a well-defined normal and a well-defined (parametrized) shape operator. This will allow us to still establish a geometric identity (Proposition 5.4) that will express the variation of volume as the integral of a well defined $\frac{1}{4}(\delta I, I)da$ to the non-immersed points. Then after using Stokes theorem and making $r \to 1^-$, we will obtain the identity of Theorem 5.1 by verifying that all other integrals (both from the approximation by $E_{r,t}$ and Stokes theorem) go to 0.

We first address the definition of the normal and (parametrized) shape operator for $E_{r,t}$.

**Lemma 5.2.** Along each $\Omega(r), A(r), \Omega^*(r)$, on the image of $E_{r,t}(p)$ there is a well-defined vector $\vec{n}$ that is normal to the image of $DE_{r,t}$. Such normal vector $\vec{n}$ varies piecewise $C^{3,\alpha}$ on $r, t, p$, and more precisely it is $C^{3,\alpha}$ while restricting $p$ to either $\Omega(r), A(r), \Omega^*(r)$. Moreover, there is a piecewise $C^{2,\alpha}$ family of linear maps $B_{r,t}(p) : \mathbb{R}^2 \to \mathbb{R}^2$ of the shape operator of the image of $E_{r,t}$.

**Proof.** For $\Omega(r), \Omega^*(r)$ the existence of $\vec{n}$ follows from the construction of the Epstein–Poincaré map, see (3.8), and from the map $F_{r,t}$ being piecewise $C^{3,\alpha}$. For $A(r)$, each curve $E_{r,t}(\gamma \times \{r\})$ is embedded for $r$ sufficiently close to 1, as it converges to $F_t(S_1)$ as $r \to 0$. Since the segment $E_{r,t}(\{p\} \times [-r, r])$ belongs to a perpendicular of $\gamma_t$ that varies smoothly on the data, we define $\vec{n}$ as the orthogonal vector to this line and $E_{r,t}(\gamma \times \{r\})$, taken so that the third coordinate of $\vec{n}$ is positive. This makes $\vec{n}$ well-defined for $r$ sufficiently close to 1.
For $\Omega(r), \Omega^*(r)$, $B_{r,t}(p)$ is clearly defined as the shape operator in $E_{r,t}$ coordinates at points where $E_{r,t}$ is an immersion. This is not well-defined at points of $\Omega(r), \Omega^*(r)$ where the curvatures at infinity are $\pm 1$, since by Theorem 3.3 the metric

$$I(X, Y) = \frac{1}{4} I^*(X + B^* X, Y + B^* Y)$$

will vanish precisely at directions $X$ (at infinity) whenever $B^* X = -X$. In particular $X$ is an eigenvalue of $B^*$, which remains true if we rescale the metric by a constant factor. Rescale then the conformal metric by a factor $e^\varepsilon$, so now the map $E_{r,t}$ becomes an immersion and by Theorem 3.3 we have that $B^* X = -e^{-\varepsilon} X$, $BX = \frac{1+e^{-\varepsilon}}{1-e^{-\varepsilon}} X$. This implies

$$I(BX, BX) = \left( \frac{1 + e^{-\varepsilon}}{1 - e^{-\varepsilon}} \right)^2 I(X, X) = \frac{1}{4} I^* ((1 - e^{-\varepsilon})X, (1 - e^{-\varepsilon})X)$$

$$= \frac{(1 + e^{-\varepsilon})^2}{4} I^* (X, X).$$

Sending $\varepsilon \to 0$ we see that we can extend $B_{r,t} X$ as a vector of norm 1 orthogonal to $\vec{n}$ for $|X|^2 = 1$.

For the region $A(r)$ we can define $B_{r,t}$ by observing that the map $E_{r,t}$ is the composition of a smooth map into the horizontal lines described in step (C2). The union of these lines are immersed for $r$ sufficiently close to 1, and hence have a well-defined shape operator. Hence we define $B_{r,t}$ as the pullback of such shape operator by $E_{r,t}$.

It is clear from the definitions that $\vec{n}$ and $B_{r,t}$ are piecewise $C^{3,\alpha}, C^{2,\alpha}$ respectively.\[\square\]

**Remark 5.3.** While $E_{r,t}$ may fail in general to be a piecewise immersion, it is an immersion while restricted to the edge locus $\partial A(r)$. Moreover, from the definition of the normal vector $\vec{n}$ we have that the dihedral angles are well-defined and vary $C^{2,\alpha}$ along $t$ and the base point. When appropriate, we will simplify notation by dropping $r, t$ sub-indices.

The following proposition generalizes the key formula to prove the differential Schläfli formula (see [30, Proposition 5]). Let $\left. \frac{\partial}{\partial t} \right|_{t=0} E_{r,t} = \xi$ be the piecewisely defined vector field by the first order variation on $t$, and let $\nabla$ denote the Levi–Civita connection of $\mathbb{H}^3$.

**Proposition 5.4.** For any $p \in \hat{C} \setminus \gamma_t$ and $u, v \in \mathbb{R}^2$ we have

$$\langle \nabla_\xi (Bu), DE_p v \rangle = -\langle \nabla_{DE_p v} \nabla_\xi \vec{n}, DE_p u \rangle + \langle R(\xi, DE_p u) \vec{n}, DE_p v \rangle \tag{5.1}$$

where we follow the convention $R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X, Y]} Z$.

**Proof.** Let us verify first that we have the equality

$$\langle B_{r,t}(u), DE_{r,t} v \rangle = -\langle \nabla_{DE_{r,t} u} \vec{n}, DE_{r,t} v \rangle$$

where $E_{r,t}$ is an immersion. This follows from the relation between the shape operator and the second fundamental form. In directions where $DE_{r,t}$ fails to be injective both sides vanish. Taking then derivative in $t$ we have

$$\langle \nabla_\xi Bu, DE v \rangle + \langle Bu, \nabla_\xi v \rangle = -\langle \nabla_\xi \nabla_{DE_{r,t} u} \vec{n}, DE_{r,t} v \rangle - \langle \nabla_{DE_{r,t} u} \nabla_\xi \vec{n}, DE_{r,t} v \rangle$$

$$= -\langle \nabla_{DE_{r,t} u} \nabla_\xi \vec{n}, DE_{r,t} v \rangle + \langle R(\xi, DE_{r,t} u) \vec{n}, DE_{r,t} v \rangle - \langle \nabla_{DE_{r,t} u} \nabla_\xi \vec{n}, DE_{r,t} v \rangle \tag{5.2}$$

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If $E$ is an immersion we have that $\langle Bu, \nabla \xi v \rangle = -\langle \nabla_{DE,\xi u} \tilde{n}, \nabla_{E} DE_{r,t} v \rangle$. Since we can extend the equality by continuity, we have then

$$\langle \nabla_{\xi} Bu, DE v \rangle = -\langle \nabla_{DEodel} \nabla_{\xi} \tilde{n}, DE v \rangle + \langle R(\xi, DE_{p} u) \tilde{n}, DE_{p} v \rangle$$

as claimed. \(\square\)

**Remark 5.5.** At points where $E$ is an immersion, we can write $\langle \nabla_{\xi} Bu, DE v \rangle$ as

$$\langle \nabla_{\xi} Bu, DE v \rangle = \langle B'(DE u), DE v \rangle + \langle \nabla_{Bu} \xi, DE v \rangle$$

which is the formula appearing in [30, Proposition 5], where $B'$ is the derivative of the shape operator in the immersed surface image.

The next step involves tracing the formula (5.1), namely $\langle \nabla_{\xi}(B\cdot), DE_{p} \rangle$, with respect to the metric in the image of $E_{r,t}$ and multiply it by its area form (both induced from $\mathbb{H}^{3}$). Note that the trace of $\langle R(\xi, DE_{p}) \tilde{n}, DE_{p} \rangle$ is $-2\langle \xi, \tilde{n} \rangle E^{*} da$, which is a multiple of the 2-form that appears in the variational formula for the volume enclosed by the maps $E_{r,t}$.

The remaining terms in (5.1) lead to the Schläfli formula we are interested in. Hence our next concern is how to perform this trace when $E_{r,t}$ is not an immersion. Let us address first $E_{r,t}$ in $\Omega(r), \Omega^{*}(r)$.

