Universal Liouville action as a renormalized volume and its gradient flow

Martin Bridgeman[*], Kenneth Bromberg[†], Franco Vargas Pallete[‡], Yilin Wang[§]

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Abstract

The universal Liouville action (also known as the Loewner energy for Jordan curves) is a Kähler potential on the Weil–Petersson universal Teichmüller space which is identified with the family of Weil–Petersson quasicircles via conformal welding. Our main result shows that, under regularity assumptions, the universal Liouville action equals the renormalized volume of the hyperbolic 3-manifold bounded by the two Epstein–Poincaré surfaces associated with the quasicircle. We also study the gradient descent flow of the universal Liouville action with respect to the Weil–Petersson metric and show that the flow always converges to the origin (the circle). This provides a bound of the Weil–Petersson distance to the origin by the universal Liouville action.

Contents

[*] bridgem@bc.edu Boston College, Chestnut Hill, MA, USA

[†] bromberg@math.utah.edu University of Utah, Salt Lake City, UT, USA

[‡] franco.vargaspallete@yale.edu Yale University, New Haven, CT, USA

[§] yilin@ihes.fr Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France

1 Introduction

The Riemann sphere $\hat{\mathbb{C}}$ is the conformal boundary of the hyperbolic 3-space $\mathbb{H}^3$. In [15] C. Epstein gave a natural way to associate to each conformal metric on $\hat{\mathbb{C}}$ a surface in $\mathbb{H}^3$. In more recent work, these Epstein surfaces have been used to define the renormalized volume of a hyperbolic 3-manifold which has deep connections to Teichmüller theory of Riemann surfaces and Liouville theory in mathematical physics [21,22,38]. We will recall the basics on Epstein surfaces in Section 3.

In this work we define and study the renormalized volume for the universal Teichmüller space, which can be identified with the set of quasicircles on the Riemann sphere up to conformal automorphisms.

For this, consider a Jordan curve $\gamma \subset \hat{\mathbb{C}}$, we let Ω and Ω^* be the two connected components of $\hat{\mathbb{C}} \smallsetminus \gamma$. Let $\mathrm{Ep}_\Omega : \Omega \to \mathbb{H}^3$ be the Epstein map associated with the Poincaré (hyperbolic) metric ρ_Ω in Ω, similarly for $\mathrm{Ep}_{\Omega^*} : \Omega^* \to \mathbb{H}^3$. The maps Ep_Ω, Ep_{Ω^*} are smooth, extend continuously to the identity map on γ, and are immersions almost everywhere. We call their images as the Epstein–Poincaré surfaces Σ_Ω and Σ_{Ω^*}. In particular, we note that, unlike in the cases previously considered (see [22], [7]), these Epstein–Poincaré surfaces are non-compact and not necessarily embedded and have infinite area. We show the following results.

Proposition 1.1 (See Proposition 4.2)**.** *If γ is not a circle, then the two Epstein–Poincaré surfaces Σ_Ω and Σ_{Ω^*} are disjoint.*

It follows directly from the definition of Epstein–Poincaré map that if γ is a circle, then both Σ_Ω and Σ_{Ω^*} are the totally geodesic plane bounded by γ with opposite orientation. See Lemma 3.7.

Proposition 1.2 (See Corollary 3.14)**.** *When γ is asymptotically conformal (see Theorem 3.10 for the equivalent definitions), there is a neighborhood of γ in $\hat{\mathbb{C}}$ on which the Epstein–Poincaré maps* Ep_Ω *and* Ep_{Ω^*} *are immersions and embeddings which extend to the identity map on γ.*

Quasicircles are in natural correspondence with points in the universal Teichmüller space $T(1)$, where we identify a quasicircle with its conformal welding homeomorphism. We are interested in a special class of quasicircles, i.e., Weil–Petersson quasicircles, which corresponds to the Weil–Petersson universal Teichmüller space $T_0(1)$. This space has been studied extensively for it being the connected component of the *unique* homogeneous Kähler metric on $T(1)$ (i.e., the Weil–Petersson metric) [39], and has a large number of equivalent descriptions from very different perspectives, see, e.g., [5, 13, 17, 25, 34, 42, 43].

Weil–Petersson quasicircles are asymptotically conformal, so Propositions 1.1 and 1.2 allow us to define the signed volume between Σ_Ω and Σ_{Ω^*}. A priori, this volume takes value in $(-\infty, \infty]$ (see Section 4.2 for more details). However, we show the following result.

Theorem 1.3. *If γ is a Weil–Petersson quasicircle, then the signed volume between the two Epstein–Poincaré surfaces, denoted as $V(\gamma)$, is finite.*

See Proposition 4.4 for the proof for smooth Jordan curves. The result for general Weil–Petersson quasicircles is obtained via an approximation argument, see Corollary 5.18.

Since $T_0(1)$ has a *unique* homogeneous Kähler structure, its Kähler potential is of critical importance. Takhtajan and Teo defined the *universal Liouville action* $\mathbf{S}$ on $T_0(1)$ and showed it to be such a Kähler potential [39]. In this work, we will consider the universal Liouville action as defined for Jordan curves (see Section 2.3), and denote it as $\tilde{\mathbf{S}}$ for clarity. The functional $\tilde{\mathbf{S}}(\gamma)$ can actually be defined for arbitrary Jordan curves, but it is finite if and only if γ is a Weil–Petersson quasicircle. Moreover, $\tilde{\mathbf{S}}$ is invariant under the action of Möbius transformations on $\hat{\mathbb{C}}$ (i.e., under the $\mathsf{PSL}_2(\mathbb{C})$ action). As the $\mathsf{PSL}_2(\mathbb{C})$ action extends to orientation preserving isometries of $\mathbb{H}^3$, it is very natural to search for a characterization of the class of Weil–Petersson quasicircles and an expression of $\tilde{\mathbf{S}}$ in terms of geometric quantities in $\mathbb{H}^3$.

A pioneering work of C. Bishop [5] shows that the class of Weil–Petersson quasicircles can be characterized as Jordan curves bounding minimal surfaces in $\mathbb{H}^3$ with finite total curvature. We obtain the following similar characterization in terms of Epstein–Poincaré surfaces. See also Section 7 where we compare Epstein–Poincaré surfaces to minimal surfaces and the convex core, answering a question of Bishop [4].

In fact, the Epstein maps come with a well-defined unit normal $\vec{n}$ pointing away from Ω and from Ω^* respectively. The mean curvature $H := \mathrm{tr}(B)/2$ is defined using the shape operator $B(v) := -\nabla_v \vec{n}$.

Theorem 1.4 (See Corollary 3.6)**.** *We have for all Jordan curves,*

$$\int_{\Sigma_\Omega} H \,\mathrm{d}a = \int_{\Sigma_\Omega} |H \,\mathrm{d}a| = \int_{\Sigma_\Omega} |\det B \,\mathrm{d}a| = \int_{\mathbb{D}} |\mathscr{S}(f)(z)|^2 \frac{(1-|z|^2)^2}{4} \mathrm{d}^2 z$$

where $f : \mathbb{D} \to \Omega$ is any conformal map, $\mathscr{S}(f) = f'''/f' - (3/2)(f''/f')^2$ is the Schwarzian derivative of f, $\mathrm{d}a$ *is the area form induced from $\mathbb{H}^3$ and the Epstein maps, and* $\mathrm{d}^2 z$ *is the Euclidean area form.*

In particular, Σ_Ω has finite total mean curvature (and finite total curvature) if and only if *γ is a Weil–Petersson quasicircle.*

Prior to this work, no exact identity between the Kähler potential and geometric quantity in $\mathbb{H}^3$ was known. The main result of this work is to provide such an identity.

Definition 1.5. Let γ be a Weil–Petersson quasicircle. We define the renormalized volume (or W-volume) associated with γ as

$$V_R(\gamma) := V(\gamma) - \frac{1}{2}\int_{\Sigma_\Omega \cup \Sigma_{\Omega^*}} H \mathrm{d}a \in (-\infty, \infty).$$

The definition is reminiscent to the renormalized volume[1] for quasi-Fuchsian manifolds [22, 38]. But we emphasize again that Σ_Ω and Σ_{Ω^*} are non-compact so the analysis involves additional technicalities.

Theorem 1.6 (See Corollary 5.12 and Theorem 5.13)**.** *If γ is a $C^{5,\alpha}$ Jordan curve with $\alpha > 0$, we have*

$$\tilde{\mathbf{S}}(\gamma) = 4V_R(\gamma). \tag{1.1}$$

If γ is a Weil–Petersson quasicircle, then we have $\tilde{\mathbf{S}}(\gamma) \geq 4V_R(\gamma)$.

Let us comment briefly on the proof of this theorem. It is easy to check that when γ is a circle, both sides of (1.1) are zero. We show under regularity assumptions that the first variation of both sides are equal. The variation of $\tilde{\mathbf{S}}$ was proved in [39], which we recall in Theorem 2.1. The first variation of V_R is more laborious since the Epstein–Poincaré surfaces are not compact and are immersed only almost everywhere. After administering appropriate truncation (where we make use of the regularity assumption), we re-derive the Schläfli formula which expresses the variation of V_R in terms of the mean curvature H, the metric I and the second fundamental form $\mathrm{I\!I}$ on Epstein surfaces (Theorem 5.2 and Theorem 5.7, with some of the technical details in Section 8), then translate the variation formula into quantities defined directly on $\Omega, \Omega^* \subset \hat{\mathbb{C}}$ (Theorem 5.9 and Corollary 5.11). For a general Weil–Petersson quasicircle γ we use an approximation by equipotentials (they are analytic curves and the universal Liouville action increases to that of γ). We believe the identity (1.1) also holds for a general Weil–Petersson quasicircle. However, our approximation argument only implies the inequality due to the lack of tightness for the volume between the Epstein–Poincaré surfaces, see Section 5.5.

The second topic of this work concerns the gradient descent flow of $\mathbf{S}$ with respect to the Weil–Petersson metric. We proceed similarly as in Bridgeman–Brock–Bromberg [7]. For $[\mu] \in T(1)$ we have a natural isomorphism $T_{[\mu]}T(1) \simeq \Omega^{-1,1}(\mathbb{D}^*)$.

[1]Renormalized volume of a convex co-compact hyperbolic 3-manifold is referred to the difference between the volume and half of the boundary area defined through a foliation near the ends. Our formula is similar to the definition of the *W-volume.* However, in the convex co-compact case, they only differ by a multiple of Euler characteristics of the boundary [22, Lem. 4.5].

Theorem 1.7 (See Theorem 6.1)**.** *The negative gradient of* $\mathbf{S}$ *on* $T_0(1)$ *with respect to the Weil–Petersson metric is the vector field*

$$V_{[\mu]} := -4\frac{\overline{\mathscr{S}(g_\mu)}}{\rho_{\mathbb{D}^*}} \in \Omega^{-1,1}(\mathbb{D}^*).$$

Moreover, the gradient descent flow of $\mathbf{S}$ *starting from any point in* $T_0(1)$ *converges to the origin* $[0]$ *which corresponds to the round circle.*

In fact, we also show that the flow starting from any point in $T(1)$ using the vector field V exists for all time. But here, V cannot be interpreted as the gradient of $\mathbf{S}$ if $[\mu] \notin T_0(1)$, and we do not know the limit and think it is an interesting question. Using the gradient flow, we also obtain bounds of the Weil–Petersson distance on $T_0(1)$ in terms of the universal Liouville action.

Theorem 1.8 (See Theorem 6.3)**.** *There exist universal positive constants* c *and* K *such that for all* $[\mu] \in T_0(1)$*, we have* $c(\mathrm{dist}_{\mathrm{WP}}([\mu],[0]) - Kc) \le \mathbf{S}([\mu])$*.*

Finally, let us make a few remarks on the motivation behind this work and additional comments on the relation with previous works.

Rohde and the last author introduced the *Loewner energy* for Jordan curves [30, 41] which is originally motivated from the large deviation theory of random fractal curves Schramm-Loewner evolutions (SLE) [41, 44]. It is shown in [12] that the Loewner energy is the *Onsager–Machlup (or the action) functional* of the SLE loop measure. It turns out quite surprisingly that the Loewner energy equals exactly $\mathbf{S}/\pi$ as proved in [42]. Since we will not make use of Loewner theory but only the fact of $\mathbf{S}$ is a Kähler potential on $T_0(1)$, we adopt the terminology of *universal Liouville action* here. SLEs play a central role in the emerging field of two-dimensional random conformal geometry. In particular, they provide a mathematical description of the geometric patterns in the scaling limits of 2D critical lattice models [23, 33, 35] and 2D conformal field theory (CFT) [2, 14, 18, 27]. On the other hand, $\mathbb{H}^3$ is the Riemannian analog of AdS_3 space. Our main result Theorem 1.6 can be interpreted as the *holographic principle* for the Loewner energy that is reminiscent of the conjectural $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence pioneered by Maldacena [24] (see also, e.g., [26, 45]). The authors are not aware of a (even conjectural) holographic principle for SLE nor for random conformal geometry in general, this work may be a first step towards this direction. We also mention [19] gives a holographic expression for determinants of discrete Dirac operator on periodic bipartite isoradial graphs.

Renormalized volume as a Liouville action has been previously studied for convex co-compact group actions in $\mathbb{H}^3$ (see work by Takhtajan–Teo [38] and Krasnov–Schlenker [22]), or equivalently, for conformally compact hyperbolic metrics. A set of applications of this study are bounds for the hyperbolic volume of mapping tori of pseudo-Anosov maps in term of their Weil–Petersson translation length (by Brock [11]) or their entropy (by Kojima–McShane [20]). This uses a bound (by Schlenker [32]) for renormalized volume in terms of Weil–Petersson distance by studying the gradient of the Liouville action, similar to our bound in Theorem 6.3. Moreover, we show in Theorem 6.1 that every flowline of the gradient converges to the absolute minimum, in analogy to the result done by the first three authors [10] for the relatively acylindrical case. This builds on work by the first two authors and Brock [7], where they used the gradient flow to find the minimum of renormalized volume for a boundary incompressible hyperbolic 3-manifold.

The paper is organized as follows: In Section 2 we collect the basics about universal Teichmüller space, its Kähler geometry, characterizations of the Weil–Petersson universal Teichmüller space, and the universal Liouville action. In Section 3 we recall the definition of Epstein surfaces and the correspondence between geometric quantities on the surface and those on the conformal boundary. We also prove the immersion and embeddedness of the Epstein–Poincaré surfaces associated with an asymptotically conformal Jordan curve. In Section 4 we study the relation between the two Epstein–Poincaré surfaces associated with the same curve. We show that they are disjoint (except for a circle), and that if the curve is regular enough, the signed volume between the Epstein–Poincaré surfaces is finite. In Section 5, we prove the variational formula for the renormalized volume and prove Theorem 1.6. Section 6 is independent from Sections 3, 4, and 5 and deals with the gradient flow of the universal Liouville action. Similarly, Section 7 describes the relative position of Epstein–Poincaré surfaces with respect to minimal surfaces and convex core. Section 8 collects the technical details and proves the Schläfli formula for the volume bounded by non-immersed Epstein–Poincaré surfaces.

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2 Universal Weil–Petersson Teichmüller space

2.1 Universal Teichmüller space

We first briefly recall a few equivalent descriptions of the universal Teichmüller space $T(1)$. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{D} = \{z \ , \ |z| < 1\}$, $\mathbb{D}^* = \hat{\mathbb{C}} - \overline{\mathbb{D}}$ and $\mathbb{S}^1 = \partial\mathbb{D}$. The group of orientation preserving conformal automorphism of $\hat{\mathbb{C}}$ is

$$\text{Möb}(\hat{\mathbb{C}}) = \mathsf{PSL}_2(\mathbb{C}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, \, ad - bc = 1 \right\}_{/A \sim -A}$$

which acts on $\hat{\mathbb{C}}$ by Möbius transformations $z \mapsto \dfrac{az+b}{cz+d}$. The subgroup preserving $\mathbb{S}^1$ is

$$\text{Möb}(\mathbb{S}^1) = \mathsf{PSU}_{1,1} = \left\{ A = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \, |\alpha|^2 - |\beta|^2 = 1 \right\}_{/A \sim -A}$$

which is isomorphic to $\mathsf{PSL}_2(\mathbb{R})$. There are a number of equivalent descriptions of $T(1)$ that we will use.

Quasisymmetric maps: We write $\mathrm{QS}(\mathbb{S}^1)$ for the group of sense preserving quasisymmetric homeomorphisms of $\mathbb{S}^1$. The *universal Teichmüller space* is

$$T(1) := \mathrm{Möb}(\mathbb{S}^1)\backslash \mathrm{QS}(\mathbb{S}^1) \simeq \{\varphi \in \mathrm{QS}(\mathbb{S}^1),\, \varphi \text{ fixes } -1, -\mathfrak{i} \text{ and } 1\}.$$

$T(1)$ is endowed with a group operation given by the composition and the origin is the identity map $\mathrm{Id}_{\mathbb{S}^1}$.

Beltrami Differentials: Given a Beltrami differential

$$\mu \in L_1^\infty(\mathbb{D}^*) = \{\mu \in L^\infty(\mathbb{D}^*),\, ||\mu||_\infty < 1\},$$

we extend it to $\hat{\mathbb{C}}$ by reflection, i.e., define for $z \in \mathbb{D}$,

$$\mu(z) = \overline{\mu\left(\frac{1}{\overline{z}}\right)}\frac{z^2}{\overline{z}^2}.$$

Let $w_\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the solution to the Beltrami equation $\partial_{\overline{z}} w_\mu = \mu \partial_z w_\mu$ fixing $-1, -\mathfrak{i}$ and 1. Then w_μ preserves $\mathbb{S}^1$ and $w_\mu|_{\mathbb{S}^1} \in \mathrm{QS}(\mathbb{S}^1)$. Since every quasisymmetric circle homeomorphism can be extended to a quasiconformal self-map of $\overline{\mathbb{D}}$, we have

$$T(1) = L_1^\infty(\mathbb{D}^*)/_\sim$$

where $\mu \sim \nu$ if and only if $w_\mu|_{\mathbb{S}^1} = w_\nu|_{\mathbb{S}^1}$. We denote by $\Phi : L_1^\infty(\mathbb{D}^*) \to T(1)$ the projection $\mu \mapsto [\mu]$. Here the origin corresponds to $[0]$.

Univalent maps: If instead we extend μ by 0 on $\mathbb{D}$ and let w^μ be the unique solution to $w^\mu_{\overline{z}} = \mu w^\mu_z$ fixing $-1, -\mathfrak{i}$ and 1, then w^μ is conformal on $\mathbb{D}$. The map $[\mu] \mapsto w^\mu|_{\mathbb{D}}$ identifies $T(1)$ with

$$\{f : \mathbb{D} \to \hat{\mathbb{C}},\text{ univalent fixing } -1, -\mathfrak{i} \text{ and } 1, \text{ extendable to q.c. map of } \hat{\mathbb{C}}\}, \tag{2.1}$$

since $\mu \sim \nu$ if and only if $w^\mu = w^\nu$ on $\mathbb{D}$. The origin corresponds to $\mathrm{Id}_{\mathbb{D}}$.

Quasicircles: By Riemann mapping theorem, the previous identification also gives

$$T(1) \simeq \{\gamma \text{ quasicircle passing through } -1, -\mathfrak{i}, \text{ and } 1\} \tag{2.2}$$

by the map $[\mu] \mapsto \gamma_\mu := w^\mu(\mathbb{S}^1)$. The origin corresponds to $\gamma_\mu = \mathbb{S}^1$. We can recover the quasisymmetric circle homeomorphism from γ_μ via conformal welding. Let Ω (resp. Ω^*) denote the connected components of $\hat{\mathbb{C}} \smallsetminus \gamma_\mu$ where $-1, -\mathfrak{i}, 1$ are in the counterclockwise direction of $\partial\Omega$ (resp. clockwise direction of $\partial\Omega^*$). Let $f_\mu = w^\mu|_{\mathbb{D}} \colon \mathbb{D} \to \Omega$ and $g_\mu \colon \mathbb{D}^* \to \Omega^*$ be the conformal maps fixing $-1, -\mathfrak{i}, 1$. Then,

$$w_\mu|_{\mathbb{S}^1} = g_\mu^{-1} \circ f_\mu|_{\mathbb{S}^1}$$

since $g_\mu = w^\mu \circ w_\mu^{-1}|_{\mathbb{D}^*}$. We call $g_\mu^{-1} \circ f_\mu|_{\mathbb{S}^1}$ the *welding homeomorphism* of the quasicircle γ_μ passing through $-1, -\mathfrak{i}, 1$.

2.2 Kähler Structure and Weil–Petersson Teichmüller space

We first define the following spaces,

$$
\begin{aligned}
A_\infty(\mathbb{D}^*) &= \{\phi : \mathbb{D}^* \to \mathbb{C} \text{ holomorphic, } \sup_{\mathbb{D}^*} |\phi|\rho_{\mathbb{D}^*}^{-1} < \infty\},\\
A_2(\mathbb{D}^*) &= \{\phi : \mathbb{D}^* \to \mathbb{C} \text{ holomorphic, } \int_{\mathbb{D}^*} |\phi|^2\rho_{\mathbb{D}^*}^{-1}\,\mathrm{d}^2 z < \infty\} \subset A_\infty(\mathbb{D}^*),
\end{aligned}
$$

where $\rho_{\mathbb{D}^*}(z) = 4/(1-|z|^2)^2$ is the hyperbolic density function and $\mathrm{d}^2 z = \mathrm{d}x \wedge \mathrm{d}y$ if $z = x + \mathrm{i}y$. The inclusion is shown in [39, Lem. I.2.1]. We define the similar spaces $A_\infty(\mathbb{D})$ and $A_2(\mathbb{D})$ (and also $A_\infty(\Omega)$ and $A_2(\Omega)$). We will also use the spaces of harmonic Beltrami differentials defined as

$$
\begin{aligned}
\Omega^{-1,1}(\mathbb{D}^*) &= \{\dot\nu \in L^\infty(\mathbb{D}^*), \dot\nu = \rho_{\mathbb{D}^*}^{-1}\,\overline{\phi},\ \phi \in A_\infty(\mathbb{D}^*)\};\\
H^{-1,1}(\mathbb{D}^*) &= \{\dot\nu \in L^\infty(\mathbb{D}^*), \dot\nu = \rho_{\mathbb{D}^*}^{-1}\,\overline{\phi},\ \phi \in A_2(\mathbb{D}^*)\} \subset \Omega^{-1,1}(\mathbb{D}^*).
\end{aligned}
$$

The universal Teichmüller space $T(1)$ has a canonical complex structure such that $\Phi : L_1^\infty(\mathbb{D}^*) \to T(1)$ is a holomorphic submersion. The holomorphic tangent space at the origin is

$$T_{[0]}T(1) = L^\infty(\mathbb{D}^*)/\ker(D_0\Phi) \simeq \Omega^{-1,1}(\mathbb{D}^*)$$

where

$$\ker(D_0\Phi) = \mathfrak{N}(\mathbb{D}^*) := \{\dot\nu \in L^\infty(\mathbb{D}^*) \colon \int_{\mathbb{D}^*} \dot\nu\phi = 0,\ \forall\phi \text{ holomorphic and} \int_{\mathbb{D}^*} |\phi|\mathrm{d}^2 z < \infty\}$$

is the space of infinitesimally trivial Beltrami differentials.

The space $L^\infty(\mathbb{D}^*)$ has a natural group structure given by the associated quasiconformal maps. We define $\lambda = \nu \star \mu^{-1}$ if $w_\lambda = w_\nu \circ w_\mu^{-1}$. Thus

$$\lambda = \left(\frac{\nu-\mu}{1-\overline{\mu}\nu}\frac{\partial_z w_\mu}{\overline{\partial_z w_\mu}}\right) \circ w_\mu^{-1}.$$

We define R_μ to be right multiplication by μ on $L^\infty(\mathbb{D}^*)$. This descends to give a map $R_\mu : T(1) \to T(1)$. Furthermore, the complex structure on $T(1)$ is right-invariant. Therefore, $D_0 R_{[\mu]} : T_{[0]}T(1) \to T_{[\mu]}T(1)$ is a complex linear isomorphism between holomorphic tangent spaces, and we obtain the identification of $T_{[\mu]}T(1) \simeq \Omega^{-1,1}(\mathbb{D}^*)$.

To define a Kähler metric on $T(1)$, one needs to endow $T(1)$ with a Hilbert manifold structure. It is known since [6] that on the subspace $\mathscr{M} = \mathrm{M\ddot{o}b}(\mathbb{S}^1)\backslash \mathrm{Diff}(\mathbb{S}^1)$ there is a unique Kähler metric up to a scalar multiple. However, $\mathscr{M}$ is not complete under the Kähler metric. Takhtajan and Teo extend the Hilbert manifold structure on $T(1)$ by defining the Hermitian metric on the distribution $\mathscr{D}([\mu]) = D_0 R_{[\mu]}(H^{-1,1}(\mathbb{D}^*)) \subset T_{[\mu]}T(1)$ induced from $H^{-1,1}(\mathbb{D}^*)$:

$$\langle\dot\mu, \dot\nu\rangle := \int_{\mathbb{D}^*} \dot\mu\overline{\dot\nu}\rho_{\mathbb{D}^*}\mathrm{d}^2 z, \quad \forall\dot\mu, \dot\nu \in H^{-1,1}(\mathbb{D}^*).$$

They prove that this distribution is integrable and define $T_0(1)$ to be the connected component containing [0] which is called the *Weil–Petersson Teichmüller space*. The

Hermitian metric defined above is called the *Weil–Petersson metric.* (One may draw the similarity with the Weil–Petersson metric on Teichmüller spaces of a Fuchsian group Γ where the integral is over $\mathbb{D}^*/\Gamma$.) In terms of the four equivalent definitions of $T(1)$, the subspace $T_0(1)$ is characterized as follows:

Quasisymmetric maps: Y. Shen [34] showed $\varphi \in T_0(1)$ if and only if φ is absolutely continuous with respect to the arclength measure, and $\log \varphi' \in H^{1/2}(\mathbb{S}^1)$, namely the fractional Sobolev space of functions u such that

$$\|u\|_{H^{1/2}}^2 := \iint_{\mathbb{S}^1 \times \mathbb{S}^1} \left| \frac{u(\zeta) - u(\xi)}{\zeta - \xi} \right|^2 \mathrm{d}\zeta \mathrm{d}\xi < \infty. \tag{2.3}$$

Beltrami Differentials: It is shown in [39] that $[\mu] \in T_0(1)$ if and only if it has a representative $\mu \in L_1^\infty(\mathbb{D}^*)$ such that

$$\int_{\mathbb{D}^*} |\mu(z)|^2 \rho_{\mathbb{D}^*}(z) \mathrm{d}^2 z < \infty.$$

Univalent maps: It is shown in [39, Thm. II.1.12] (see also [13]) that a univalent function $f : \mathbb{D} \to \hat{\mathbb{C}}$ fixing $-1, -\mathfrak{i}, 1$ and extendable to a quasiconformal map of $\hat{\mathbb{C}}$, corresponds to an element of $T_0(1)$ via the identification (2.1) if and only if the Schwarzian derivative

$$\mathscr{S}(f) := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

satisfies

$$\int_{\mathbb{D}} |\mathscr{S}(f)|^2 \rho_{\mathbb{D}}^{-1} \, \mathrm{d}^2 z < \infty. \tag{2.4}$$

In other words, the *Bers' embedding* $\beta([\mu]) := \mathscr{S}(f) \in A_2(\mathbb{D})$.
Furthermore, let $\tilde{f} = A \circ f$ where A is a Möbius map sending $\Omega = f(\mathbb{D})$ to a bounded domain (as a priori, $\overline{\Omega}$ may contain ∞). Then $f \in T_0(1)$ if and only if

$$\int_{\mathbb{D}} |\mathscr{N}(\tilde{f})|^2 \mathrm{d}^2 z < \infty \tag{2.5}$$

where $\mathscr{N}(\tilde{f}) = \tilde{f}''/\tilde{f}'$ is the *nonlinearity* of $\tilde{f}$. We note that the expression in (2.4) is invariant under the transformation $f \to A \circ f \circ B$, for all $A \in \mathsf{PSL}_2(\mathbb{C})$ and $B \in \mathsf{PSU}_{1,1}$ but the expression in (2.5) is not invariant under such transformations.

Quasicircles: A quasicircle passing through $-1, -\mathfrak{i}, 1$ which corresponds via (2.2) to an element of $T_0(1)$ is called a *Weil–Petersson quasicircle.* It is easy to see that if γ and $\tilde{\gamma}$ are two quasicircles passing through $-1, -\mathfrak{i}, 1$ and $\tilde{\gamma} = A(\gamma)$ for some $A \in \mathsf{PSL}_2(\mathbb{C})$, then $\tilde{\gamma}$ is Weil–Petersson if and only if γ is Weil–Petersson. Therefore, we may extend the definition to say that a Jordan curve γ is Weil–Petersson if and only if it is $\mathsf{PSL}_2(\mathbb{C})$-equivalent to a Weil–Petersson quasicircle passing through $-1, -\mathfrak{i}, 1$.

2.3 Universal Liouville action

Takhtajan and Teo introduced the *universal Liouville action* $\mathbf{S}$ on $T_0(1)$ and showed it to be a Kähler potential on $T_0(1)$. See [39, Thm. II.4.1]. We will consider it as a functional on the space of Weil–Petersson quasicircles.