For $U \subseteq C, \varphi \in C^{\infty}(U)$ denote by $E_{\varphi}, E_{\varphi}$ the Epstein map and Epstein Gauss map for $e^{\varphi}|dz|^{2}$, respectively. We say that a 2-tensor $T : C^{\infty}(U) \rightarrow \Lambda^{2,0}(U)$ is compatible if $T$ is a differentiable map so that for any $\varphi \in C^{\infty}(U)$, $x \in U, v \in \mathbb{R}^{2}$ so that $\langle (DE_{\varphi}) x', (DE_{\varphi}) x v \rangle \equiv 0$, we have that $T(\varphi) x,(v) \equiv 0$. Examples of compatible maps are each summand in formula (5.1).

For any compatible $T$ we define $\text{Tr}(T) \in \Omega^{2}(U)$ as

$$\text{Tr}(T)(x) := \lim_{\varepsilon \to 0} E_{\varphi+\varepsilon}^{*} \left( \text{tr}((E_{\varphi+\varepsilon}^{-1})^{*} T(\varphi + \varepsilon)) da_{\varepsilon} \right), \quad (5.3)$$

where $tr, da_{\varphi+\varepsilon}$ are respectively the trace and area form on the orthogonal complement of $E_{\varphi+\varepsilon}$. To see that this limit is well defined, observe that for $\varepsilon \neq 0$ sufficiently small we have that $E_{\varphi+\varepsilon}$ is an immersion at $x \in U$. In particular, the limit agrees with $E_{\varphi}^{*}(\text{tr}((E_{\varphi}^{-1})^{*} T) da)$ if $E_{\varphi}$ is an immersion at $x \in U$. As an application of Theorem 3.3 we can take orthonormal $u, v \in \mathbb{R}^{2}$ so that $u_{\varepsilon} := (DE_{\varphi+\varepsilon}) x u, v_{\varepsilon} := (DE_{\varphi+\varepsilon}) x v \in T^{1}\mathbb{H}^{3}$ are orthogonal for all $\varepsilon$. Hence

$$E_{\varphi+\varepsilon}^{*} \left( \text{tr}((E_{\varphi+\varepsilon}^{-1})^{*} T(\varphi + \varepsilon)) da_{\varepsilon} \right) = \left( \frac{1}{|u_{\varepsilon}|^{2}} T(\varphi + \varepsilon)(u,u) + \frac{1}{|v_{\varepsilon}|^{2}} T(\varphi + \varepsilon)(v,v) \right) |u_{\varepsilon}|.|v_{\varepsilon}| \, dx \, dy$$

and the limit (5.3) exists even if either or both $|u_{\varepsilon}|, |v_{\varepsilon}|$ go to 0 linearly with $\varepsilon$, since in that case we have that the respective $T(\varphi)(u,u), T(\varphi)(v,v)$ vanishes and the corresponding $\frac{1}{|u_{\varepsilon}|^{2}} T(\varphi + \varepsilon)(u,u), \frac{1}{|v_{\varepsilon}|^{2}} T(\varphi + \varepsilon)(v,v)$ converges to a derivative of $T$.

For the terms in (5.1) we can make this computation explicit for $-\langle \nabla_{DE_{p} u} \nabla_{\xi} \tilde{n}, DE_{p} u \rangle$ and $\langle R(\xi, DE_{p} u) \tilde{n}, DE_{p} u \rangle$. Observe that at points where $E$ is an immersion we have that $\text{Tr}(\langle \nabla_{DE_{p} u} \nabla_{\xi} \tilde{n}, DE_{p} u \rangle)$ is equal to $E^{*}(\text{div}(\nabla_{\xi} \tilde{n}) da) = E^{*}(d(\xi, \nabla_{\xi} \tilde{n})) = -d(\xi \nabla_{\xi} \tilde{n})$. 27
where \( i\nabla_{\xi}\bar{n} \) is the 1-form defined by \( u \mapsto \langle DE_p u, \nabla \xi \bar{n} \rangle \). Hence for all points we get
\[
\text{Tr}(-\langle \nabla DE_p v \nabla \xi \bar{n}, DE_p u \rangle) = -d(i\nabla_{\xi}\bar{n}).
\]
For \( \langle R(\xi, DE_p u)\bar{n}, DE_p v \rangle = -\langle \xi, \bar{n} \rangle \langle DE_p u, DE_p v \rangle \), we can see that this symmetric tensor is the pullback of a symmetric tensor in \( \bar{n}^\perp \). Then \( \text{Tr}((R(\xi, DE_p u)\bar{n}, DE_p v) = -2\langle \xi, \bar{n} \rangle E^* \text{d}a \), which vanishes if \( E \) fails to be an immersion.

**Lemma 5.6** (See [30, Eq. (3.1) and Prop. 5]). At points where \( E \) is an immersion, the form \( \frac{1}{2} \text{Tr}(\langle \nabla \xi(B^\cdot), DE_p v \rangle) \) agrees with the pullback by \( E \) of the form \( (\delta H + \frac{1}{4}\langle \delta I, \mathcal{I} \rangle) \text{d}a \).

We now relate the variation of the volume \( V_2 \) with \( \text{Tr}(\langle \nabla \xi(B^\cdot), DE_p v \rangle) \), see (5.5). Following (5.3) and (5.1) we obtain
\[
\text{Tr}(\langle \nabla \xi(B^\cdot), DE_p v \rangle) = -d(i\nabla_{\xi}\bar{n}) - 2\langle \xi, \bar{n} \rangle E^* \text{d}a.
\]

(5.4)

For \( E_{r,t} \) in \( A(r) \), we can establish and trace (5.1) in the embedded surface that contains the image of \( E_{r,t} \) (for \( r \) sufficiently close to 1) and then take the pullback by \( E_{r,t} \).

Let \( V_2(r, t) \) be defined as the volume bounded by \( E_{r,t} \). Namely, extend \( E_{r,t} : S^2 = \hat{C} \rightarrow \mathbb{H}^3 \) as a map from the closed ball \( B^3 \) and define
\[
V_2(r, t) := \int_{B^3} E_{r,t}^*(\text{vol}_{B^3}).
\]

By Stokes, this definition does not depend on the specific extension of \( E_{r,t} \) to \( B^3 \).

Since \( E_{r,t} \) vary \( C^{3,0} \), as piecewisely defined map from \( \Omega(r), \Omega^*(r), A(r) \), we can take the extension to vary \( C^{3,0} \) on \( t \) and check that \( \partial_t V_2(r, t) \) is given by
\[
\partial_t V_2 = \left( \int_{\Omega(r)} + \int_{\Omega^*(r)} + \int_{A(r)} + \frac{1}{2} \text{Tr}(\langle \nabla \xi(B^\cdot), DE_p v \rangle) + \frac{1}{2} d(i\nabla_{\xi}\bar{n}) \right)
\]

where \( \xi = \partial_t E_{r,0} \), \( \bar{n} \) is the normal vector described in Lemma 5.2 and \( \text{d}a \) is the area form of the orthogonal plane to \( \bar{n} \). The negative sign is due to the fact that we are taking normal vector \( \bar{n} \) pointing inward the region bounded by \( E_{r,t} \).

Applying (5.4) we have then
\[
\partial_t V_2 = \left( \int_{\Omega(r)} + \int_{\Omega^*(r)} + \int_{A(r)} + \frac{1}{2} \text{Tr}(\langle \nabla \xi(B^\cdot), DE_p v \rangle) + \frac{1}{2} d(i\nabla_{\xi}\bar{n}) \right).
\]

Applying Stokes theorem for \( \frac{1}{2} d(i\nabla_{\xi}\bar{n}) \) yields the integral of \( \frac{1}{2} i\nabla_{\xi}\bar{n} \) over each boundary component. Since \( E^\cdot \) is embedded along \( \partial A(r) \), then as in [30] we have that along \( \partial A(r) \), we have
\[
\begin{align*}
i\nabla_{\xi}(\nu_{\Omega(r)}) + i\nabla_{\xi}(\nu_{A(r)}) &= \frac{\partial \theta^+}{\partial t} E^* \text{d}\ell \\
i\nabla_{\xi}(\nu_{\Omega^*(r)}) + i\nabla_{\xi}(\nu_{A(r)}) &= \frac{\partial \theta^-}{\partial t} E^* \text{d}\ell
\end{align*}
\]

where \( \theta^+(x) \) (respectively \( \theta^-(x) \)) is the exterior dihedral angle of the planes orthogonal to \( \nu_{\Omega(r)}, \nu_{A(r)} \) at \( E(x) \) (respectively \( \nu_{\Omega^*(r)}, \nu_{A(r)} \) at \( E(x) \)), and \( \text{d}\ell \) is the length form in \( \mathbb{H}^3 \).
Applying then Stokes for $\partial_t V_2$ we get

$$
\partial_t V_2 = \left( \int_{\Omega(r)} + \int_{\Omega^r(r)} + \int_{A(r)} \frac{1}{2} \mathrm{Tr}(\langle \nabla \xi(B\cdot), DE_p\rangle) \right) + \frac{1}{2} \left( \int_{\partial\Omega(r)} \frac{\partial \theta^+}{\partial t} E^* d\ell + \int_{\partial\Omega^r(r)} \frac{\partial \theta^-}{\partial t} E^* d\ell \right). \tag{5.5}
$$