Indeed, let γ be a Jordan curve which does not pass through ∞. Let D and D^* be respectively the bounded and unbounded connected component of $\hat{\mathbb{C}} \smallsetminus \gamma$, $f : \mathbb{D} \to D$ and $g : \mathbb{D}^* \to D^*$ be *any* conformal maps such that $g(\infty) = \infty$ (note that D might not be Ω, it can also be Ω^*, and f and g are different from the canonical maps f_μ and g_μ). Define

$$\tilde{\mathbf{S}}(\gamma) := \int_{\mathbb{D}} |\mathscr{N}(f)|^2 \,\mathrm{d}^2 z + \int_{\mathbb{D}^*} |\mathscr{N}(g)|^2 \,\mathrm{d}^2 z + 4\pi \log |f'(0)/g'(\infty)| \tag{2.6}$$

and is $\mathsf{PSL}_2(\mathbb{C})$-invariant (it can be seen via the identity with π times the Loewner energy of γ [42]) and finite if and only if γ is a Weil–Petersson quasicircle. The universal Liouville action $\mathbf{S}([\mu])$ for $[\mu] \in T_0(1)$ is defined as $\tilde{\mathbf{S}}(A(\gamma_\mu))$ where γ_μ is the Weil–Petersson quasicircle passing through $-1, -\mathfrak{i}, 1$ corresponding to $[\mu]$ via the identification (2.2) and $A \in \mathsf{PSL}_2(\mathbb{C})$ is any Möbius transformation such that $A(\gamma_\mu)$ is bounded. The universal Liouville action $\mathbf{S}$ satisfies the following properties:

- $\mathbf{S}([\mu]) \geq 0$ for all $[\mu] \in T_0(1)$ (see, e.g., [42, Thm. 1.4]);
- $\tilde{\mathbf{S}}(\gamma) = 0$ if and only if γ is a circle, or equivalently, $[\mu] = [0]$.

The first variation formula of $\mathbf{S}$ from [39] will be a key ingredient in our proofs. We now restate it for $\tilde{\mathbf{S}}$. Let γ be the Weil–Petersson quasicircle passing through $-1, -\mathfrak{i}, 1$ corresponding to an element $[\mu]$ of $T_0(1)$. Let Ω and Ω^* be the connected components of $\hat{\mathbb{C}} \smallsetminus \gamma$ as in Section 2.1. Let $f_\mu : \mathbb{D} \to \Omega$ and $g_\mu : \mathbb{D}^* \to \Omega^*$ be the conformal maps fixing $-1, -\mathfrak{i}, 1$. Let $\dot\nu \in H^{-1,1}(\mathbb{D}^*) \simeq T_{[\mu]} T_0(1)$, $t \in (-\|\dot\nu\|_\infty^{-1}, \|\dot\nu\|_\infty^{-1})$, $w_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the solution fixing $-1, -\mathfrak{i}, 1$ to the Beltrami equation

$$\frac{\partial_{\bar z} w_t}{\partial_z w_t}(z) = \begin{cases} 0 & z \in \Omega, \\ t (g_\mu)_* \,\dot\nu(z) & z \in \Omega^* \end{cases}$$

where

$$(g_\mu)_* \,\dot\nu(z) = \dot\nu \circ g_\mu^{-1} \,\frac{\overline{(g_\mu^{-1})'}}{(g_\mu^{-1})'}.$$

We let $\gamma_t = w_t(\gamma)$ which is a small deformation of γ.

Theorem 2.1 ([39, Cor. II.3.9])**.** *The universal Liouville action satisfies the following first variation formula. Let $\dot\nu \in H^{-1,1}(\mathbb{D}^*) \simeq T_{[\mu]} T_0(1)$,*

$$(\mathrm{d}\mathbf{S})_{[\mu]}(\dot\nu) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{\mathbf{S}}(\gamma_t) = 4 \,\mathrm{Re} \int_{\mathbb{D}^*} \dot\nu \mathscr{S}(g_\mu) \mathrm{d}^2 z = -4 \,\mathrm{Re} \int_{\Omega^*} ((g_\mu)_* \dot\nu) \,\mathscr{S}(g_\mu^{-1}) \,\mathrm{d}^2 z.$$

Remark 2.2. We note that compared to the formula in [39], we take the derivative of $\mathbf{S}$ in the real tangent space (which is canonically isomorphic to the holomorphic tangent space) while [39] takes derivative in the holomorphic tangent space and both derivatives are related by

$$(\mathrm{d}\mathbf{S})_{[\mu]}(\dot\nu) = 2 \,\mathrm{Re}\, \partial_{\dot\nu} \,\mathbf{S}([\mu]).$$

The last equality in Theorem 2.1 follows from a change of variable and the chain rule for Schwarzian derivatives which shows

$$\mathscr{S}(g^{-1}) = -\mathscr{S}(g) \circ g^{-1} (g^{-1\prime})^2.$$

See another proof of Theorem 2.1 in [37] using the Loewner theory and when $\dot\nu$ is compactly supported.

Remark 2.3. We choose $\dot\nu$ to be *harmonic* Beltrami differential as $H^{-1,1}(\mathbb{D}^*)$ is isomorphic to $T_{[\mu]}T_0(1)$. Clearly, the variational formula also holds for $\dot\nu \in H^{-1,1}(\mathbb{D}^*) \oplus \mathfrak{N}(\mathbb{D}^*)$ if $\int |\mathscr{S}(g)| \mathrm{d}^2 z < \infty$, which is the case, e.g., whenever the curve γ is $C^{3,\alpha}$ for $\alpha > 0$.

3 Preliminaries on Epstein surfaces

3.1 Definition of general Epstein surfaces

In [15], Epstein associated to a smooth conformal metric ρ on a domain $\Omega \subseteq \mathbb{S}^n$ a surface $\mathrm{Ep}_\rho : \Omega \to \mathbb{H}^{n+1}$ given by taking an envelope of horospheres based at points of Ω with size determined by the conformal metric. Explicitly, for $x \in \mathbb{H}^{n+1}$ in the ball model, we let ν_x be the hyperbolic visual measure on the unit sphere $\mathbb{S}^n = \partial \mathbb{H}^{n+1}$ from x (namely, the pull-back of the round metric on $\mathbb{S}^n$ by any isometry of $\mathbb{H}^{n+1}$ sending x to the origin), then for $z \in \Omega$ we define

$$\mathfrak{H}(z,\rho) = \{x \in \mathbb{H}^{n+1} \mid \nu_x(z) = \rho(z)\}.$$

Then the set $\mathfrak{H}(z,\rho)$ is a horosphere based at z. The *Epstein map* Ep_ρ is the solution to the envelope equation of these horospheres. More precisely, there exists a (unique) smooth map

$$\widetilde{\mathrm{Ep}}_\rho \colon \Omega \to T^1\mathbb{H}^{n+1}$$

such that $\widetilde{\mathrm{Ep}}_\rho(z)$ is an outward pointing normal to $\mathfrak{H}(z,\rho)$ and

$$\mathrm{Ep}_\rho \colon \Omega \to \mathbb{H}^{n+1}$$

is the composition of $\widetilde{\mathrm{Ep}}_\rho$ with the projection $T^1\mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$ and that the image of the tangent maps of Ep_ρ at z is orthogonal to $\widetilde{\mathrm{Ep}}_\rho(z)$. We call the image of Ep_ρ the *Epstein surface associated with* ρ and denote it by Σ_ρ.

These *Epstein surfaces* generalize surfaces such as the convex hull boundary and have had numerous applications in hyperbolic geometry, complex analysis and the study of univalent functions. We record some basic facts about Epstein maps. The following property follows directly from the definition.

Lemma 3.1 (Naturality of Epstein map)**.** *If $x \in \mathbb{H}^{n+1}$, $h \in \mathrm{Isom}_+(\mathbb{H}^{n+1})$ we have that $h^*(\nu_{h(x)}) = \nu_x$. Hence it follows*

$$\mathrm{Ep}_\rho = h \circ \mathrm{Ep}_{h^*\rho} \tag{3.1}$$

where $h^*\rho$ is the pull-back metric of ρ under h.

Theorem 3.2 (See [8, 22]). *Let Ω be a domain in $\mathbb{S}^n$ and ρ a smooth conformal metric on Ω. Let $\rho_t = e^{2t}\rho$ for some $t \in \mathbb{R}$.*

1. *The value of* $\mathrm{Ep}_\rho(z)$ *is determined by ρ and its first derivatives at z.*
2. *We let $\mathfrak{g}_t : T_1\mathbb{H}^3 \to T_1\mathbb{H}^3$ be time t geodesic flow. We have $\widetilde{\mathrm{Ep}}_{\rho_t} = \mathfrak{g}_{-t} \circ \widetilde{\mathrm{Ep}}_\rho$.*
3. *Let $\mathfrak{g}_{-\infty} : T_1\mathbb{H}^3 \to \hat{\mathbb{C}}$ be the* hyperbolic gauss map *sending a tangent vector to the endpoint of the associated geodesic ray as $t \to -\infty$. Then $g_{-\infty}(\widetilde{\mathrm{Ep}}_\rho(z)) = z$.*
4. *For each $z \in \Omega$ there are at most two values of t where* Ep_{ρ_t} *is not an immersion at z.*

Whenever Ep_ρ is an immersion we pullback the fundamental forms $\mathrm{I}, \mathrm{II}, \mathrm{III}$ of Σ_ρ with respect to Ep_ρ to Ω to obtain

$$\mathrm{II}(X,Y) = \mathrm{I}(BX,Y) \qquad \mathrm{III}(X,Y) = \mathrm{II}(BX,Y) = \mathrm{I}(BX,BY)$$

where B is the pullback of the shape operator of Σ_ρ, namely,

$$D\,\mathrm{Ep}_\rho(BX) = -\nabla_{D\,\mathrm{Ep}_\rho X}\widetilde{\mathrm{Ep}}_\rho$$

since $\widetilde{\mathrm{Ep}}_\rho$ defines a unit normal vector field on Σ_ρ. The eigenvalues $\{k_+, k_-\}$ of B are the *principal curvatures* of the surface Σ_ρ. The *mean curvature* H is defined as $\mathrm{tr}(B)/2$. If I_t is the pullback of the metric on Σ_{ρ_t} under Ep_{ρ_t}, then by [22][2],

$$\mathrm{I}_t(X,Y) = \mathrm{I}(\cosh(t)X + \sinh(t)BX, \cosh(t)Y + \sinh(t)BY).$$

Expanding out we have

$$\mathrm{I}_t = \frac{1}{4}(e^{2t}\hat{\mathrm{I}} + 2\hat{\mathrm{II}} + e^{-2t}\hat{\mathrm{III}})$$

where

$$\begin{aligned}
\hat{\mathrm{I}} &= \mathrm{I} + 2\mathrm{II} + \mathrm{III} = \mathrm{I}((\mathrm{id} + B)\cdot, (\mathrm{id} + B)\cdot)\\
\hat{\mathrm{II}} &= \mathrm{I} - \mathrm{III} = \mathrm{I}((\mathrm{id} + B)\cdot, (\mathrm{id} - B)\cdot)\\
\hat{\mathrm{III}} &= \mathrm{I} - 2\mathrm{II} + \mathrm{III} = \mathrm{I}((\mathrm{id} - B)\cdot, (\mathrm{id} - B)\cdot).
\end{aligned}$$

These are called the *fundamental forms at infinity* $\hat{\mathrm{I}}, \hat{\mathrm{II}}, \hat{\mathrm{III}}$ and it is natural then to define the *shape operator at infinity* by $\hat{B} = (\mathrm{id} + B)^{-1}(\mathrm{id} - B)$ which satisfies

$$\hat{\mathrm{II}}(X,Y) = \hat{\mathrm{I}}(\hat{B}X,Y) \qquad \hat{\mathrm{III}}(X,Y) = \hat{\mathrm{II}}(\hat{B}X,Y) = \hat{\mathrm{I}}(\hat{B}X,\hat{B}Y).$$

Further we define the *mean curvature at infinity* is $\hat{H} = \mathrm{tr}(\hat{B})/2$. These formulas can be inverted with $B = (\mathrm{id} + \hat{B})^{-1}(\mathrm{id} - \hat{B})$ and

$$\mathrm{I} = \frac{1}{4}(\hat{\mathrm{I}} + 2\hat{\mathrm{II}} + \hat{\mathrm{III}}) = \frac{1}{4}\hat{\mathrm{I}}((\mathrm{id} + \hat{B})\cdot, (\mathrm{id} + \hat{B})\cdot)$$

[2] We note that our convention for Epstein maps is slightly different from the one in [22], that our foliation Σ_{ρ_t} coincides with their foliation $S_{t-(\log 2)/2}$. Our choice is such that when ρ is the Poincaré metric on the unit disk, the Epstein surface is exactly the totally geodesic plane bounded by the unit circle. See Lemma 3.7.

$$\begin{aligned} \mathrm{I\!I} &= \frac{1}{4}(\hat{\mathrm{I}} - \hat{\mathrm{I\!I\!I}}) = \frac{1}{4}\hat{\mathrm{I}}((\mathrm{id}+\hat{B})\cdot, (\mathrm{id}-\hat{B})\cdot) \\ \mathrm{I\!I\!I} &= \frac{1}{4}(\hat{\mathrm{I}} - 2\hat{\mathrm{I\!I}} + \hat{\mathrm{I\!I\!I}}) = \frac{1}{4}\hat{\mathrm{I}}((\mathrm{id}-\hat{B})\cdot, (\mathrm{id}-\hat{B})\cdot). \end{aligned}$$

If $\Phi = g(z)\,\mathrm{d}z^2$ is a quadratic differential (g not necessarily holomorphic) and ρ a conformal metric then we define the norm of Φ with respect to ρ by

$$\|\Phi(z)\|_\rho = \frac{|\Phi(z)|}{\rho(z)}. \tag{3.2}$$

Epstein (see [15, Section 5]) gave the following description of the fundamental forms at infinity.

Theorem 3.3. *Let $\rho = e^\varphi|\mathrm{d}z|^2$ be a conformal metric on Ω. Then*

- $\hat{\mathrm{I}} = \rho$.
- *The Epstein map Ep_ρ is an immersion on $\{z \in \Omega \mid -1 \notin \{k_+, k_-\}\}$.*
- *If $\hat{K}$ is the Gaussian curvature of $\hat{\mathrm{I}}$ and $\vartheta = (\varphi_{zz} - \frac{1}{2}\varphi_z^2)\,\mathrm{d}z^2$, then*

$$\hat{\mathrm{I\!I}} = \vartheta + \overline{\vartheta} - \hat{K}\rho.$$

- *The eigenvalues of $\hat{B}$ are*

$$\hat{k}_\pm = \frac{1-k_\pm}{1+k_\pm} = -\hat{K} \pm 2\|\vartheta\|_\rho.$$

We note that we have the equation

$$\rho = \mathrm{I}((\mathrm{id}+B)\cdot, (\mathrm{id}+B)\cdot) = (\mathrm{id}+B)^*\mathrm{I}. \tag{3.3}$$

We let $\mathrm{d}\hat{a}$ be the area form for $\hat{\mathrm{I}} = \rho$, then we have the area measure on Σ_ρ satisfies

$$\mathrm{d}A = \frac{1}{4}|\det(\mathrm{id}+\hat{B})||\mathrm{d}\hat{a}|.$$

We define the *signed area* of Σ_ρ, denoted be $\mathrm{d}a$, as the area form with induced orientation by $\widetilde{\mathrm{Ep}}_\rho$, which satisfies

$$\mathrm{d}a = \frac{1}{4}\det(\mathrm{id}+\hat{B})\mathrm{d}\hat{a}.$$

Thus $\mathrm{d}A = |\mathrm{d}a|$.