**Proof of Theorem 5.1.** Following Lemma 5.6 and Equation (5.5), we only need to prove that

$$
\lim_{r \to 0} \int_{A(r)} \frac{1}{2} \mathrm{Tr}(\langle \nabla \xi(B\cdot), DE_p\rangle) = 0 \quad \text{and}
$$

$$
\lim_{r \to 0} \frac{1}{2} \left( \int_{\partial\Omega(r)} \frac{\partial \theta^+}{\partial t} E^* d\ell + \int_{\partial\Omega^r(r)} \frac{\partial \theta^-}{\partial t} E^* d\ell \right) = 0.
$$

For the first term, observe that $A(r)$ belongs to the surface described in (C2). These families of surfaces can be described by

$$(r, s) \rightarrow (x(r, s, t), y(r, s, t), z(r, s, t)),
$$

where $r, s$ parametrize the surface as in (C2) for $\gamma_t$. This parametrization extends smoothly for $r = 1$ towards the boundary of $H^3$ by making $z(1, s, t) \equiv 0$. Moreover, given (C1) and Lemma 3.12 we have that $z(r, s, t) = O((1 - r))$.

Hence the first and second fundamental form (as well as their first order variations) and the inverse of the first fundamental form are of order at most $(1 - r)^{-2}$, from which the terms $H, \delta H, \langle \delta I, \Pi \rangle$ are uniformly bounded. As the area of $A(r)$ decays at least of the order of $1 - r$, we have that

$$
\lim_{r \to 1^-} \int_{A(r)} \frac{1}{2} \mathrm{Tr}(\langle \nabla \xi(B\cdot), DE_p\rangle) = 0.
$$

Likewise, the functions $\theta^\pm$ that take each $(r, s, t)$ to the angle between $\Omega(r), \Omega^r(r)$ and $A(r)$ at $(x(r, s, t), y(r, s, t), z(r, s, t))$, extend smoothly to $r = 1$ as right angles. Hence in particular $\frac{\partial \theta^\pm}{\partial t} = O((1 - r))$. This is not enough for the desired limit, as the curves $\partial\Omega(r), \partial\Omega^r(r)$ have length comparable to $(1 - r)^{-1}$. What we can rather do is use again that the Epstein–Poincaré surfaces agree up to second order to use parametrizations $\gamma^\pm_\varepsilon(s)$ satisfying $\varepsilon = (1 - r))$

$$
\left\| \frac{d\gamma^+_\varepsilon(s)}{ds} - \frac{d\gamma^-_\varepsilon(s)}{ds} \right\| \leq C\varepsilon^2 \tag{5.6}
$$

$$
|\theta'(\gamma^+_\varepsilon(s))| + |\theta'(\gamma^-_\varepsilon(s))| \leq C\varepsilon^2
$$

for some uniform constant $C > 0$. Then since the last coordinate of $\gamma^\pm_\varepsilon$ is $O(\varepsilon)$, we have that for some uniform constant $C > 0$

$$
\left| \int_{\partial\Omega(r)} \frac{\partial \theta^+}{\partial t} E^* d\ell + \int_{\partial\Omega^r(r)} \frac{\partial \theta^-}{\partial t} E^* d\ell \right|
$$

$$
\leq C \int_{S^1} \left| \frac{\theta'(\gamma^+_\varepsilon(s))}{\varepsilon} \right| \left\| \frac{d\gamma^+_\varepsilon}{ds} \right\| + \left| \frac{\theta'(\gamma^-_\varepsilon(s))}{\varepsilon} \right| \left\| \frac{d\gamma^-_\varepsilon}{ds} \right\| ds \tag{5.7}
$$

$$
\leq \frac{1}{\varepsilon} \int_{S^1} \left| \theta'(\gamma^+_\varepsilon(s)) \right| \left\| \frac{d\gamma^+_\varepsilon}{ds}(s) - \frac{d\gamma^-_\varepsilon}{ds}(s) \right\| + \left| \theta'(\gamma^+_\varepsilon(s)) + \theta'(\gamma^-_\varepsilon(s)) \right| \left\| \frac{d\gamma^-_\varepsilon}{ds} \right\| ds
$$

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goes to 0 as $\varepsilon \to 0$ uniformly in $t$.

We define then $V_2(r, t)$ using the parameters of Proposition 4.6, so that $V(\gamma_t) = V_1(r, t) + V_2(r, t)$ for any $r$ sufficiently small. For the parametrized region in $V_2(r, t)$ we can see that the $t$ derivatives of the functions $f, g$ in the proof of Proposition 4.6 agree as well up to order 2 (in $z$ variable), so by the same argument we have that $\lim_{r \to 0} \partial_t V_1(r, 0) = 0$.

As for any $r$ small we have that $\partial_t V(\gamma_t) = \partial_t V_1(r, t) + \partial_t V_2(r, t)$, we send $r$ to 0 on the right hand side to obtain

$$\frac{\partial}{\partial t} V(\gamma_t) = \int_{\Omega} + \int_{\Omega^*} \frac{1}{2} \text{Tr}(\langle \nabla \xi(B), DEp \rangle)$$

which completes the proof by Lemma 5.6.

### 5.2 Variation of mean curvature and Schlafli formula

The following result is proved by Krasnov–Schlenker, see [17, Cor. 6.2], for the renormalized volume of convex co-compact manifolds. We adapt it to the renormalized volume associated with a smooth Jordan curve.

**Theorem 5.7.** We have the first order variation of the $V_R$

$$\delta V_R(\gamma) = -\frac{1}{4} \int_{\Omega, \Omega^*} \delta H^* + \frac{1}{2} \langle \delta I^*, I I^* \rangle \, da^*$$

where $I I^* = \partial \, dz^2 + \bar{\partial} \, d\bar{z}^2$ is the traceless part of $I^*$, $\langle A, B \rangle$ stands for $\text{tr}[A^{-1}B^{-1}]$.

**Proof.** By Definition 4.7 and Remark 4.8, we can express $\delta V_R$ as the integral of smooth 2-forms in $\Omega, \Omega^*$, so that at points where the respective Epstein–Poincaré maps are immersions these forms are given by the pullback of the form

$$(\delta H + \frac{1}{4} \langle \delta I, I \rangle) \, da - \frac{1}{2} (\delta H da - H \delta(da))$$

by the respective Epstein–Poincaré map. Following [17, Section 6] this pullback is expressed precisely as $-\frac{1}{4} \langle \delta H^* + \frac{1}{2} \langle \delta I^*, I I^* \rangle \rangle \, da^*$. As points where the Epstein–Poincaré maps are immersions are dense in $\Omega, \Omega^*$ and all forms discussed are continuous, the result follows.

More explicitly, we can write the variation of $V_R$ in terms of the Beltrami differentials. We consider a $C^{5,\alpha}$ family of Jordan curves $(\gamma_t)$ as in the previous section and let $F_t$ be the corresponding homeomorphism of $\hat{\mathbb{C}}$ which maps $\Omega_0$ conformally onto $\Omega_t$ and a diffeomorphism from $\Omega_0^*$ to $\Omega_t^*$. For $z \notin \gamma_0$, let

$$\mu_t := \frac{\partial_{\bar{z}} F_t}{\partial z F_t} = t \nu + O(t^2).$$

We have in particular, $\hat{F} := \frac{d}{dt} F_t |_{t=0}$ satisfies

$$\partial_{\bar{z}} \hat{F} = \nu, \quad F_t(z) = z + t \hat{F}(z) + O(t^2).$$

Since $F_t$ is conformal in $\Omega_0$, $\hat{\nu}|_{\Omega_0} \equiv 0$. 

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Lemma 5.8. We have $\|\dot{\nu}\|_\infty < \infty$. Moreover, $\dot{\nu}|_{\Omega^*} \in H^{-1,1}(\Omega^*) \oplus \mathfrak{H}(\Omega^*)$.

Proof. On $\Omega$ we have that the 1-parameter family $F_t$ is conformal, while in $\overline{\Omega}^*$ we can write $F_t$ as the composition $g_t \circ H_t \circ g_0$, where $H_t$ is a $C^{3,\alpha}$ family of maps that agree with $h_t^{-1}$ near $\partial \mathbb{D}$. As $g_t, H_t, g_0$ extend to the boundary of their respective domains and are $C^{3,\alpha}$ as a family of maps, the $L^\infty$ bound of $\dot{\nu}$ follows from the compactness of the domains.

For the second claim, as $(\gamma_t)$ corresponds to a differentiable path in $T_0(1)$, the projection of $\dot{\nu}$ onto harmonic Beltrami differentials $\Omega^{-1,1}(\Omega^*)$ parallel to $\mathfrak{H}(\Omega^*)$ lies in $H^{-1,1}(\Omega^*)$. This completes the proof. \hfill \square

Corollary 5.9. The first variation of the renormalized volume associated with the family of deformed Jordan curves $(\gamma_t := F_t(\gamma_0))$ is given by

$$\delta V_R(\gamma) = - \operatorname{Re} \int_{\Omega^*} \dot{\nu} \mathcal{S}[g^{-1}] d^2z,$$

where we recall $g : \mathbb{D}^* \to \Omega^*$ is any conformal map.

Proof. As $\dot{\nu} \in L^\infty(\Omega^*)$ and $\mathcal{S}[g^{-1}]$ is continuous functions up to the boundary. The integrals in (5.8) are absolutely convergent. We only need to check the pointwise identity

$$\left(\frac{1}{4} \delta H^* + \frac{1}{8} (\delta \mathcal{I}^*, \mathbb{I}_0^*)\right) da^* = \dot{\nu} \mathcal{S}[g^{-1}] d^2z$$

on $\Omega^*$. We have

$$dF_t(z) = dz + t\partial_\nu \bar{F} \, dz + t\partial_z \bar{F} \, d\bar{z} + O(t^2) = dz + t\partial_z \bar{F} \, dz + t\nu \, d\bar{z} + O(t^2)$$

and in the $dz, d\bar{z}$ coordinates

$$dF_t(z)d\overline{F_t(z)} = \left(\frac{1}{2} (1 + 2t \operatorname{Re}(\partial_z \bar{F})) \frac{t\nu}{t\nu} \right) + O(t^2).$$

Therefore, the hyperbolic metric in $\Omega_t$ is

$$e^\varphi (1 + 2ts + O(t^2)) dF_t(z)d\overline{F_t(z)} = I^* + te^\varphi \left(\frac{\dot{\nu}}{\operatorname{Re}(\partial_z \bar{F}) + s} \begin{pmatrix} \dot{\nu} & \operatorname{Re}(\partial_z \bar{F}) + s \end{pmatrix} \begin{pmatrix} \dot{\nu} \\ \operatorname{Re}(\partial_z \bar{F}) + s \end{pmatrix} \right) + O(t^2).$$

where $s$ is some smooth function on $\Omega$ and

$$I^* = e^\varphi dz d\bar{z} = \frac{1}{2} \begin{pmatrix} 0 & e^\varphi \\ e^\varphi & 0 \end{pmatrix}. $$

We obtain

$$\delta I^* = e^\varphi \left(\begin{pmatrix} \dot{\nu} \\ \operatorname{Re}(\partial_z \bar{F}) + s \end{pmatrix} \begin{pmatrix} \dot{\nu} \\ \operatorname{Re}(\partial_z \bar{F}) + s \end{pmatrix} \right).$$

Recall that

$$\mathbb{I}_0^* = \begin{pmatrix} \vartheta & 0 \\ 0 & \overline{\vartheta} \end{pmatrix} = \begin{pmatrix} \mathcal{S}[g^{-1}] & 0 \\ 0 & \mathcal{S}[g^{-1}] \end{pmatrix}. $$

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we have (using the complexified inner product \( \langle A, B \rangle = \text{Re} \, \text{Tr} \left[ (I^*)^{-1} A (I^*)^{-1} B \right] \))

\[
\langle \delta I^*, I^*_0 \rangle = 8e^{-\phi} \text{Re} \left( \hat{\nu} \mathcal{J} [g^{-1}] \right).
\]

We obtain the claimed variation formula from Corollary 3.5 which shows \( H^* = -K^* \equiv 1 \) and which implies \( \delta H^* \equiv 0 \).

\[\text{Corollary 5.10.} \quad \text{We have for all } C^{5,\alpha} \text{ Jordan curves } \gamma, \text{ we have}
\]

\[\tilde{S}(\gamma) = 4V_R(\gamma).
\]

\[\text{Proof.} \quad \text{When } \gamma \text{ is a circle, we have } \tilde{S}(\gamma) = 0 \text{ and } V_R(\gamma) = 0 \text{ since both Epstein surfaces are the geodesic plane bounded by } \gamma. \text{ Given a smooth Jordan curve } \gamma. \text{ The variational formula Proposition 2.4 and Corollary 5.9 show that}
\]

\[\tilde{S}(\gamma) = 4V_R(\gamma)
\]

by taking a smooth deformation from \( \gamma \) to a circle.

\[\square
\]

5.3 Approximation of general WP curve

The goal of the section is to prove the following theorem.

\[\text{Theorem 5.11.} \quad \text{We have for any Weil–Petersson quasicircle } \gamma,
\]

\[\tilde{S}(\gamma) \geq 4V_R(\gamma).
\]

\[\text{Remark 5.12.} \quad \text{We have already proved the equality when } \gamma \text{ is } C^{5,\alpha}. \text{ We also believe the equality holds for arbitrary Weil–Petersson quasicircle but are only able to prove the inequality.}
\]

For the inequality, we will use the approximation using equipotential curves. Let \( \gamma \) be a Weil–Petersson quasicircle, \( f : \mathbb{D} \to \Omega \) be a conformal map as before. Up to post-composing \( f \) by a Möbius map, we may assume that \( f(0) = 0 \), \( f'(0) = 1 \) and \( f''(0) = 0 \). The equipotentials

\[\gamma_n = f_n(S^1), \quad \text{where } f_n(z) := \frac{n}{n-1} f \left( \frac{n-1}{n} z \right)
\]

form a family of analytic Jordan curves. The map \( f_n \) satisfies the same normalization as \( f \) at 0. We let \( \Omega^*_n := \hat{\mathbb{C}} \setminus f_n(\mathbb{D}) \) (resp. \( \Omega^* := \hat{\mathbb{C}} \setminus f(\mathbb{D}) \)) and \( g_n \) (resp. \( g \)) be an arbitrary conformal map \( \mathbb{D}^* \to \Omega^*_n \) (resp, \( \mathbb{D}^* \to \Omega^* \)). Apart from the analyticity, the family of equipotentials is nice because of the following theorem.

\[\text{Theorem 5.13 (See [33, Cor. 1.5]).} \quad \text{Along the family of equipotentials the universal Liouville action converges and is non-decreasing. We have}
\]

\[\lim_{n \to \infty} \uparrow \tilde{S}(\gamma_n) = \tilde{S}(\gamma).
\]

\[\text{If } \gamma \text{ is not a circle, then } \tilde{S}(\gamma_{n+1}) > \tilde{S}(\gamma_n).
\]
Lemma 5.14. We have
\[
\int_{\Sigma \Omega} H \, da \xrightarrow{n \to \infty} \int_{\Sigma \Omega^*} H \, da. \tag{5.9}
\]

Proof. It follows from [32, Cor. A.4., Cor. A.6] that the element \([\mu_n]\) in \(T_0(1)\) associated with \(\gamma_n\) converges to \([\mu]\) which is associated with \(\gamma\). In particular, [32, Chap. I, Thm. 2.13, Thm. 3.1] implies that
\[
\int_D \|S(f_n)\|^2 \rho_D \, d^2 z = \int_D |S(f_n)|^2 \rho_D^{-1} \, d^2 z \xrightarrow{n \to \infty} \int_D |S(f)|^2 \rho_D \, d^2 z.
\]
As \(T_0(1)\) is a topological group, we have \([\mu_n]^{-1}\) converges to \([\mu]^{-1}\) which implies
\[
\int_{D^*} \|S(g_n)\|^2 \rho_{D^*} \, d^2 z = \int_{D^*} |S(g_n)|^2 \rho_{D^*}^{-1} \, d^2 z \xrightarrow{n \to \infty} \int_{D^*} |S(g)|^2 \rho_{D^*} \, d^2 z.
\]
Using (3.11) the proof is completed. \(\square\)

Lemma 5.15. Recall that \(V_2(\gamma)(\varepsilon)\) denotes the signed volume between \(E_{p\Omega}\) and \(E_{p\Omega^*}\) above level \(\varepsilon\). We have \(V_2(\gamma_n)(\varepsilon)\) converges to \(V_2(\gamma)(\varepsilon)\) for all \(\varepsilon > 0\).