Corollary 3.4. *Let Σ_ρ be the Epstein surface for $\rho = e^\varphi|\mathrm{d}z|^2$. Then at places where Ep_ρ is an immersion, we have*

- $\hat{H} = -\hat{K}$,
- $H\mathrm{d}a = \left(\frac{1-\hat{K}^2}{4} + \|\vartheta\|_\rho^2\right) d\hat{a}$,
- $\det(B)\,\mathrm{d}a = \left(\frac{(1+\hat{K})^2}{4} - \|\vartheta\|_\rho^2\right) d\hat{a}$.

Proof. As the eigenvalues of $\hat{B}$ are $\hat{k}_\pm = -\hat{K} \pm 2\|\vartheta\|_\rho$ we have

$$\operatorname{tr}(\hat{B}) = -2\hat{K} \qquad \det(\hat{B}) = \hat{K}^2 - 4\|\vartheta\|_\rho^2.$$

Thus $\hat{H} = \operatorname{tr}(\hat{B})/2 = -\hat{K}$. We also have

$$\begin{aligned} H &= \frac{1}{2}(k_1 + k_2) = \frac{1}{2}\left(\frac{1-\hat{k}_1}{1+\hat{k}_1} + \frac{1-\hat{k}_2}{1+\hat{k}_2}\right) = \frac{1-\det(\hat{B})}{\det(\mathrm{id}+\hat{B})} \\ \det(B) &= k_1 k_2 = \left(\frac{1-\hat{k}_1}{1+\hat{k}_1}\right)\left(\frac{1-\hat{k}_2}{1+\hat{k}_2}\right) = \frac{\det(\mathrm{id}-\hat{B})}{\det(\mathrm{id}+\hat{B})}. \end{aligned}$$

From this, we obtain

$$H\mathrm{d}a = \frac{1}{4}(1 - \det(\hat{B}))\mathrm{d}\hat{a} = \left(\frac{1-\hat{K}^2}{4} + \|\vartheta\|_\rho^2\right)\mathrm{d}\hat{a}$$

and

$$\det(B)\mathrm{d}a = \frac{1}{4}\det(\mathrm{id}-\hat{B})\mathrm{d}\hat{a} = \left(\frac{(1+\hat{K})^2}{4} - \|\vartheta\|_\rho^2\right)\mathrm{d}\hat{a}$$

as claimed. □

3.2 Epstein–Poincaré surface

We now consider the Epstein map associated with the Poincaré metric ρ_Ω (namely, complete and $\hat{K} \equiv -1$) on a simply connected domain $\Omega \subsetneq \mathbb{C}$, that we call the *Epstein–Poincaré map* Ep_Ω. We write similarly the Epstein–Poincaré surface as Σ_Ω. There are two connected components of the complement of a Jordan curve γ in $\hat{\mathbb{C}}$, we will study the relation between the two Epstein–Poincaré maps later in Section 4 which will be crucial to defining renormalized volume. However, let us first record some properties of a single Epstein–Poincaré map.

As the Euclidean diameter of the horosphere of $z \in \Omega$ associated with ρ_Ω goes to 0 as $z \to \partial\Omega$, the Epstein map extends to the identity map on $\partial\Omega$ (and Ep_Ω meets $\hat{\mathbb{C}}$ along $\partial\Omega$). Epstein showed that in this case, $\vartheta = \mathscr{S}(f^{-1})$ the Schwarzian quadratic differential of f^{-1}, where $f : \mathbb{D} \to \Omega$ is any conformal map. It follows from above that $\hat{B}$ has eigenvalues $1 \pm 2\|\mathscr{S}(f^{-1})\|_\Omega$ where $\|\cdot\|_\Omega = \|\cdot\|_{\rho_\Omega}$ is the norm with respect to the hyperbolic metric ρ_Ω as defined in (3.2). Thus inverting we have principal curvatures

$$k_\pm = -\frac{\|\mathscr{S}(f^{-1})\|_\Omega}{\|\mathscr{S}(f^{-1})\|_\Omega \pm 1}. \tag{3.4}$$

Applying Corollary 3.4 above we have the following result.

Theorem 3.5. *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain. Then* Ep_Ω *is an immersion on $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_\Omega \neq 1\}$. Furthermore,*

$$H\mathrm{d}a = -\det B\,\mathrm{d}a = \|\mathscr{S}(f^{-1})(z)\|_\Omega^2\,\mathrm{d}\hat{a} = |H\mathrm{d}a|.$$

Observe that since $\|\mathscr{S}(f^{-1})(z)\|_\Omega^2 \,\mathrm{d}\hat{a}$ is a smooth form defined for all points $z \in \Omega$ and the set $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_\Omega \neq 1\}$ is dense and has full measure, we can uniquely extend $H\mathrm{d}a$ to Ω as $\|\mathscr{S}(f^{-1})(z)\|_\Omega^2 \,\mathrm{d}\hat{a}$ and obtain the following corollary from the characterization (2.4).

Corollary 3.6. *A Jordan curve γ is a Weil–Petersson quasicircle if and only if*

$$\int_\Sigma H \mathrm{d}a = -\int_\Sigma \det(B)\,\mathrm{d}a < \infty.$$

From the formula of the principal curvatures (3.4), we obtain immediately the following explicit example of Epstein–Poincaré surface.

Lemma 3.7. *The Epstein–Poincaré surface associated with $\mathbb{D}$ (take $f = \mathrm{id}_\mathbb{D}$) is the totally geodesic plane bounded by $\partial\mathbb{D}$ and $\mathrm{Ep}_\mathbb{D}(0) = (0,0,1)$ in the upper half-space model. From the naturality of Epstein map, if M is a Möbius transformation (which extends to an isometry of $\mathbb{H}^3$ as we explain below), we have $\mathrm{Ep}_{M(\mathbb{D})} \circ M = M \circ \mathrm{Ep}_\mathbb{D}$.*

Remark 3.8. In particular, the Euclidean radius of the horosphere associated with $\rho = 4|\mathrm{d}z|^2$ is $1/2$. From this, we obtain that more generally, the Euclidean radius of the horosphere associated with ρ is $1/\sqrt{\rho}$.

Although totally geodesic planes are trivial examples of Epstein–Poincaré surfaces, the other Epstein–Poincaré surfaces have an elegant description in terms of these maps associated with the geodesic planes and osculating Möbius transformations (Lemma 3.9).

Let us first recall the classical result about extending a Möbius transformation to an isometry of $\mathbb{H}^3$. For this, we use the upper half-space model and use quaternions to parametrize $\mathbb{H}^3 = \mathbb{C} \oplus \mathfrak{j}\mathbb{R}_+ = \{z + \mathfrak{j}t \mid z = x + \mathfrak{i}y \in \mathbb{C}, t > 0\}$ (so that (x,y,t) in the upper-half space is identified with $z + \mathfrak{j}t$). A Möbius transformation $z \mapsto \frac{az+b}{cz+d}$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{PSL}_2(\mathbb{C})$ extends to the isometry of $\mathbb{H}^3$ by

$$Z \mapsto (aZ+b)(cZ+d)^{-1}, \quad \forall Z = z + \mathfrak{j}t$$

using the multiplication on quaternions, see [1, Sec. 2.1] for more details.

Since the Epstein map depends on the metric and its derivatives at infinity (Theorem 3.2), the Epstein–Poincaré map $\mathrm{Ep}_\Omega \circ f(z_0)$ depends only on the two-jet of f at z_0 (namely the values of $f(z_0), f'(z_0)$, and $f''(z_0)$). There exists a unique Möbius transformation M_{f,z_0} with the same two-jet as f at z_0, called the *osculating Möbius transformation of f at z_0*. Therefore,

$$\mathrm{Ep}_\Omega \circ f(z_0) = \mathrm{Ep}_{M_{f,z_0}(\mathbb{D})} \circ M_{f,z_0}(z_0).$$

From the naturality of the Epstein map (Lemma 3.1) we have

$$\mathrm{Ep}_{M_{f,z_0}(\mathbb{D})} \circ M_{f,z_0}(z_0) = M_{f,z_0} \circ \mathrm{Ep}_\mathbb{D}(z_0).$$

Summarizing, we have proved (assuming $z_0 = 0$) and using Lemma 3.7:

Lemma 3.9. *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and $f : \mathbb{D} \to \Omega$ be a univalent map. We have $\mathrm{Ep}_\Omega \circ f(0) = M_{f,0} \circ \mathrm{Ep}_\mathbb{D}(0) = M_{f,0}(\mathfrak{j})$.*

We now state equivalent conditions for a curve to be asymptotically conformal.

Theorem 3.10. *(See [29, Thm. 11.1]) Let f be a conformal map from $\mathbb{D}$ onto a domain bounded by a Jordan curve γ. The following are equivalent:*

- *(AC1)* γ *is asymptotically conformal;*
- *(AC2)* $\lim_{|\zeta|\to 1-} \frac{f''(\zeta)}{f'(\zeta)}(1-|\zeta|^2) = 0$;
- *(AC3)* $\lim_{|\zeta|\to 1-} \|\mathscr{S}(f)\|_{\mathbb{D}}(\zeta) = 0$.

From now on, we will assume that the boundary of Ω is asymptotically conformal and use a few classical results from geometric function theory.

Example 3.11. Weil–Petersson quasicircles satisfy AC3 (see [39, Corollary II.1.4]) and are therefore asymptotically conformal.

The following is a simple consequence of the Koebe 1/4 theorem.

Theorem 3.12. *(See [29, Cor. 1.4]) Let γ be a Jordan curve bounding Ω and ρ_Ω the hyperbolic metric on Ω. Then*

$$\frac{1}{2d(z,\gamma)} \leq \sqrt{\rho_\Omega(z)} \leq \frac{2}{d(z,\gamma)}$$

where $d(\cdot,\cdot)$ denotes the Euclidean distance in $\mathbb{R}^2$.

Using this we obtain the following control over the behavior of the Epstein map.

Corollary 3.13. *Let γ be a Jordan curve bounding Ω. Then Ep_Ω extends continuously to the identity on γ with*

$$(\sqrt{5}-2)d(z,\gamma) \leq d(\mathrm{Ep}_\Omega(z),\gamma) \leq 5d(z,\gamma)$$

where $d(\cdot,\cdot)$ denotes the Euclidean distance in $\mathbb{R}^3$. Furthermore if $\mathrm{Ep}_\Omega(z) = (Z(z),\xi(z)) \in \mathbb{C}\times\mathbb{R}_+$, then

$$\frac{1}{5}d(z,\gamma) \leq \xi(z) \leq 4d(z,\gamma).$$

Proof. Let $s = d(z,\gamma)$. Since $\mathrm{Ep}_\Omega(z)$ is on the boundary of a horosphere of Euclidean radius $r = 1/\sqrt{\rho(z)}$ based at z by Remark 3.8, Theorem 3.12 shows that

$$s/2 \leq r \leq 2s.$$

We let $z_i \to z \in \gamma$. Then

$$d(\mathrm{Ep}_\Omega(z_i), z) \leq 2r + d(z_i,z) \leq 4d(z_i,\gamma) + d(z_i,z) \leq 5d(z_i,z).$$

Therefore $\mathrm{Ep}_\Omega(z_i) \to z$ giving Ep_Ω extends continuously to the identity on γ.
By the triangle inequality we also have

$$\sqrt{s^2+r^2} - r \leq d(\mathrm{Ep}_\Omega(z),\gamma) \leq 2r + s.$$

Thus

$$(\sqrt{5}-2)s \leq d(\mathrm{Ep}_\Omega(z), \gamma) \leq 5s.$$

To bound ξ, let $z_0 \in \Omega$ and $f : \mathbb{D} \to \Omega$ a uniformizing map with $f(0) = z_0$. Further by post-composition by a translation, it suffices to consider $z_0 = 0$. Thus the osculating Möbius map M of f at 0 (i.e. the Möbius map with the same 2-jet at 0) is

$$M(\zeta) = \frac{\alpha\zeta}{\beta\zeta + 1/\alpha}$$

where $\alpha^2 = f'(0)$ and $2\alpha\beta = -f''(0)/f'(0)$. By Lemma 3.9 we have

$$\mathrm{Ep}_\Omega \circ f(0) = M(\mathfrak{j}) = \alpha\mathfrak{j}(\beta\mathfrak{j} + 1/\alpha)^{-1} = \frac{\alpha\overline{\beta} + \mathfrak{j}}{|\beta|^2 + \frac{1}{|\alpha|^2}}.$$

Thus

$$\xi(z_0) = \frac{|\alpha|^2}{|\alpha\beta|^2 + 1} = \frac{|f'(0)|}{1 + \left|\frac{f''(0)}{2f'(0)}\right|^2}. \tag{3.5}$$

As f is univalent, by the Bieberbach theorem (see [3]) then $|f''(0)| \leq 4|f'(0)|$. We also know that $1/r^2 = \rho(z_0) = 4/|f'(0)|^2$. Thus we have $|f'(0)| = 2r$ and

$$\frac{s}{5} \leq \frac{2r}{5} = \frac{|f'(0)|}{5} \leq \xi(z_0) \leq |f'(0)| = 2r \leq 4s.$$

The result follows. □

Corollary 3.14. *If Ω is bounded by an asymptotically conformal curve γ then Ep_Ω is an immersion and embedding in a neighborhood of $\partial\Omega$.*

Proof. We note that $\|\mathscr{S}(f^{-1})(f(\zeta))\|_\Omega = \|\mathscr{S}(f)(\zeta)\|_{\mathbb{D}}$. Since γ is asymptotically conformal, Theorem 3.5 and (AC3) imply that Ep_Ω is an immersion in a neighborhood of $\partial\Omega$.

By Corollary 3.13 Ep_Ω extends to the identity on $\partial\Omega$. If Ep_Ω is not an embedding in a neighborhood of $\partial\Omega$ then there exists sequences $x_i \to u, y_i \to v$ with $x_i \neq y_i$ but $\mathrm{Ep}_\Omega(x_i) = \mathrm{Ep}_\Omega(y_i)$ and $u, v \in \partial\Omega$. Then we have $u = v$.

We now obtain our contradiction to $\mathrm{Ep}_\Omega(x_i) = \mathrm{Ep}_\Omega(y_i)$ for all i. We let $A_\delta := \{f(\zeta) \in \Omega\colon 1 - |\zeta| < \delta\}$. Given any $\epsilon > 0$ we can choose δ such that $\|\mathscr{S}(f^{-1})\| < \epsilon$ on A_δ. For sufficiently large i we have that the geodesic arc γ_i joining x_i to y_i is in A_δ. Further for any fixed $r_0 > 0$, for sufficiently large i the r_0-neighborhood of the geodesic arc γ_i is in A_δ. Therefore as $\|\mathscr{S}(f^{-1})\| < \epsilon$ on A_δ by [9, Lemma 3.5], then for $r_0 \leq 1/2$ the curve $\mathrm{Ep}_\Omega \circ \gamma_i$ has geodesic curvature less than $\kappa = \frac{3\epsilon}{2r_0(1-\epsilon)^2}$ in $\mathbb{H}^3$. A standard fact about hyperbolic space is that any smooth curve with geodesic curvature bounded above by 1 is embedded (see, for example [9, Lemma 3.6]). It follows that by choosing ϵ, r_0 such that $\kappa \leq 1$ then $\mathrm{Ep}_\Omega(x_i) \neq \mathrm{Ep}_\Omega(y_i)$, a contradiction. □

3.3 Explicit expression of Epstein maps in the upper-space model

For concreteness, we also mention that in the upper-space model, Epstein maps have explicit expressions as derived in [15, 22]. We collect them here for the readers' convenience. We choose to include the simple derivations or examples to be specific about our conventions which is slightly different from [22] as we mentioned before.

Let $\rho = e^{\varphi}|\mathrm{d}z|^2$ be a smooth conformal metric on an open set $U \subset \mathbb{C}$. The Epstein map $\mathrm{Ep}_\rho : z \in U \mapsto (Z, \xi) \in \mathbb{C} \times \mathbb{R}_+ = \mathbb{H}^3$ is given explicitly by

$$\xi = \frac{2e^{-\varphi/2}}{1 + |\varphi_{\bar z}|^2 e^{-\varphi}}, \qquad Z = z + \frac{2\varphi_{\bar z} e^{-\varphi}}{1 + |\varphi_{\bar z}|^2 e^{-\varphi}} = z + \xi \cdot \psi, \tag{3.6}$$

where

$$\psi := \varphi_{\bar z} e^{-\varphi/2}, \qquad \varphi_{\bar z} = \partial_{\bar z}\varphi.$$

The Epstein Gauss map is $\widetilde{\mathrm{Ep}}_\rho \colon U \subset \mathbb{C} \to T_1\mathbb{H}^3$ such that the base point is Ep_ρ and the vector component is $\xi \overrightarrow{\eta}$ where

$$\overrightarrow{\eta} = \left(\frac{2\varphi_{\bar z} e^{-\varphi/2}}{1 + |\varphi_{\bar z}|^2 e^{-\varphi}}, \frac{1 - |\varphi_{\bar z}|^2 e^{-\varphi}}{1 + |\varphi_{\bar z}|^2 e^{-\varphi}} \right) = \left(\frac{2\psi}{1 + |\psi|^2}, \frac{1 - |\psi|^2}{1 + |\psi|^2} \right) \tag{3.7}$$

is a Euclidean normal vector. It is straightforward to check that the geodesic flow $\mathfrak{g}_t(\widetilde{\mathrm{Ep}}_\rho(z)) \in T_1\mathbb{H}^3$ satisfies

$$\mathfrak{g}_{-t}(\widetilde{\mathrm{Ep}}_\rho(z)) = \widetilde{\mathrm{Ep}}_{e^{2t}\rho}(z),$$

and the base point of $\mathfrak{g}_{-t}(\widetilde{\mathrm{Ep}}_\rho(z))$ tends to z as $t \to \infty$.

Example 3.15.

- If $\varphi \equiv 2t$, then for all z,

$$\mathrm{Ep}_\varphi(z) = (z, 2e^{-t}) \qquad \overrightarrow{\eta} = (0, 1).$$

- If $e^{\varphi} = \frac{4}{(1+|z|^2)^2}$, then for all $z \in \mathbb{C}$, $(Z, \xi) = (0, 1)$.
- If $\varphi = \log 4 - 2\log(1 - |z|^2)$, i.e., $\rho = e^{\varphi}|\mathrm{d}z|^2$ is the hyperbolic metric in $\mathbb{D}$, then for $z = re^{\mathrm{i}\theta} \in \mathbb{D}$,

$$\mathrm{Ep}_\rho(re^{\mathrm{i}\theta}) = \left(\frac{2r}{1 + r^2} e^{\mathrm{i}\theta}, \frac{1 - r^2}{1 + r^2} \right) = \overrightarrow{\eta}.$$

  This is consistent with Lemma 3.7 (and one of the advantage of choosing this convention is) that the Epstein–Poincaré map Ep_ρ maps $\mathbb{D}$ onto the totally geodesic plane in $\mathbb{H}^3$ bounded by $\partial\mathbb{D}$.

More generally, we have the following explicit formula for the Epstein–Poincaré map associated with a simply connected domain Ω.

Lemma 3.16. *Let $f : \mathbb{D} \to \Omega$ be a conformal map and $\mathrm{Ep}_\Omega : \Omega \to \Sigma_\Omega$ the Epstein–Poincaré map. For $z = f(\zeta)$, $\zeta \in \mathbb{D}$, we have*

$$\begin{aligned}
\psi(z) = \varphi_{\overline{z}} e^{-\varphi/2} &= \frac{|f'(\zeta)|}{f'(\zeta)} \left(-\overline{\frac{f''(\zeta)}{f'(\zeta)}} \frac{(1-|\zeta|^2)}{2} + \zeta \right), \\
e^{-\varphi(z)/2} &= \frac{1}{2} |f'(\zeta)| (1-|\zeta|^2) \\
\xi(z) = \frac{2e^{-\varphi/2}}{1+|\psi|^2} &= \frac{|f'(\zeta)|(1-|\zeta|^2)}{1 + |-\frac{f''(\zeta)}{f'(\zeta)} \frac{(1-|\zeta|^2)}{2} + \overline{\zeta}|^2}, \\
Z(z) = z + \xi \cdot \psi = f(\zeta) &+ \frac{\left(-\overline{\frac{f''(\zeta)}{f'(\zeta)}} \frac{(1-|\zeta|^2)}{2} + \zeta \right) f'(\zeta)(1-|\zeta|^2)}{1 + |-\frac{f''(\zeta)}{f'(\zeta)} \frac{(1-|\zeta|^2)}{2} + \overline{\zeta}|^2}.
\end{aligned}$$

We verify from the explicit formulas that the expression of $\mathrm{Ep}_\Omega(f(0))$ coincides with the one in (3.5) and that the Epstein map Ep_Ω extends continuously to γ as the identity map using the bounds

$$\left| \frac{(1-|\zeta|^2)}{2} \frac{f''(\zeta)}{f'(\zeta)} - \overline{\zeta} \right| \leq 2, \qquad (1-|\zeta|^2) \left| f'(\zeta) \right| \leq 4 \operatorname{dist}(f(\zeta), \gamma). \tag{3.8}$$

See [29, Prop. 1.2, Cor. 1.4].

4 Renormalized volume for a Jordan curve

In this section, let $\gamma \subset \hat{\mathbb{C}}$ be a Jordan curve and Ω, Ω^* be the connected component of $\hat{\mathbb{C}} \smallsetminus \gamma$. Let Ep_Ω (resp. Ep_{Ω^*}) be the Epstein–Poincaré map associated with Ω (resp, Ω^*). We write as before Σ_Ω and Σ_{Ω^*} for their images.

4.1 Disjoint Epstein–Poincaré surfaces

When γ is a circle, both Epstein surfaces coincide with the geodesic plane bounded by γ by Lemma 3.7. We now show that in all other cases, the two Epstein surfaces of a Jordan curve are disjoint. We need the following special case of Grunsky's inequality, see, e.g., [28, Thm. 4.1, (21)] for the proof.

Lemma 4.1 (Consequence of Grunsky inequality)**.** *Suppose that $f : \mathbb{D} \to \mathbb{C}$ and $g : \mathbb{D}^* \to \hat{\mathbb{C}}$ are univalent functions on $\mathbb{D}$ and $\mathbb{D}^*$ such that $f(0) = 0$ and $g(\infty) = \infty$, and $f(\mathbb{D}) \cap g(\mathbb{D}^*) = \emptyset$. Then we have*

$$\int_{\mathbb{D}} \left| \frac{f'(z)}{f(z)} - \frac{1}{z} \right|^2 \mathrm{d}^2 z + \int_{\mathbb{D}^*} \left| \frac{g'(z)}{g(z)} - \frac{1}{z} \right|^2 \mathrm{d}^2 z \leq 2\pi \log \left| \frac{g'(\infty)}{f'(0)} \right|$$

where $g'(\infty) = \lim_{z\to\infty} g'(z)$. Equality holds if $\mathbb{C} \smallsetminus \{f(\mathbb{D}) \cup g(\mathbb{D}^*)\}$ has zero Lebesgue measure.

Applying this we obtain the following result.

Proposition 4.2. *If γ is not a circle, then Σ_Ω and Σ_{Ω^*} are disjoint.*

Proof. We consider $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ univalent maps. We show that for $z \in \mathbb{D}, w \in \mathbb{D}^*$ the horospheres at $f(z), g(w)$ associated with the metrics ρ_Ω and ρ_{Ω^*} respectively are disjoint. By pre-composition and post-composition by Möbius transformations we can assume $z = 0, w = \infty$ and $f(0) = 0, g(\infty) = \infty$. By Remark 3.8, the Euclidean diameter of the horosphere at $f(0) = 0$ is $|f'(0)|$ and the horosphere at $g(\infty) = \infty$ is the plane at Euclidean height $|g'(\infty)|$ (this can be seen by considering the map $z \mapsto 1/g(1/z)$). Thus they are disjoint if $|g'(\infty)| > |f'(0)|$. This follows from Lemma 4.1 above. $\square$

4.2 Volume between the Epstein–Poincaré surfaces

Let $\gamma \subset \hat{\mathbb{C}}$ be an asymptotically conformal Jordan curve. We now define the volume between Σ_Ω and Σ_{Ω^*}. Without loss of generality, we assume that γ does not contain $\infty \in \hat{\mathbb{C}}$ and use the upper half-space model of $\mathbb{H}^3$. We cautiously note that both Epstein–Poincaré surfaces are non-compact and may not be embedded everywhere. For this reason we use an approximation to compute the volume. For $\varepsilon > 0$, let

$$\mathrm{vol}_\varepsilon = \mathbf{1}_{\xi \geq \varepsilon} \frac{\mathrm{vol}_{eucl}}{\xi^3}$$

where vol_{eucl} is the Euclidean volume form.

Let φ_γ be a continuous map $\overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$, such that $\varphi_\gamma|_\Omega = \mathrm{Ep}_\Omega$, $\varphi_\gamma|_{\Omega^*} = \mathrm{Ep}_{\Omega^*}$, and $\varphi_\gamma|_{\mathbb{H}^3}$ is differentiable. This is possible since Ep_Ω and Ep_{Ω^*} extend to the identity map on γ. We define

$$V_2(\gamma)(\varepsilon) := \int_{\mathbb{H}^3} \varphi_\gamma^* \,\mathrm{vol}_\varepsilon \,.$$

This is the *signed volume between the Epstein surfaces* bounded by γ above level ε.

Since the boundary values of φ_γ are determined and $\varphi_\gamma(\overline{\mathbb{H}^3}) \cap \{(Z, \xi) \colon \xi \geq \varepsilon\}$ is compact, we have $V_2(\gamma)(\varepsilon)$ is finite and independent of the choice of φ_γ. Since both Epstein surfaces are disjoint (unless γ is a circle) by Proposition 4.2 and embedded near the boundary by Corollary 3.14, without loss of generality, we assume further more that the Jacobian of φ_γ is positive in a neighborhood U_γ of γ in $\mathbb{H}^3$. (If γ is a circle, then we choose φ_γ such that the Jacobian is zero.) The limit

$$V(\gamma) := \lim_{\varepsilon \to 0+} V_2(\gamma)(\varepsilon) \in (-\infty, \infty] \tag{4.1}$$

exists since $\int_{U_\gamma} \varphi_\gamma^* \,\mathrm{vol}_\varepsilon$ increases as $\varepsilon \to 0+$ and $\int_{\mathbb{H}^3 \smallsetminus U_\gamma} \varphi_\gamma^* \,\mathrm{vol}_\varepsilon$ is constant for small enough ε. The monotonicity and (3.1) also show that the limit is invariant under actions of elements in $\mathsf{PSL}_2(\mathbb{C})$ which do not send any point of γ to $\infty \in \hat{\mathbb{C}}$.

Definition 4.3. For an asymptotically conformal Jordan curve $\gamma \subset \hat{\mathbb{C}}$, we define *the signed volume between the Epstein–Poincaré surfaces* $V(\gamma)$ to be the limit in (4.1) applied to the curve $A(\gamma)$, where A is any element in $\mathsf{PSL}_2(\mathbb{C})$ such that $A(\gamma)$ does not passes through ∞.

The above definition is clearly $\mathsf{PSL}_2(\mathbb{C})$-invariant.

4.3 Volume for smooth Jordan curves

In this subsection we will see if the Jordan curve γ is sufficiently smooth, then the map Ep_Ω extends not only continuously to γ but also osculates to the totally geodesic plane bounded by the circle osculating to γ in $\mathbb{C}$. This will be useful later to prove that the volume between Ep_Ω and Ep_{Ω^*} is finite, if γ is sufficiently smooth.

If γ is $C^{4,\alpha}$ for some $0<\alpha<1$, Kellogg's theorem (see, e.g., [16, Thm. II.4.3]) implies that the conformal map $f:\mathbb{D}\to\Omega$ extends to a $C^{4,\alpha}$ homeomorphism $\overline{\mathbb{D}}\to\overline{\Omega}$. Hence by Lemma 3.16 the Epstein map Ep_Ω extends to $\partial\Omega$ as a $C^{2,\alpha}$ map. We define the *osculating circle* $\mathscr{C}_\gamma(\theta)$ at $\gamma(e^{\mathrm{i}t})$ to be the circle tangent to γ at $\gamma(e^{\mathrm{i}t})$ and with the same curvature as γ at $\gamma(e^{\mathrm{i}t})$. We then define he *osculating plane* $\mathscr{P}_\gamma(\theta)$ to be the geodesic plane in $\mathbb{H}^3$ so that the boundary of $\mathscr{P}_\gamma(\theta)$ is $\mathscr{C}_\gamma(\theta)$.