Proof. For this, we denote for \(\varepsilon > 0\),
\[
K_{\varepsilon,n} := \{\zeta \in \mathbb{D} : \xi_n \circ f_n(\zeta) \geq \varepsilon\}, \quad K_{\varepsilon} := \{\zeta \in \mathbb{D} : \xi \circ f(\zeta) \geq \varepsilon\},
\]
where \((y_n, \xi_n)\) is the Epstein–Poincaré map on the domain \(\Omega_n = f_n(\mathbb{D})\). By (3.17), we have for all \(n\),
\[
\text{dist}(f_n(\zeta), \gamma_n) \leq |\xi_n \circ f_n(\zeta)| \leq 4 \text{dist}(f_n(\zeta), \gamma_n)
\]
which implies for all \(\zeta \in K_{\varepsilon,n}\),
\[
\text{dist}(f_n(\zeta), \gamma_n) \geq \varepsilon/4.
\]
It is not hard to see that \(f_n\) converges uniformly to \(f\) on \(\overline{\mathbb{D}}\) from the explicit expression. However, it holds more generally for any sequence of normalized conformal maps representing converging sequence in \(T_0(1)\). In fact, we extend \(f_n\) to a \(K\)-quasiconformal map of \(\hat{\mathbb{C}}\), where \(K\) is independent of \(n\) since a converging sequence in \(T_0(1)\) is also bounded in \(T(1)\). The family of \(K\)-quasiconformal maps, normalized as \(f_n\), is a normal family and converges uniformly along subsequences on all compact sets of \(\hat{\mathbb{C}}\). As the limit on \(\overline{\mathbb{D}}\) is \(f\), the convergence is thus along the whole sequence when restricted to \(\overline{\mathbb{D}}\). Moreover, the derivatives of \(f_n\) converges to the derivatives of \(f\) uniformly on compact sets of \(\mathbb{D}\) by Cauchy’s integral formula.

Hence, there exists \(n_0\) such that for all \(n \geq n_0\), we have
\[
\|f_n - f\|_{\infty, \overline{\mathbb{D}}} < \varepsilon/16.
\]
This implies
\[
\text{dist}(f(\zeta), \gamma) \geq \varepsilon/8 \quad \text{and} \quad \xi \circ f(\zeta) \geq \varepsilon/40.
\]
Summarizing, we have for all \( n \geq n_0 \),

\[
K_{\varepsilon,n} \subset K_{\varepsilon/40}.
\]

Since \( K_{\varepsilon/40} \) is a compact set in \( \mathbb{D} \) independent of \( n \), we have that all derivatives of \( f_n \) converge uniformly to the derivatives of \( f \) on \( K_{\varepsilon/40} \). As the Epstein–Poincaré map only depends on \( f, f', \) and \( f'' \), \( \text{Ep}_{\Omega_n} \circ f_n \) converges uniformly to \( \text{Ep}_{\Omega} \circ f \) uniformly on \( K_{\varepsilon/40} \). Similarly argument applies to the Epstein–Poincaré maps \( \text{Ep} \Omega_n^{*} \circ g_n \). We obtain that \( V_2(\gamma_n)(\varepsilon) \) converges to \( V_2(\gamma)(\varepsilon) \).

We obtain the following corollary.

**Corollary 5.16.** If \( \gamma \) is a Weil–Petersson curve, then

\[
V(\gamma) \leq \frac{1}{4} \tilde{S}(\gamma) + \frac{1}{2} \int_{\Sigma_{\Omega} \cup \Sigma_{\Omega}^{*}} H da < \infty.
\]

**Proof.** For small enough \( \varepsilon > 0 \),

\[
V_2(\gamma)(\varepsilon) = \lim_{n \to \infty} V_2(\gamma_n)(\varepsilon)
\]

\[
\leq \lim_{n \to \infty} \frac{1}{4} \tilde{S}(\gamma_n) + \frac{1}{2} \int_{\Sigma_{\Omega_n} \cup \Sigma_{\Omega_n}^{*}} H da = \frac{1}{4} \tilde{S}(\gamma) + \frac{1}{2} \int_{\Sigma_{\Omega} \cup \Sigma_{\Omega}^{*}} H da
\]

by Theorem 5.13. We obtained the inequality by taking \( \varepsilon \to 0 \).

Theorem 5.11 follows immediately from this corollary.

6 Gradient flow of the universal Liouville action

Following Bridgeman–Brock–Bromberg [5] and Bridgeman–Bromberg–Vargas-Pallete [6], we introduce the following flow on \( T(1) \). For \( [\mu] \in T(1) \), we have a natural isomorphism \( T_{[\mu]} T(1) \simeq \Omega^{-1,1}(\mathbb{D}^*) \). We therefore define the vector field

\[
V_{[\mu]} := -4 \frac{\mathcal{F}(g_\mu)}{\rho_{\mathbb{D}^*}} \in \Omega^{-1,1}(\mathbb{D}^*).
\]

**Theorem 6.1.** The vector field \( V \) has flowlines that exist for all time on \( T(1) \). The flow restricts to a flow on \( T_0(1) \) and is the (negative) Weil–Petersson gradient of the Liouville functional \( S \). Furthermore all flowlines on \( T_0(1) \) converges to the origin \( [0] \) which corresponds to the round circle.

**Proof.** By the Nehari bound we have that in the Teichmüller metric on \( T(1) \), \( ||V||_\infty \leq 6 \).

Thus as \( T(1) \) is complete in the Teichmüller metric, the flow under \( V \) exists for all time on \( T(1) \). If \( [\mu] \in T_0(1) \) then by the characterization (2.4) we have

\[
\int_{\mathbb{D}^*} |\mathcal{F}(g_\mu)|^2 \rho_{\mathbb{D}^*}^{-1} < \infty.
\]
Thus \( V_{[\mu]} \in H^{-1,1}(\mathbb{D}^*) \simeq T_{[\mu]}T_0(1) \) and therefore by integrability the flow preserves \( T_0(1) \).

Furthermore if \( \dot{\nu} \in H^{-1,1}(\mathbb{D}^*) \simeq T_{[\mu]}T_0(1) \) then by Theorem 2.1

\[
(dS)_{[\mu]}(\dot{\nu}) = 4\Re \int_{\mathbb{D}^*} \dot{\nu} \mathcal{S}(g_{[\mu]}) = -\Re \int_{\mathbb{D}^*} \dot{\nu} V_{[\mu]} \rho_{\mathbb{D}^*} = -\langle V_{[\mu]}, \dot{\nu} \rangle_{\text{WP}}.
\]

Therefore \( \nabla_{\text{WP}} S = -V \) and

\[
dS(V) = -||V||^2_{\text{WP}}.
\]

We consider the flowline \( \mathbb{R}_+ \to T_0(1) : t \mapsto \alpha(t) \) for \( V \) starting at a point \([\mu] = \alpha(0) \in T_0(1)\). Since \( S \geq 0 \), for all \( T > 0 \),

\[
0 \leq \int_0^T ||V(\alpha(t))||^2 \, dt = S([\mu]) - S(\alpha(T)) \leq S([\mu]).
\]

Thus

\[
\int_0^\infty ||V(\alpha(t))||^2_{\text{WP}} \, dt < \infty.
\]

We therefore have a sequence \( t_n \to \infty \) such that

\[
\lim_{n \to \infty} ||V(\alpha(t_n))||_{\text{WP}} = 0.
\]

By [32, Ch. I, Lem. 2.1], we have for all \( \phi \in A_\infty(\mathbb{D}) \),

\[
||\phi|| \leq \sqrt{\frac{3}{4\pi}} ||\phi||_2.
\]

Therefore

\[
\lim_{n \to \infty} ||V(\alpha(t_n))||_{\infty} = 0.
\]

Thus the conformal maps \( g_{\alpha(t_n)} \) have Schwarzian \( \mathcal{S}(g_{\alpha(t_n)}) \to 0 \). By normalcy, we obtain a subsequence \( g_{\alpha(t_{n_i})} \) converging uniformly on compact sets to a Möbius map preserving \(-1,1,i\). Therefore \( \lim_{i \to \infty} \alpha(t_{n_i}) = [0] \), the origin of \( T_0(1) \).