Proposition 4.4. *Let γ be a $C^{4,\alpha}$ Jordan curve in $\mathbb{C}$ for some $0<\alpha<1$. Then Ep_Ω and $\mathscr{P}_\gamma(\theta)$, viewed as surfaces in $\mathbb{R}^3$, are tangent at $(\gamma(e^{\mathrm{i}t}),0)$ and agree up to second order.*

Proof. We pick a point $z_0=x_0+\mathrm{i}y_0\in\gamma=\partial\Omega$. By post-composing by a Möbius transformation, we can assume that $z_0=0$. Let $f:\mathbb{H}\to\Omega$ be a conformal map, where $\mathbb{H}=\{\zeta=u+\mathrm{i}v\colon v>0\}$ denotes the half-plane. Without loss of generality, we assume that $f(0)=0$, $f'(0)=1$ and $f''(0)=0$. Thus the plane bounded by the osculating circle (which is the line $\{y=0\}\subset\mathbb{C}$) at z_0 is the plane $\{y=0\}\subset\mathbb{C}\times\mathbb{R}_+=\{x+\mathrm{i}y+\mathrm{j}\xi\colon\xi>0\}$. We now show that $\mathrm{Ep}_\Omega(f(\zeta))=u+\mathrm{j}v+O(|\zeta|^3)$ for $|\zeta|$ small, which would imply $\mathrm{Ep}_\Omega(x+\mathrm{i}y)=x+\mathrm{j}y+O(|x+\mathrm{i}y|^3)$ and thus the statement follows.

We take $\zeta=u+\mathrm{i}v\in\mathbb{H}$. We let $g_\zeta:\mathbb{D}\to\Omega$ uniformize Ω with $g_\zeta(0)=f(\zeta)$. The osculating Möbius transformation M_ζ for g_ζ at 0 is

$$M_\zeta(w)=f(\zeta)+\frac{\alpha w}{\beta w+1/\alpha}$$

with $\alpha^2=g_\zeta'(0)$ and $2\alpha\beta=-g_\zeta''(0)/g_\zeta'(0)$. Then by Lemma 3.9 we have

$$\mathrm{Ep}_\Omega(f(\zeta))=M_\zeta(\mathrm{j})=f(\zeta)+\frac{\alpha\overline{\beta}+\mathrm{j}}{|\beta|^2+\frac{1}{|\alpha|^2}}=f(\zeta)+\frac{\alpha^2(\overline{\alpha\beta})+|\alpha|^2\mathrm{j}}{|\alpha\beta|^2+1}.$$

We choose $g_\zeta=f\circ\varphi_\zeta$ where $\varphi_\zeta:\mathbb{D}\to\mathbb{H}$ is given by

$$\varphi_\zeta(w)=u+\mathrm{i}v\left(\frac{1-w}{1+w}\right).$$

Thus as $\varphi_\zeta(0)=\zeta$, $\varphi_\zeta'(0)=-2\mathrm{i}v$, $\varphi_\zeta''(0)=4\mathrm{i}v$,

$$\begin{aligned}
\alpha^2 &= g_\zeta'(0)=f'(\zeta)\varphi_\zeta'(0)=-2\mathrm{i}vf'(\zeta)=-2\mathrm{i}v+O(|\zeta|^3)\\
g_\zeta''(0) &= f'(\zeta)\varphi_\zeta''(0)+f''(\zeta)\varphi_\zeta'(0)^2=4\mathrm{i}v(f'(\zeta)+\mathrm{i}vf''(\zeta))=4\mathrm{i}v+O(|\zeta|^3)\\
\alpha\beta &= -\frac{g_\zeta''(0)}{2g_\zeta'(0)}=1+\mathrm{i}v\frac{f''(\zeta)}{f'(\zeta)}=1+O(|\zeta|^2).
\end{aligned}$$

As $f(\zeta)=\zeta+O(|\zeta|^3)$ then a straightforward computation shows

$$\mathrm{Ep}_\Omega(f(\zeta))=\zeta+\frac{1}{2}(-2\mathrm{i}v+2\mathrm{j}v)+O(|\zeta|^3)=u+\mathrm{j}v+O(|\zeta|^3)$$

as claimed. $\square$

Next we show that for sufficiently regular curves γ this volume is in fact finite.

Proposition 4.5. *Let γ be a $C^{5,\alpha}$ Jordan curve in $\mathbb{C}$ for some $0<\alpha<1$. Then $V(\gamma)$ is finite.*

Proof. Without loss of generality, we assume that γ is parametrized by arc-length as a function $\mathbb{S}^1\to\mathbb{C}$. Take φ_γ some continuous map $\overline{\mathbb{H}^3}\to\overline{\mathbb{H}^3}$ as before, i.e., $\varphi_\gamma|_\Omega=\mathrm{Ep}_\Omega$, $\varphi_\gamma|_{\Omega^*}=\mathrm{Ep}_{\Omega^*}$, and $\varphi_\gamma|_{\mathbb{H}^3}$ is differentiable. We take the following $C^{4,\alpha}$ parametrization of a neighbourhood U of γ in $\overline{\mathbb{H}^3}$, denoted $G:\mathbb{S}^1_s\times\overline{\mathbb{H}^2}_{(a,b)}\to\overline{\mathbb{H}^3}$, by

$$G(s,a,b)=\gamma(s)+\mathrm{i}a\gamma'(s)+\mathrm{j}b. \tag{4.2}$$

It is a straightforward calculation to see that the hyperbolic metric in G-coordinates is given by

$$\frac{(1-ak(s))^2}{b^2}\mathrm{d}s^2+\frac{1}{b^2}\mathrm{d}a^2+\frac{1}{b^2}\mathrm{d}b^2,$$

where $k(s)$ is the signed curvature of γ given by $\gamma''(s)=\mathrm{i}k(s)\gamma'(s)$. Hence the volume form is given by

$$\frac{(1-ak(s))}{b^3}\,\mathrm{d}s\,\mathrm{d}a\,\mathrm{d}b.$$

If we assume that γ is $C^{5,\alpha}$, then the Epstein–Poincaré surfaces are $C^{3,\alpha}$ up to the boundary. This means that there are $C^{3,\alpha}$ functions $a_\Omega,a_{\Omega^*}:\mathbb{S}^1_s\times[0,\varepsilon_0]_b\to\mathbb{R}$ so that the Epstein–Poincaré surfaces in the neighbourhood U of γ are parametrized by $G(s,a_\Omega(s,b),b)$, $G(s,a_{\Omega^*}(s,b),b)$. And since by Proposition 4.4 the Epstein–Poincaré surfaces agree up to second order at γ, then there exists a constant $C>0$ so that $|a_\Omega(s,b)-a_{\Omega^*}(s,b)|\le Cb^3$.

Hence for small enough neighborhood U of γ, the integral $\int_U\varphi_\gamma^*\,\mathrm{vol}$ will be bounded by

$$V_1(\gamma)(\varepsilon_0):=\int_{\mathbb{S}^1}\int_0^{\varepsilon_0}\int_{a_\Omega(s,z)}^{a_{\Omega^*}(s,z)}\frac{(1-ak(s))}{b^3}\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}s.$$

This integral is well-defined and convergent since

$$\left|\int_{a_\Omega(s,b)}^{a_{\Omega^*}(s,b)}\frac{(1-ak(s))}{b^3}\mathrm{d}a\right|=\frac{1}{b^3}\Big|a_{\Omega^*}(s,b)-a_\Omega(s,b)\Big|.\left|\left(\frac{a_\Omega(s,b)+a_{\Omega^*}(s,b)}{2}\right)k(s)-1\right|$$

is bounded by a constant independent of (s,b). Hence $V(\gamma)=\lim_{\varepsilon\to0+}V_2(\gamma)(\varepsilon)$ is a finite real value. $\square$

Definition 4.6. Let γ be a Weil–Petersson quasicircle in $\mathbb{C}$. Then we define $V_R(\gamma)$, the renormalized volume of γ, as

$$\begin{aligned}V_R(\gamma)&:=V(\gamma)-\frac12\int_{\Sigma_\Omega\cup\Sigma_{\Omega^*}}H\mathrm{d}a\\&=V(\gamma)-\frac12\int_\Omega\|\mathscr{S}(f^{-1})\|^2(z)\rho_\Omega(z)\mathrm{d}^2z-\frac12\int_{\Omega^*}\|\mathscr{S}(g^{-1})\|^2(z)\rho_{\Omega^*}(z)\mathrm{d}^2z.\end{aligned}\tag{4.3}$$

Remark 4.7. The second identity follows from Theorem 3.5. A priori, $V_R(\gamma) \in (-\infty, \infty]$ as $V(\gamma) \in (-\infty, \infty]$ and the integrals of mean curvature are finite by Corollary 3.6. Proposition 4.5 shows that if γ is $C^{5,\alpha}$, then $V_R(\gamma) < \infty$. From the $\mathsf{PSL}_2(\mathbb{C})$-invariance of each summand in (4.3) we can easily see that V_R is $\mathsf{PSL}_2(\mathbb{C})$-invariant.

5 Universal Liouville action as renormalized volume

Our objective in this section is to prove that the renormalized volume in Definition 4.6 agrees up to a constant with the Loewner energy for $C^{5,\alpha}$ curves.

5.1 Schläfli formula for the variation of the volume

We say that $(\gamma_t)_{t\in[0,1]}$ is a $C^{k,\alpha}$ family of Jordan curves in $\mathbb{C}$ for some $k \geq 1$, $0 < \alpha < 1$, if γ_t is $C^{k,\alpha}$ for all $t \in [0,1]$, and we can choose the conformal maps $f_t : \mathbb{D} \to \Omega_t$ to be jointly $C^{k,\alpha}$ for $(t,z) \in [0,1] \times \overline{\mathbb{D}}$ and $g_t : \mathbb{D}^* \to \Omega_t^*$ to be $C^{k,\alpha}$ for $(t,z) \in [0,1] \times \overline{\mathbb{D}^*}$.

Example 5.1. Let $\gamma = \gamma_1$ be a $C^{k,\alpha}$ Jordan curve in $\mathbb{C}$ for some $k \geq 1$, $0 < \alpha < 1$, and D be a small round disk in Ω and $A = \Omega \smallsetminus D$. Then there is $0 < r < 1$ such that the round annulus $A_r = \{z \in \mathbb{C} \,|\, r < |z| < 1\}$ is conformally equivalent to A (sending $\mathbb{S}^1$ onto γ and $r\mathbb{S}^1$ onto ∂D). It is easy to see using Kellogg's theorem that any conformal map $F : A_r \to A$ is $C^{k,\alpha}$ on $\overline{A_r}$. The family of Jordan curves $(\gamma_t = F((r + t(1-r))\mathbb{S}^1))_{t\in[0,1]}$ is a $C^{k,\alpha}$ family of Jordan curves. This can be seen using [31].

Theorem 5.2. *Let $(\gamma_t)_{t\in[0,1]}$ be a $C^{5,\alpha}$ family of Jordan curves for some $0 < \alpha < 1$. Then the first derivative of the volume $V(\gamma_t)$ is computed by*

$$\frac{\partial}{\partial t}\bigg|_{t=0} V(\gamma_t) = \int_\Omega \mathrm{Ep}_\Omega^* \left(\delta H + \frac{1}{4}\langle \delta \mathrm{I}, \mathrm{I\!I}\rangle \,\mathrm{d}a\right) + \int_{\Omega^*} \mathrm{Ep}_{\Omega^*}^* \left(\delta H + \frac{1}{4}\langle \delta \mathrm{I}, \mathrm{I\!I}\rangle \,\mathrm{d}a\right)$$

where $\mathrm{Ep}_\Omega, \mathrm{Ep}_{\Omega^*}$ are the Epstein–Poincaré maps of Ω, Ω^* (respectively); $\mathrm{I}, \mathrm{I\!I}, H, \mathrm{d}a$ are the metric, second fundamental form, mean curvature and area form on Σ_Ω and Σ_{Ω^*}; and δ denotes first order variation.

The proof of this theorem when the Epstein–Poincaré surfaces are immersions everywhere is completed in Theorem 5.7. We postpone the proof of the non-immersed case to Section 8.

In order to prove this, we will decompose the region between the Epstein surfaces into two subregions and analyse the variation on both. This will require us to use a Schläfli formula — the key ingredient — for the variation of regions with piecewise smooth boundary.

We consider $F : \overline{\mathbb{B}} \to \mathbb{H}^3$ a parametrization of a region R by the closed unit 3-ball $\overline{\mathbb{B}}$ such that F is an immersion in $\mathbb{B}$ and the boundary map $E : \hat{\mathbb{C}} \to \mathbb{H}^3$ is piecewise smooth. We let V be the hyperbolic volume of R defined as $\int_{\mathbb{B}} F^* \mathrm{vol}_{hyp}$. For our purposes, we can assume that E is piecewise smooth on two disjoint smooth simply connected domains Ω, Ω^* and the annulus A between them. Further wherever E is an immersion we let B be the pullback of the shape operator by E. Now we consider a smooth variation of E by maps F_t, E_t with the same decomposition of $\hat{\mathbb{C}}$ and let $\xi = \partial_t E_t|_{t=0}$ be the vector field on ∂R describing this variation.

We have the following version of Schläfli.

Theorem 5.3 (See [36, Thm 4])**.** *Let F_t, E_t be variations of F, E as above such that E is an immersion on Ω, Ω^*, and A. Then the variation of volume satisfies*

$$2\partial_t V|_{t=0} = \int_{\Omega\cup\Omega^*\cup A} \operatorname{tr}\langle \nabla_\xi(B\cdot), DE\cdot\rangle\, E^*\mathrm{da} + \left(\int_{\partial\Omega} \frac{\partial\theta}{\partial t} E^*\mathrm{d}\ell + \int_{\partial\Omega^*} \frac{\partial\theta^*}{\partial t} E^*\mathrm{d}\ell\right)$$

where tr *denotes the trace of a* 2*-tensor with respect to the induced metric,* da *denotes the induced area form, θ is the exterior dihedral angle between the regions Ω, A, θ^* is the exterior dihedral angle between the regions Ω^*, A, and* dℓ *is the length measure of $\partial\Omega$.*

The referenced variational formula in [36] differs from the above in the first term on the right-hand side. We obtain our formula above using the following lemma.

Lemma 5.4 (See [36, Eq. (3.1) and Prop. 5])**.** *At points where E is an immersion, the form $\frac{1}{2}\operatorname{tr}\langle \nabla_\xi(B\cdot), DE\cdot\rangle\, E^*\mathrm{da}$ agrees with the pullback by E of the form $(\delta H + \frac{1}{4}\langle \delta\mathrm{I}, \mathrm{I\!I}\rangle)\,\mathrm{da}$.*

We now describe the decomposition we will use for $C^{5,\alpha}$ curves.

5.2 Decomposition

We consider a $C^{5,\alpha}$ family $(\gamma_t)_{t\in[0,1]}$ of Jordan curves ($\alpha > 0$). We will define a parametrization of the Epstein surfaces that allows us to compute the derivative $\frac{\partial}{\partial t}|_{t=0} V(\gamma_t)$. Since scalar multiplications are isometries of $\mathbb{H}^3$, we can assume without loss of generality that all curves γ_t have Euclidean arclength 2π. Furthermore, for small ε we have that $V(\gamma_t) = V_1(\gamma_t)(\varepsilon) + V_2(\gamma_t)(\varepsilon)$, where V_2 is defined in Section 4.2. Moreover, we also know that $V_2(\gamma_t)(\varepsilon) \xrightarrow{\varepsilon\to 0} V(\gamma_t)$ converges uniformly in t by the proof of Proposition 4.5.

Let $f_t : \mathbb{D} \to \Omega_t$, $g_t : \mathbb{D}^* \to \Omega_t^*$ be univalent functions that extend to $C^{k,\alpha}$ functions on $[0,1]\times\overline{\mathbb{D}}$ and $[0,1]\times\overline{\mathbb{D}^*}$ respectively. Consider ε sufficiently small so that for $z \in \overline{\mathbb{D}}$ with $|z| > 1-\varepsilon$ we have that $\mathrm{Ep}_{\Omega_t}(f_t(z))$ belongs to the parametrized neighbourhood U_{γ_t} from Proposition 4.5 for all $t \in [0,1]$. Take the horizontal line $L_{t,z}$ (horocycle centered at $\infty \in \hat{\mathbb{C}}$) obtained by varying the second G-coordinate (4.2) in U_{γ_t} starting from $\mathrm{Ep}_{\Omega_t}(f_t(z))$, and define $h_t(z) \in \overline{\mathbb{D}^*}$ to be the point such that $\mathrm{Ep}_{\Omega_t^*}(g_t(h_t(z)))$ is the first point of intersection of the horizontal line with $\mathrm{Ep}_{\Omega_t^*}$. See Figure 1 for an illustration.

Clearly along $\partial\mathbb{D}$ the map h_t agrees with $g_t^{-1}\circ f_t$, and from the regularity of f_t, g_t and the G-coordinates of U_{γ_t} we can see that the 1-parameter family of functions h_t is $C^{3,\alpha}$ in $1-\varepsilon < |z| \le 1$ and t-parameters. Moreover, h_t is a diffeomorphism.

For $1-\varepsilon < r < 1$, define the cylindrical neighbourhood of γ_0 as

$$A(r) = f_0(\{r \le |z| \le 1\}) \cup g_0(h_0(\{r \le |z| \le 1\})),$$

which we parametrize by $\mathbb{S}^1 \times [r, 1/r]$, sending (p,s) to $f_0(sp)$ if $s \le 1$ and sending (p,s) to $g_0(h_0(\frac{p}{s}))$ if $s \ge 1$. These cylindrical neighbourhoods are nested as r grows, and their intersection as $r \to 1^-$ is γ_0. Define as well $\Omega(r), \Omega^*(r)$ to be the components of $\mathbb{C} \smallsetminus A(r)$ in Ω_0 and Ω_0^*, respectively.

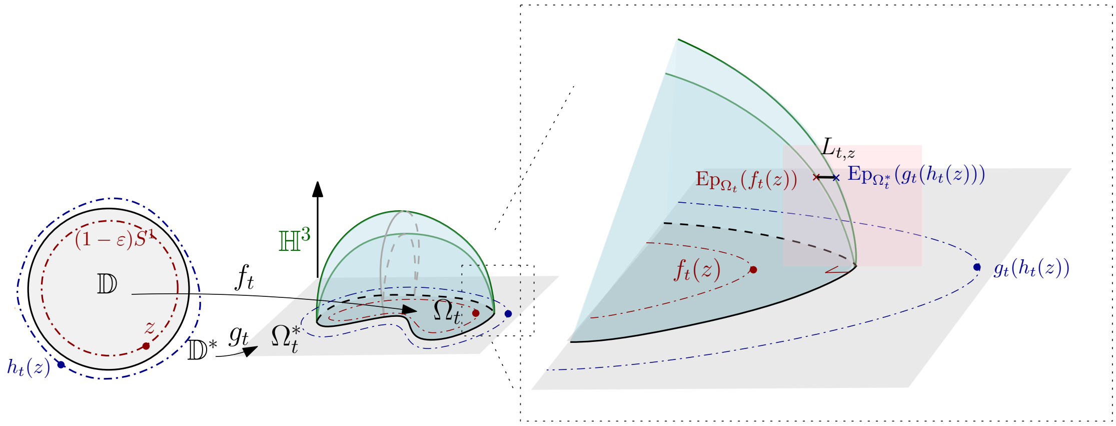

Figure 1: Illustration of the two Epstein–Poincaré surfaces associated with the two connected component of $\hat{\mathbb{C}} \smallsetminus \gamma_t$ and the map h_t.

Define a 1-parameter family of homeomorphisms $F_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ so that for $z \in \overline{\Omega_0}$ we define $F_t(z) := f_t(f_0^{-1}(z))$, for $z \in g_0(h_0(\{1-\varepsilon < |z| \le 1\}))$ we define $F_t(z) = g_t \circ h_t \circ h_0^{-1} \circ g_0^{-1}(z)$. In this way, if $\mathrm{Ep}_{\Omega_0}(u)$ and $\mathrm{Ep}_{\Omega_0^*}(v)$ are connected by the line $L_{0,z}$ (namely, $u = f_0(z)$ and $v = g_0 \circ h_0(z)$), then $\mathrm{Ep}_{\Omega_t}(F_t(u))$ and $\mathrm{Ep}_{\Omega_t^*}(F_t(v))$ are connected by the line $L_{t,z}$. We extend F_t to the rest of Ω_0^* as a $C^{3,\alpha}$ map in both $\overline{\Omega_0^*}$ and t parameters. Let us also fix F_0 to be the identity. It follows then that $F_t|_{\Omega_0}$ is a conformal map between Ω_0 and Ω_t, and $F_t|_{\gamma_0}$ parametrizes γ_t. For $1-\varepsilon < r < 1$, we construct the family of piecewise smooth maps $E_{r,t} : \hat{\mathbb{C}} \to \mathbb{H}^3$ satisfying the following properties:

(C1) In $\Omega(r), \Omega^*(r)$ the map $E_{r,t}$ is defined as the composition of the Epstein–Poincaré maps $\mathrm{Ep}_{\Omega_t}, \mathrm{Ep}_{\Omega_t^*}$ with F_t.

(C2) Considering the parametrization of $A(r)$, for each $p \in \mathbb{S}^1$ we have that $E_{r,t}(\{p\} \times [r, 1/r])$ is the straight horizontal segment $L_{t,rp}$.

The map $E_{r,\cdot}(\cdot)$ is piecewise smooth, and it is $C^{3,\alpha}$ while restricted to $[0,1] \times \Omega(r)$, $[0,1] \times A(r)$, $[0,1] \times \Omega^*(r)$.

Lemma 5.5. *Along each $\Omega(r), A(r), \Omega^*(r)$, on the image of $E_{r,t}(p)$ there is a well-defined unit vector $\vec{n}$ that is normal to the image of $DE_{r,t}$. On $\Omega(r)$ and $\Omega^*(r)$, $\vec{n}$ coincides with $\widetilde{\mathrm{Ep}}_{\Omega_t}$ and $\widetilde{\mathrm{Ep}}_{\Omega_t^*}$ respectively and on $A(r)$, we choose $\vec{n}$ to have positive vertical component. The corresponding Euclidean unit vector $\vec{\eta}$ varies piecewise $C^{3,\alpha}$ on $\{(r,t,p) \,|\, r \in (1-\varepsilon, 1], t \in [0,1], p \in \Omega(r)$ or $A(r)$ or $\Omega^*(r)\}$, and when $r = 1$, $p \in A(r) = \gamma$, $\vec{\eta} \equiv (0,0,1)$.*

Proof. The regularity of $\vec{n}$ on $\Omega(r), \Omega^*(r)$ follows from the construction of the Epstein–Poincaré map, see (3.7). The regularity of $\vec{n}$ in $A(r)$ for $1-\varepsilon < r < 1$ can be seen using the G-coordinates parametrizing Σ_{Ω_t} and $\Sigma_{\Omega_t^*}$.

To obtain the regularity of the Euclidean unit vector $\vec{\eta}$ on $A(r)$ up to $r = 1$ and its value $(0,0,1)$, we use the G-coordinates and the expression of ξ in Lemma 3.16. □

Remark 5.6. When appropriate, we will simplify notation by dropping r, t sub-indices.

5.3 Proof of Schläfli formula

We first prove Theorem 5.2 under the added assumption that the Poincare-Epstein surfaces are immersions.

Theorem 5.7. *Let γ_t be a $C^{5,\alpha}$ family of Jordan curves ($\alpha > 0$) such that the Poincare-Epstein surfaces are* immersions. *Then the first derivative of the volume $V(\gamma_t)$ is computed by*

$$\frac{\partial}{\partial t}\bigg|_{t=0} V(\gamma_t) = \int_{\Omega} \mathrm{Ep}_{\Omega}^* \left(\delta H + \frac{1}{4}\left\langle \delta \mathrm{I}, \mathrm{I\!I}\right\rangle \mathrm{d}a\right) + \int_{\Omega^*} \mathrm{Ep}_{\Omega^*}^* \left(\delta H + \frac{1}{4}\left\langle \delta \mathrm{I}, \mathrm{I\!I}\right\rangle \mathrm{d}a\right).$$

Remark 5.8. To remove the assumption of immersion and to prove Theorem 5.2 will require some technical analysis which we leave to a later section (see Section 8). As by Theorem 3.5 the map Ep_{Ω} (respectively for Ep_{Ω^*}) is an immersion in $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_{\Omega} \neq 1\}$, in particular we will extend continuously the right-hand side of the formula in Theorem 5.2 to the locus $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_{\Omega} = 1\}$ as smooth differential forms so the variation of volume formula still holds.

Proof. For r close to 1 we define $V_2(r,t)$ as the volume bounded by $E_{r,t}$. Similarly, we define $V_1(r,t)$ as the volume of the region between $\Sigma_{\Omega}, \Sigma_{\Omega^*}$ outside of $E_{r,t}$. Then we have that $V(\gamma_t) = V_1(r,t) + V_2(r,t)$.

We first show that

$$\lim_{r\to 1^-} \frac{\partial}{\partial t}\bigg|_{t=0} V_2(r,t) = \int_{\Omega} \mathrm{Ep}_{\Omega}^* \left(\delta H + \frac{1}{4}\left\langle \delta \mathrm{I}, \mathrm{I\!I}\right\rangle \mathrm{d}a\right) + \int_{\Omega^*} \mathrm{Ep}_{\Omega^*}^* \left(\delta H + \frac{1}{4}\left\langle \delta \mathrm{I}, \mathrm{I\!I}\right\rangle \mathrm{d}a\right).$$

Combining Theorem 5.3 and Lemma 5.4, we only need to prove that

$$\lim_{r\to 1^-} \int_{A(r)} \frac{1}{2}\operatorname{tr}\left\langle \nabla_{\xi}(B\cdot), DE_p\cdot\right\rangle = 0 \qquad \text{and}$$

$$\lim_{r\to 1^-} \frac{1}{2}\left(\int_{\partial\Omega(r)} \frac{\partial\theta}{\partial t} E^* \mathrm{d}\ell + \int_{\partial\Omega^*(r)} \frac{\partial\theta^*}{\partial t} E^* \mathrm{d}\ell\right) = 0.$$

For the first term, observe that $A(r)$ belongs to the surface described in (C2). These families of surfaces can be described by

$$\begin{aligned}\{p \in \mathbb{S}^1, s \in [r, 1/r], r \in (1-\varepsilon, 1], t \in [0,1]\} &\to \overline{\mathbb{H}^3} \subset \mathbb{R}^3 \\ (p,s,r,t) &\mapsto (x(p,s,r,t), y(p,s,r,t), z(p,s,r,t)),\end{aligned}$$

where p, s parametrize the surface as in (C2) for γ_t. This parametrization extends $C^{3,\alpha}$ for $r = 1$ towards the boundary of $\mathbb{H}^3$ by making $z(p,s,1,t) \equiv 0$. Moreover, given (C1) and Lemma 3.16 we have that $z(p,s,r,t) = O(1-r)$ uniformly for all other parameters. Hence the first and second fundamental form (as well as their first order variations) are of order at most $(1-r)^{-2}$, and the inverse of the first fundamental form has order $(1-r)^2$. This follows from the expression of these fundamental forms in terms of the derivatives up to third order of the parametrization into $\mathbb{R}^3$ and the conformal factor $z(p,s,r,t) = O(1-r)$. Thus it follows that the terms $H, \delta H, \langle \delta I, \mathrm{I\!I}\rangle$ are uniformly bounded.

As by Proposition 4.4 $E_{r,t}(A(r))$ has euclidean area $O((1-r)^3)$ so the hyperbolic area of $E_{r,t}(A(r))$ is of order $O(1-r)$, we have that

$$\lim_{r\to 1^-} \int_{A(r)} \frac{1}{2} \operatorname{tr} \langle \nabla_\xi (B\cdot), DE_p \cdot \rangle = 0.$$

Likewise, the exterior dihedral angle θ that takes each (p,r,t) to the angle between $E_{r,t}(\Omega(r))$ and $E_{r,t}(A(r))$ at $(x(p,r,r,t), y(p,r,r,t), z(p,r,r,t)) =: \gamma_r(p)$, extends smoothly to $r=1$ as right angles. Similarly for θ^* the exterior dihedral angle between $E_{r,t}(\Omega^*(r))$ and $E_{r,t}(A(r))$ along γ_r^*, where $\gamma_r^*(p) := (x(p,1/r,r,t), y(p,1/r,r,t), z(p,1/r,r,t))$. We use again that the Epstein–Poincaré surfaces agree up to second order (Proposition 4.4) to use parametrizations $\gamma_r(p)$ and $\gamma_r^*(p)$ satisfying

$$\begin{aligned} \left\| \frac{\mathrm{d}\gamma_r}{\mathrm{d}p}(p) - \frac{\mathrm{d}\gamma_r^*}{\mathrm{d}p}(p) \right\| &\le C(1-r)^2 \\ \left| \frac{\mathrm{d}\theta(p)}{\mathrm{d}t} + \frac{\mathrm{d}\theta^*(p)}{\mathrm{d}t} \right| &\le C(1-r)^2 \end{aligned} \tag{5.1}$$

for some uniform constant $C>0$. Then since the last coordinate of γ_r and γ_r^* is $O(1-r)$, we have that for some uniform constant $C>0$

$$\begin{aligned} &\Big| \int_{\mathrm{d}\Omega(r)} \frac{\mathrm{d}\theta}{\mathrm{d}t} E^* \mathrm{d}\ell + \int_{\mathrm{d}\Omega^*(r)} \frac{\mathrm{d}\theta^*}{\mathrm{d}t} E^* \mathrm{d}\ell \Big| \\ &\le C \int_{\mathbb{S}^1} \left| \frac{1}{1-r} \frac{\mathrm{d}\theta(\gamma_r(p))}{\mathrm{d}t} \right| . \left\| \frac{\mathrm{d}\gamma_r}{\mathrm{d}p} \right\| + \left| \frac{1}{1-r} . \frac{\mathrm{d}\theta^*(\gamma_r^*(p))}{\mathrm{d}t} \right| . \left\| \frac{\mathrm{d}\gamma_r^*}{\mathrm{d}p} \right\| \mathrm{d}p \\ &\le \frac{1}{1-r} \int_{\mathbb{S}^1} \left| \frac{\mathrm{d}\theta(\gamma_r(p))}{\mathrm{d}t} \right| . \left\| \frac{\mathrm{d}\gamma_r}{\mathrm{d}p}(p) - \frac{\mathrm{d}\gamma_r^*}{\mathrm{d}p}(p) \right\| + \left| \frac{\mathrm{d}\theta(\gamma_r(p))}{\mathrm{d}t} + \frac{\mathrm{d}\theta^*(\gamma_r^*(p))}{\mathrm{d}t} \right| . \left\| \frac{\mathrm{d}\gamma_r^*}{\mathrm{d}p} \right\| \mathrm{d}p \end{aligned} \tag{5.2}$$

goes to 0 as $r \to 1^-$ uniformly in t.