To show that the flow line converges to \([0]\), we observe that \([0]\) is the unique global minimum for \( S \) on \( T_0(1) \). Therefore there is a neighborhood of \([0]\) \in \( T_0(1) \) which is an attractor. By the above, \( \alpha \) enters this neighborhood and therefore it converges to \([0]\). \( \square \)

Using the gradient flow we may bound the Weil–Petersson distance between \([\mu]\) and \([0]\) by the universal Liouville action. We first recall some results proved by Takhtajan and Teo that we summarize in the lemma below.

**Lemma 6.2** ([32, Ch.I, Lem. 2.5, Rem. 2.4, Cor. 2.6]). There exists \( 0 < \delta < 1 \) such that for all \( \mu \in \Omega^{-1,1}(\mathbb{D}^*) \) with \( ||\mu||_\infty < \delta \),

\[
\left| \frac{|\partial_z w_\mu(z)|^2}{(1 - |w_\mu(z)|^2)^2} - \frac{1}{(1 - |z|^2)^2} \right| < \frac{1}{(1 - |z|^2)^2}.
\]

Moreover, for such \( \mu \), the map \( D_0(\beta \circ R_{[\mu]}): H^{-1,1}(\mathbb{D}^*) \to A_2(\mathbb{D}) \) is a bounded linear isomorphism with

\[
||D_0(\beta \circ R_{[\mu]})(\nu)||_2 \leq 24||\nu||_2, \quad ||\nu||_2 \leq K||D_0(\beta \circ R_{[\mu]})(\nu)||_2
\]

where \( K = \sqrt{2}/(1 - \delta)^2 \).
Theorem 6.3. With the same constants $\delta$ and $K$ as in Lemma 6.2. Let $c < 2\delta \sqrt{4\pi/3}$ then for $[\mu] \in T_0(1)$, we have

$$c(\text{dist}_{WP}([\mu], [0]) - Kc) \leq S([\mu]).$$

Proof. We let $t \mapsto \alpha(t)$ be the gradient flow line starting at $[\mu]$ and $\tau$ be the first time $\|V(\alpha(t))\|_{WP} = c$. Then $\|V(\alpha(t))\|_{WP} > c$ for all $t < \tau$. Thus

$$S([\mu]) - S(\alpha(\tau)) = \int_0^\tau \|V(\alpha(t))\|^2_{WP} \, dt \geq c \int_0^\tau \|V(\alpha(t))\|_{WP} \, dt \geq c \text{dist}_{WP}([\mu], \alpha(\tau)).$$

We have therefore

$$S([\mu]) \geq c(\text{dist}_{WP}([\mu], [0]) - \text{dist}_{WP}(\alpha(\tau), [0])).$$

As $\|V(\alpha(\tau))\|_{WP} = c$ then by (6.1), $\|V(\alpha(\tau))\|_{\infty} \leq \sqrt{3/4\pi}c < 2\delta$. Therefore

$$\|\hat{\beta}(\alpha(\tau))\|_{\infty} = \|\mathcal{J}(g_{\alpha(\tau)})\|_{\infty} < \delta/2 < 1/2$$

where $\hat{\beta}$ is the Bers embedding $T(1) \to A_{\infty}(\mathbb{D}^*)$. As $\hat{\beta}(T_0(1)) = \hat{\beta}(T(1)) \cap A_2(\mathbb{D}^*)$ the linear path

$$\gamma(s) := [s\hat{\mu}], \text{ where } \hat{\mu} = -\frac{2}{z^4} \mathcal{J}(g_{\alpha(\tau)}) \left(\frac{1}{z}\right) \text{ satisfies } \|\hat{\mu}\|_{\mathbb{D},\infty} < \delta$$

for $s \in [0, 1]$ from 0 to $\alpha(\tau)$ is in the ball of radius $\delta$ of $T(1)$, and also in $T_0(1)$ since by Ahlfors-Weil theorem

$$\hat{\beta}([s\hat{\mu}]) = s\mathcal{J}(g_{\alpha(\tau)}) \in A_2(\mathbb{D}^*).$$

In the $L_2$ metric on $A_2(\mathbb{D}^*)$ this path has length $\|V(\alpha(\tau))\|_{WP} \leq c$. By Lemma 6.2 we have that the preimage of this path by $\hat{\beta}$ has therefore length less than $Kc$. \hfill \Box

7 Comparisons to minimal surfaces and convex core

The following Lemma shows that even at degenerate point the Epstein–Poincaré map behaves like a parametrized surface, which follows mainly from its analytic expansion. Lemma 7.1 will be used in Proposition 7.5, which together with Proposition 7.2 and Proposition 7.4 answer a question of Bishop [2] about how minimal surfaces and convex cores relate to Epstein–Poincaré maps.

Lemma 7.1. For any totally geodesic plane $P \subset \mathbb{H}^3$ containing $\overline{E_{P}\zeta}$, there exist points arbitrarily close to $\zeta$ so that their image by the map $E_{P\zeta}$ lie in each component of $\mathbb{H}^3 \setminus P$.

Proof. Let $f : \mathbb{D} \to \Omega$ be a conformal map, and without loss of generality let us assume that $\zeta = 0$ and $f(0) = 0, f'(0) = 1, f''(0) = 0$ (after composing with a Möbius transformation if needed). Observe that by Lemma 3.12 this is equivalent to composing by an isometry of $\mathbb{H}^3$ so that $E_{P\zeta}(0)$ lies at top of the associated horosphere tangent at 0. Hence it will suffice to prove that the $y$ coordinate of $E_{P\zeta}(0)$ (following Lemma 3.12) lies
in either side of any real line in \( \mathbb{C} \) passing through 0 = \( y(0) \) for arbitrarily small values of \( \zeta \). As the Lemma follows immediately if \( EP_0 \) is an immersion at 0, we can assume that 1 = \(|\mathcal{N}(f^{-1})|(0) = \frac{1}{2}|\mathcal{N}(f^{-1})(0)| = \frac{1}{2}|f''(0)| \). This means we can assume

\[
f(\zeta) = \zeta + a\zeta^3 + o(|\zeta|^3),
\]

for \(|a| = \frac{2}{3}\). Moreover, by conjugating \( f \) by a rotation and further expand its power series, we can assume

\[
f(\zeta) = \zeta + \frac{2}{3}\zeta^3 + b\zeta^4 + c\zeta^5 + o(|\zeta|^5),
\]

for \( b, c \in \mathbb{C} \).

For this expansion of \( f \) we will show that for any line \( L \subset \mathbb{C} \) passing through the origin there exists arbitrarily close to 0 values of \( \zeta \) so that

\[
y(f(\zeta)) = f(\zeta) + \left( -\frac{f''(\zeta)}{f'(\zeta)} \frac{1-|\zeta|^2}{2} + \zeta \right) f'(\zeta)(1 - |\zeta|^2) \frac{1}{1 + \left| -\frac{f''(\zeta)}{f'(\zeta)} \frac{1-|\zeta|^2}{2} + \zeta \right|^2}
\]

has image on each component of \( \mathbb{C} \setminus L \). The strategy is to deduce that from the power expansion of \( y \), which we are separating in the following cases

1. \( b \neq 0 \): In this case we will consider the second order expansion of \( y \). As we have \( f'(\zeta) = 1 + 3\zeta^2 + o(|\zeta|^2) \) and \( f''(\zeta) = 4\zeta + 12b\zeta^2 + o(|\zeta|^2) \), then

\[
-\frac{f''(\zeta)}{f'(\zeta)} \frac{1-|\zeta|^2}{2} + \zeta = \zeta - \left( 4\zeta + 12b\zeta^2 \right) \frac{1-|\zeta|^2}{2} + o(|\zeta|^2)
\]

\[
= \zeta - 2\zeta - 6b\zeta^2 + o(|\zeta|^2)
\]

Thus

\[
\left| -\frac{f''(\zeta)}{f'(\zeta)} \frac{1-|\zeta|^2}{2} + \zeta \right|^2 = -2\zeta^2 + 5|\zeta|^2 - 2\zeta^2 + o(|\zeta|^2)
\]

and subsequently

\[
\frac{\left( -\frac{f''(\zeta)}{f'(\zeta)} \frac{1-|\zeta|^2}{2} + \zeta \right)}{1 + \left| -\frac{f''(\zeta)}{f'(\zeta)} \frac{1-|\zeta|^2}{2} + \zeta \right|^2} = \zeta - 2\zeta - 6b\zeta^2 + o(|\zeta|^2)
\]

Replacing in the equation for \( y(f(\zeta)) \) we obtain

\[
y(f(\zeta)) = 2\zeta - 2\zeta - 6b\zeta^2 + o(|\zeta|^2)
\]

From the linear term we can easily that that the result follows for any line \( L \) different from \( i\mathbb{R} \). For \( L = i\mathbb{R} \) we use that \( b \neq 0 \) to select small values of \( \zeta \) so that \( (6b)^2\zeta^2 \) has either strictly positive or strictly negative real part.
2. \( b = 0, \left( \frac{20}{3} - 10\pi \right) \notin \mathbb{i} \mathbb{R} \). We proceed as in the previous case, except that now we look at the third degree expansion of \( y \). As we have \( f'(\zeta) = 1 + 2\zeta^2 + o(|\zeta|^3) \) and \( f''(\zeta) = 4\zeta + 20\zeta^3 + o(|\zeta|^3) \), then

\[
-\frac{f''(\zeta)}{f'(\zeta)} \frac{(1 - |\zeta|^2)}{2} + \zeta = \zeta - (4\zeta + 20\zeta^3)(1 - 2\zeta^2) \left( \frac{1 - |\zeta|^2}{2} \right) + o(|\zeta|^3)
\]

\( = \zeta - 2\zeta + (4 - 10\pi)\zeta^3 + 2\zeta|\zeta|^2 + o(|\zeta|^3) \) \hspace{1cm} (7.5)

Hence

\[
-\frac{f''(\zeta)}{f'(\zeta)} \frac{(1 - |\zeta|^2)}{2} + \zeta \geq -2\zeta^2 + 5|\zeta|^2 - 2\zeta^2 + o(|\zeta|^3)
\]

\( \geq -2\zeta^2 + 5|\zeta|^2 - 2\zeta^2 + o(|\zeta|^3) \) \hspace{1cm} (7.6)

and subsequently

\[
\frac{-(\frac{f''(\zeta)}{f'(\zeta)} \frac{(1 - |\zeta|^2)}{2} + \zeta)}{1 + \frac{f''(\zeta)}{f'(\zeta)} \frac{(1 - |\zeta|^2)}{2} + |\zeta|^2} = \zeta - 2\zeta + (4 - 10\pi)\zeta^3 + 2\zeta|\zeta|^2
\]

\( = (\zeta - 2\zeta)(2\zeta^2 - 5|\zeta|^2 + 2\zeta^2) + o(|\zeta|^2) \) \hspace{1cm} (7.7)

Replacing in the equation for \( y(f(\zeta)) \) we obtain

\[
y(f(\zeta)) = 2\zeta - 2\zeta + \frac{2}{3}\zeta^3 + (4 - 10\pi)\zeta^3 + 2\zeta|\zeta|^2
\]

\( + (\zeta - 2\zeta)(4\zeta^2 - 6|\zeta|^2 + 2\zeta^2) + o(|\zeta|^2) \) \hspace{1cm} (7.8)

As in the previous case, because of the linear term of \( y \) it suffices to prove the statement for \( L = \mathbb{i} \mathbb{R} \). And as \( \left( \frac{20}{3} - 10\pi \right) \notin \mathbb{i} \mathbb{R} \) then for any small purely real value of \( \zeta \) we have that \( y \) has either strictly positive or strictly negative real part.

3. \( b = 0, \left( \frac{20}{3} - 10\pi \right) \in \mathbb{i} \mathbb{R} \). We resume at the third degree expansion of the previous case and after some elementary calculations we obtain

\[
y(f(\zeta)) = (\zeta - \zeta) \left( 2 + \frac{31}{6}(\zeta - \zeta)^2 - (\zeta - \zeta)(\zeta + \zeta) + \frac{1}{2}(\zeta + \zeta)^2 \right)
\]

\( + \left( \frac{5}{6} - \frac{5}{4} \right) \left( (\zeta + \zeta)^3 - 3(\zeta + \zeta)^3(\zeta - \zeta) + 3(\zeta + \zeta)(\zeta - \zeta)^2 - (\zeta - \zeta)^3 \right)
\]

\( + o(|\zeta|^3) \). \hspace{1cm} (7.9)

As the terms \( -(\zeta - \zeta)^2(\zeta + \zeta), \left( \frac{5}{6} - \frac{5}{4} \right) \left( -3(\zeta + \zeta)^3(\zeta - \zeta) - (\zeta - \zeta)^3 \right) \) are purely real and all other terms are purely imaginary, it is easy to see that we can take arbitrarily small values of \( \zeta \) so that \( y \) has either strictly positive or strictly negative real part.
Let us extend the notation $\Sigma_\Omega$ by taking $\Sigma_\Omega(t)$ as the image of $Ep_{t^1/\rho_\Omega}$ for $t \in \mathbb{R}$. The following proposition shows that a minimal surface in $\mathbb{H}^3$ with boundary $\gamma \subset \mathbb{C}$ is in between appropriate equidistant images $\Sigma_\Omega(t), \Sigma_\Omega^*(t)$ ($\mathbb{C} \setminus \gamma = \Omega \cup \Omega^*$). We write
\[
\|\mathcal{F}(f^{-1})\|_\infty := \sup_{z \in \Omega} |\mathcal{F}(f^{-1})(z)|\rho_\Omega^{-1}(z) = \sup_{\zeta \in \mathbb{D}} |\mathcal{F}(f)(\zeta)|\rho_\mathbb{D}^{-1}(\zeta) = \|\mathcal{F}(f)\|_\infty.
\]

**Proposition 7.2.** Let $M \subset \mathbb{H}^3$ be a minimal surface so that $\partial_\infty M = \gamma$. Denote by $M_\Omega$ the closure of the component of $\mathbb{H}^3 \setminus M$ with conformal boundary $\Omega$. Given conformal map $f : \mathbb{D} \to \Omega$, denote by $t_0 = \frac{1}{2} \log \left(\max\{1, 2\|\mathcal{F}(f)\|_\infty - 1\}\right)$. Then for any $t \geq t_0$ we have that $\Sigma_\Omega(t) \subset M_\Omega$.

**Proof.** Recall that by the discussion at the start of Subsection 3.3, the principal curvatures at infinity of $\rho_\Omega$ are bounded below by $1 - 2\|\mathcal{F}(f)\|_\infty$. By taking any $\Omega' \subset \Omega$ bounded by an equipotential, we have that the same lower bound $1 - 2\|\mathcal{F}(f)\|_\infty$ holds for $\rho_{\Omega'}$.

We will start by showing that $\Sigma_{\Omega'}(t) \subset M_\Omega$ for any $t \geq t_0$.

Fixing $\Omega' \subset \Omega$, define $t' = \inf\{t \in \mathbb{R} | \Sigma_{\Omega'}(t) \subset M_\Omega\}$. As we have $\partial \Omega' \subset \Omega$, then $t' < +\infty$ and $\Sigma_{\Omega'}(t') \cap M$ is a non-empty compact subset of $\mathbb{H}^3$. If we assume by contradiction that $t' > t_0$ then $e^{2t'}\rho_{\Omega'}$ has principal curvatures at infinity bounded strictly below by $-1$. As we have that the mean curvature at infinity $H^*$ is the opposite of Gaussian curvature of the metric (see Corollary 3.5), then the principal curvatures at infinity $k_{1,2}$ of $e^{2t'}\rho$ satisfy $k_{1}^2 + k_{2}^2 = e^{-2t'} < 1$ at every point. This implies that the mean curvature vector of $\Sigma_{\Omega'}(t')$, given by $\frac{1 - k_{1}^2 - k_{2}^2}{(1+k_1^2)(1+k_2^2)}$ times the outer normal to the associated horosphere, is positively parallel to the outer normal to the horosphere. This leads to a contradiction as at the tangent point between $\Sigma_{\Omega'}(t')$ and $M$ we will have that the mean curvature vector points in the opposite direction.

As we can obtain $\Sigma_\Omega(t)$ as limits of $\Sigma_{\Omega'}(t)$, the conclusion follows for $\Omega$. \hfill \Box

**Remark 7.3.** From Proposition 7.2 we have that if $\|\mathcal{F}(f)\|_\infty, \|\mathcal{F}(g)\|_\infty < 1$ (where $f, g$ are uniformization maps for $\Omega, \Omega^*$) then the minimal surface $M$ lies in between the Epstein–Poincaré maps from $\Omega, \Omega^*$. This is an alternate proof of Proposition 4.1 under the assumption $\|\mathcal{F}(f)\|_\infty, \|\mathcal{F}(g)\|_\infty < 1$.

We can impose instead conditions on the curvatures of the minimal surface to obtain the same conclusion as in Remark 7.3.

**Proposition 7.4.** Let $M \subset \mathbb{H}^3$ be a minimal surface so that $\partial_\infty M = \gamma$. Denote by $M_\Omega$ the closure of the component of $\mathbb{H}^3 \setminus M$ with conformal boundaries $\Omega$. Assume that any point of $M$ the principal curvatures are strictly between $-1$ and $1$. Then for any $t \geq 0$ we have that $\Sigma_\Omega(t) \subset M_\Omega$.

**Proof.** For $z \in \Omega$ define $\nu_M$ as the visual metric of $M$, given by the value of the visual metric at $z$ of the first horosphere $H_z$ at $z$ that intersects $M$ (or equivalently, $H_z = \partial B_z$, where $B_z$ is the largest open horoball based at $z$ disjoint from $M$). As $z \notin \partial_\infty M$ we have that $\nu_M$ is well-defined, and by the condition on mean curvatures we have that $H_z$ is tangent to $M$ at a unique point. Indeed, $H_z \cap M$ is an isolated set, and if it is not a singleton then we would find an intrinsic geodesic segment of $M$ that on its interior
belong to $\mathbb{H}^3 \setminus \overline{B}_e$ but its endpoints lie in $H_\gamma$. Such curve will have an interior point of geodesic curvature greater than 1, but as a geodesic segment of $M$ its geodesic curvature is always less than 1, which is a contradiction.

Using that the point of tangency of $H_\gamma$ and $M$ is unique, one has that the Gaussian curvature of $\nu_M$ at $z$ is given by $-\frac{1+\lambda^2}{1-\lambda^2} \leq -1$, where $\pm \lambda$ are the principal curvatures of $M$ at the point of tangency. By the Ahlfors-Schwarz lemma we have that $\nu_M \leq e^{2t} \rho_1$ for any $t \geq 0$, which in particular implies that $\Sigma_{\Omega}(t) \subset M_\Omega$.

Finally, we will see how the Epstein–Poincaré map has image inside a neighbourhood of the convex core.

**Proposition 7.5.** Define $\epsilon$ as 0 if $\|\mathcal{F}(f)\|_{\infty}, \|\mathcal{F}(g)\|_{\infty} < 1$, and as the maximum between $\tanh^{-1}\left(\frac{\|\mathcal{F}(f)\|_{\infty}}{\|\mathcal{F}(f)\|_{\infty}+1}\right)$ and $\tanh^{-1}\left(\frac{\|\mathcal{F}(g)\|_{\infty}}{\|\mathcal{F}(g)\|_{\infty}+1}\right)$ otherwise. Then $\Sigma_{\Omega_1}, \Sigma_{\Omega_2}$ belong to the $\epsilon$ neighbourhood of the convex core $C(\gamma)$.

**Proof.** We will prove the following claim, which combined with Proposition 4.1 proves the result.

**Claim:** Take $t_0 = 0$ if $\|\mathcal{F}(f)\|_{\infty} < 1$ and $t_0 = \tanh^{-1}\left(\frac{\|\mathcal{F}(f)\|_{\infty}}{\|\mathcal{F}(f)\|_{\infty}+1}\right)$ otherwise. For any round disk $D \subset \Omega$ we will show that $\Sigma_{\Omega}$ lies in the component of $\mathbb{H}^3 \setminus \Sigma_D(t_0)$ whose boundary at infinity contains $\gamma$.

Define the (signed) distance function $d : \mathbb{H}^3 \to \mathbb{R}$ to $\Sigma_D(t_0)$ so that $d^{-1}\{t\} = \Sigma_D(t_0 + t)$. Considering $d \circ \Sigma_{\Omega}$, we define $d_0 = \sup\{d(\Sigma_{\Omega}(z)) \mid z \in \Omega\}$. The claim is equivalent to show that $d_0 \leq 0$.

As $\gamma$ lies in the exterior of $\Omega'$ we have that for $\lim_{z \in \Omega_1, z \to \gamma} d(\Sigma_{\Omega}(z)) = -\infty$, from which it follows that $d_0$ is realized at a point $z_0 \in \Omega$. By Lemma 7.1 we have that $\text{Ep}_{\Sigma_1}(z_0)$ must be perpendicular to $\Sigma_{\Omega_2}(d_0 + t_0)$, since otherwise we can produce $z_1$ close to $z_0$ so that $d(z_1) > d(z_0)$. In particular $\text{Ep}_{\Sigma_1}(z_0)$ extends to a geodesic orthogonal to $\Sigma_D(t_0)$ with $z_0$ as one of its endpoints. We finish addressing by cases:

1. $z_0 \in D$: write $\rho_1 = e^{2u} \rho_D$ near $z_0$. As $\Sigma_{\Omega}$ belongs to $d^{-1}\{(-\infty, d_0 + t_0]\}$ we have that $u(z_0) = d_0 + t_0$, but as $\text{Ep}_{\Sigma_1}(z_0)$ is perpendicular to $\Sigma_{\Omega_2}(d_0 + t_0)$ we have that $u(z_0) = d_0 + t_0$. Hence $\Delta u(z_0) \leq 0$, where $\Delta$ is the Laplacian defined by $\rho_D$. As the curvature of $\rho_D$ at is given by $-1 = e^{-2u(z_0)}(-\Delta u(z_0) - 1) \geq -e^{-2(d_0 + t_0)}$ we have that $d_0 + t_0 \leq 0$. Since by assumption we have that $t_0 \geq 0$, it follows that $d_0 \leq 0$.

2. $z_0 \notin D$: A simple calculation shows that (with respect to $\text{Ep}_{\Sigma_1}(z_0)$) $\Sigma_D(d_0 + t_0)$ is an umbilic surface with principal curvatures equal to $-\frac{\|\mathcal{F}(f)(z_0)\|_1}{\|\mathcal{F}(f)(z_0)\|_1 + 1}$ (when finite). As $\Sigma_{\Omega}$ is contained in $d^{-1}\{(-\infty, d_0 + t_0]\}$, we have that $-\frac{\|\mathcal{F}(f)(z_0)\|_1}{\|\mathcal{F}(f)(z_0)\|_1 + 1} \leq \tanh(-d_0 - t_0)$. If $\|\mathcal{F}(f)(z_0)\|_1 < 1$ and $t_0 \geq 0$ then it follows that $d_0 \leq 0$. On the other hand if $\|\mathcal{F}(f)(z_0)\|_1 \geq 1$ then we have that $t_0 \geq \tanh^{-1}\left(\frac{\|\mathcal{F}(f)(z_0)\|_1}{\|\mathcal{F}(f)(z_0)\|_1 + 1}\right)$. Hence from $-\frac{\|\mathcal{F}(f)(z_0)\|_1}{\|\mathcal{F}(f)(z_0)\|_1 + 1} \leq \tanh(-d_0 - t_0)$ it follows that $d_0 \leq 0$. 

\[\square\]
Remark 7.6. As with Remark 7.3, we have that Proposition 7.5 shows that if we have that \( \| {\mathcal{S}}(f) \|_\infty, \| {\mathcal{S}}(g) \|_\infty < 1 \) then the Epstein–Poincaré map has image inside the convex core. This in in contract with the analogous result for convex co-compact hyperbolic 3-manifolds, where the condition \( \| {\mathcal{S}}(f) \|_\infty, \| {\mathcal{S}}(g) \|_\infty < 1 \) is not required. This is because in the convex co-compact case we can take a point at infinity where the conformal factor between the Poincaré and the Thurston metric is extremized (by the co-compact action in the boundary), which makes it so that we can follow the proof of Proposition 7.5 using only case 1.

Remark 7.7. While in Propositions 7.2, 7.4, 7.5 the restrictions on the norm of the Schwarzian or the curvature are not necessarily sharp, some restriction is necessary. This can be seen for instance in the following example. Denote by \( \Omega_0 \) the complement of the real segment \([0, 1]\) in \( \mathbb{C} \) and by \( \Omega_n \) a sequence of domains bounded by equipotentials of \( \Omega_0 \) so that \( \Omega_n \xrightarrow{n \to \infty} \Omega_0 \). Since that the convex core of \( \Omega_0 \) is given by the half-plane defined by the circular arc, it is easy to see that \( \text{Ep}_{\Omega_0} \) pierces through the convex core. As \( \Omega_n \) bound equipotentials of \( \Omega_0 \) and \( \Omega_n \xrightarrow{n \to \infty} \Omega_0 \), one can verify that the stronger conclusions from Propositions 7.2, 7.4, 7.5 do not follow for \( n \) sufficiently large.

References