Using the parameters of Proposition 4.5, we can see that the t derivatives of the functions a_Ω, a_{Ω^*} in the proof of Proposition 4.5 agree as well up to order 2, so by the same argument we have that $\lim_{r\to 1^-} \partial_t V_1(r,0) = 0$.

As for any r near 1^- we have that $\partial_t V(\gamma_t) = \partial_t V_1(r,t) + \partial_t V_2(r,t)$, we send r to 1 on the right hand side to obtain the result. □

5.4 Variation of mean curvature and Schläfli formula

The goal of this section is to prove the identity between the renormalized volume and the universal Liouville action when the curve is regular enough (Corollary 5.12).

The following result is proved by Krasnov–Schlenker, see [22, Cor. 6.2], for the renormalized volume of convex co-compact manifolds. We adapt it to the renormalized volume associated with a smooth Jordan curve using Theorem 5.2.

Theorem 5.9. *We have the first order variation of the V_R*

$$\delta V_R(\gamma) = -\frac{1}{4} \int_{\Omega \cup \Omega^*} \delta \hat{H} + \frac{1}{2} \left\langle \delta \hat{\mathrm{I}}, \hat{\mathrm{I\!I}}_0 \right\rangle \mathrm{d}a^*$$

where $\hat{\mathrm{I\!I}}_0 = \vartheta\, \mathrm{d}z^2 + \bar{\vartheta}\, \mathrm{d}\bar{z}^2$ is the traceless part of $\hat{\mathrm{I\!I}}$, $\langle A, B\rangle$ stands for $\operatorname{tr}[\hat{\mathrm{I}}^{-1} A \hat{\mathrm{I}}^{-1} B]$.

Proof. By Definition 4.6, Remark 4.7 and Theorem 5.2, we can express δV_R as the integral of smooth 2-forms in Ω, Ω^*, so that at points where the respective Epstein–Poincaré maps are immersions these forms are given by the pullback of the form

$$(\delta H + \frac{1}{4}\langle \delta \mathrm{I}, \mathrm{I\!I}\rangle)\,\mathrm{d}a - \frac{1}{2}(\delta H \mathrm{d}a - H\delta(\mathrm{d}a))$$

by the respective Epstein–Poincaré map. Following [22, Section 6] this pullback is expressed precisely as $-\frac{1}{4}\left(\delta\hat{H} + \frac{1}{2}\left\langle \delta\hat{\mathrm{I}}, \hat{\mathrm{I\!I}}_0\right\rangle\right)\mathrm{d}\hat{a}$. As points where the Epstein–Poincaré maps are immersions are dense in Ω, Ω^*, and by the piecewise regularity of E and of its shape operator (Lemma 8.6) we have that all forms discussed vary continuously, we have then that the result follows. □

More explicitly, we can write the variation of V_R in terms of the Beltrami differentials. We consider a $C^{5,\alpha}$ family of Jordan curves (γ_t) as in the previous section and let F_t be the corresponding homeomorphism of $\hat{\mathbb{C}}$ which maps Ω_0 conformally onto Ω_t and a diffeomorphism from Ω_0^* to Ω_t^*, as constructed in Section 5.2. For $z \notin \gamma_0$, let

$$\mu_t := \frac{\partial_{\bar{z}} F_t}{\partial_z F_t} = t\dot{\nu} + O(t^2).$$

We have in particular, $\dot{F} := \frac{\mathrm{d}}{\mathrm{d}t} F_t|_{t=0}$ satisfies

$$\partial_{\bar{z}}\dot{F} = \dot{\nu}, \qquad F_t(z) = z + t\dot{F}(z) + O(t^2).$$

Since F_t is conformal in Ω_0, $\dot{\nu}|_{\Omega_0} \equiv 0$.

Lemma 5.10. *We have $\|\dot{\nu}\|_\infty < \infty$. Moreover, $\dot{\nu}|_{\Omega^*} \in H^{-1,1}(\Omega^*) \oplus \mathfrak{N}(\Omega^*)$.*

Proof. On Ω we have that the 1-parameter family F_t is conformal, while in $\overline{\Omega^*}$, $F_t|_{\Omega^*}$ is jointly $C^{3,\alpha}$ in (t, z). The L^∞ bound of $\dot{\nu}$ follows from the compactness of the domains (viewed in $\hat{\mathbb{C}}$).

For the second claim, as (γ_t) corresponds to a differentiable path in $T_0(1)$, the projection of $\dot{\nu}$ onto harmonic Beltrami differentials $\Omega^{-1,1}(\Omega^*)$ parallel to $\mathfrak{N}(\Omega^*)$ lies in $H^{-1,1}(\Omega^*)$. This completes the proof. □

Corollary 5.11. *The first variation of the renormalized volume associated with the family of deformed Jordan curves $(\gamma_t := F_t(\gamma_0))$ is given by*

$$\delta V_R(\gamma) = -\operatorname{Re}\int_{\Omega^*} \dot{\nu}\mathscr{S}[g^{-1}]\mathrm{d}^2 z. \tag{5.3}$$

where we recall $g : \mathbb{D}^* \to \Omega^*$ is any conformal map.

Proof. As $\dot{\nu} \in L^\infty(\Omega^*)$ and $\mathscr{S}[g^{-1}]$ is continuous functions up to the boundary. The integrals in (5.3) are absolutely convergent. From Theorem 5.9, we only need to check the pointwise identity

$$\left(\frac{1}{4}\delta\hat{H} + \frac{1}{8}\left\langle \delta\hat{\mathrm{I}}, \hat{\mathrm{I\!I}}_0\right\rangle\right)\mathrm{d}\hat{a} = \operatorname{Re}\left(\dot{\nu}\mathscr{S}[g^{-1}]\right)\,\mathrm{d}^2 z \tag{5.4}$$

on Ω^*. For this, we have

$$\mathrm{d}F_t(z) = \mathrm{d}z + t\partial_z\dot{F}\,\mathrm{d}z + t\partial_{\bar{z}}\dot{F}\,\mathrm{d}\bar{z} + O(t^2) = \mathrm{d}z + t\partial_z\dot{F}\,\mathrm{d}z + t\dot{\nu}\,\mathrm{d}\bar{z} + O(t^2)$$

and in the $\mathrm{d}z, \mathrm{d}\bar{z}$ coordinates

$$\mathrm{d}F_t(z)\mathrm{d}\overline{F_t(z)} = \begin{pmatrix} t\bar{\dot{\nu}} & \frac{1}{2}(1+2t\,\mathrm{Re}(\partial_z\dot{F})) \\ \frac{1}{2}(1+2t\,\mathrm{Re}(\partial_z\dot{F})) & t\dot{\nu} \end{pmatrix} + O(t^2).$$

Therefore, the hyperbolic metric in Ω_t^* is

$$e^{\varphi}(1+2ts+O(t^2))\,\mathrm{d}F_t(z)\mathrm{d}\overline{F_t(z)} = \hat{\mathrm{I}} + te^{\varphi}\begin{pmatrix} \bar{\dot{\nu}} & \mathrm{Re}(\partial_z\dot{F})+s \\ \mathrm{Re}(\partial_z\dot{F})+s & \dot{\nu} \end{pmatrix} + O(t^2).$$

where s is some smooth function on Ω^* and

$$\hat{\mathrm{I}} = e^{\varphi}\mathrm{d}z\mathrm{d}\bar{z} = \frac{1}{2}\begin{pmatrix} 0 & e^{\varphi} \\ e^{\varphi} & 0 \end{pmatrix}.$$

We obtain

$$\delta\hat{\mathrm{I}} = e^{\varphi}\begin{pmatrix} \bar{\dot{\nu}} & \mathrm{Re}(\partial_z\dot{F})+s \\ \mathrm{Re}(\partial_z\dot{F})+s & \dot{\nu} \end{pmatrix}.$$

Recall that

$$\hat{\mathrm{II}}_0 = \begin{pmatrix} \vartheta & 0 \\ 0 & \bar{\vartheta} \end{pmatrix} = \begin{pmatrix} \mathscr{S}(g^{-1}) & 0 \\ 0 & \overline{\mathscr{S}(g^{-1})} \end{pmatrix},$$

we have

$$\left\langle \delta\hat{\mathrm{I}}, \hat{\mathrm{II}}_0 \right\rangle = 8e^{-\varphi}\,\mathrm{Re}(\dot{\nu}\mathscr{S}[g^{-1}]).$$

Since $\mathrm{d}\hat{a} = e^{\varphi}\,\mathrm{d}^2 z$ and from Corollary 3.4, we have $\hat{H} = -\hat{K} \equiv 1$ which implies $\delta\hat{H} \equiv 0$, we obtain the claimed formula (5.4). □

Corollary 5.12. *We have for all $C^{5,\alpha}$ Jordan curves γ, we have*

$$\tilde{\mathbf{S}}(\gamma) = 4V_R(\gamma).$$

Proof. When γ is a circle, we have $\tilde{\mathbf{S}}(\gamma) = 0$ and $V_R(\gamma) = 0$ since both Epstein surfaces are the geodesic plane bounded by γ.

Given a $C^{5,\alpha}$ Jordan curve γ. We consider a $C^{5,\alpha}$ family $(\gamma_t)_{t\in[0,1]}$ of Jordan curves as in Example 5.1 such that $\gamma_0 = \partial D$ is a circle and $\gamma_1 = \gamma$. The variational formula Theorem 2.1 and Corollary 5.11 show that

$$\tilde{\mathbf{S}}(\gamma) = 4V_R(\gamma)$$

since $\tilde{\mathbf{S}}(\gamma_0) = 4V_R(\gamma_0)$. □

5.5 Approximation of general Weil–Petersson quasicircle

The goal of the section is to prove the following theorem using an approximation.

Theorem 5.13. *We have for any Weil–Petersson quasicircle γ,*

$$\tilde{\mathbf{S}}(\gamma) \geq 4V_R(\gamma).$$

Remark 5.14. We have already proved the equality when γ is a $C^{5,\alpha}$ Jordan curve. We also believe the equality holds for arbitrary Weil–Petersson quasicircle but are only able to prove the inequality.

For the inequality, we will use the approximation using equipotential curves. Let γ be a Weil–Petersson quasicircle, $f : \mathbb{D} \to \Omega$ be a conformal map. Up to post-composing f by a Möbius map, we may assume that $f(0) = 0$, $f'(0) = 1$ and $f''(0) = 0$. The equipotentials

$$\gamma_n = f_n(\mathbb{S}^1), \quad \text{where } f_n(z) := \frac{n}{n-1} f\left(\frac{n-1}{n} z\right)$$

form a family of analytic Jordan curves. The map f_n satisfies the same normalization as f at 0. We let $\Omega_n^* := \hat{\mathbb{C}} \smallsetminus \overline{f_n(\mathbb{D})}$ (resp. $\Omega^* := \hat{\mathbb{C}} \smallsetminus \overline{f(\mathbb{D})}$) and g_n (resp. g) be an arbitrary conformal map $\mathbb{D}^* \to \Omega_n^*$ (resp, $\mathbb{D}^* \to \Omega^*$). Apart from the analyticity, the family of equipotentials is nice because of the following theorem.

Theorem 5.15 (See [40, Cor. 1.5])**.** *Along the family of equipotentials the universal Liouville action converges and is non-decreasing. We have*

$$\lim_{n\to\infty} \uparrow \tilde{\mathbf{S}}(\gamma_n) = \tilde{\mathbf{S}}(\gamma).$$

If γ is not a circle, then $\tilde{\mathbf{S}}(\gamma_{n+1}) > \tilde{\mathbf{S}}(\gamma_n)$.

Lemma 5.16. *We have*

$$\int_{\Sigma_{\Omega_n} \cup \Sigma_{\Omega_n^*}} H \mathrm{da} \xrightarrow{n\to\infty} \int_{\Sigma_\Omega \cup \Sigma_{\Omega^*}} H \mathrm{da}. \tag{5.5}$$

Proof. It follows from [39, Cor. A.4., Cor. A.6] that the element $[\mu_n]$ in $T_0(1)$ associated with γ_n converges to $[\mu]$ which is associated with γ. In particular, [39, Chap. I, Thm. 2.13, Thm. 3.1] implies that

$$\int_{\mathbb{D}} \|\mathscr{S}(f_n)\|_{\mathbb{D}}^2 \, \rho_{\mathbb{D}} \, \mathrm{d}^2 z = \int_{\mathbb{D}} |\mathscr{S}(f_n)|^2 \rho_{\mathbb{D}}^{-1} \, \mathrm{d}^2 z \xrightarrow{n\to\infty} \int_{\mathbb{D}} \|\mathscr{S}(f)\|_{\mathbb{D}}^2 \, \rho_{\mathbb{D}} \, \mathrm{d}^2 z.$$

As $T_0(1)$ is a topological group, we have $[\mu_n]^{-1}$ converges to $[\mu]^{-1}$ which implies

$$\int_{\mathbb{D}^*} \|\mathscr{S}(g_n)\|_{\mathbb{D}^*}^2 \, \rho_{\mathbb{D}^*} \, \mathrm{d}^2 z = \int_{\mathbb{D}^*} |\mathscr{S}(g_n)|^2 \rho_{\mathbb{D}^*}^{-1} \, \mathrm{d}^2 z \xrightarrow{n\to\infty} \int_{\mathbb{D}^*} \|\mathscr{S}(g)\|_{\mathbb{D}^*}^2 \, \rho_{\mathbb{D}^*} \, \mathrm{d}^2 z.$$

The proof is completed using Theorem 3.5 and that $\|\mathscr{S}(f^{-1})(f(\zeta))\|_\Omega = \|\mathscr{S}(f)(\zeta)\|_{\mathbb{D}}$. $\square$

Lemma 5.17. *Recall that $V_2(\gamma)(\varepsilon)$ denotes the signed volume between* Ep_Ω *and* Ep_{Ω^*} *above Euclidean height ε. We have $V_2(\gamma_n)(\varepsilon)$ converges to $V_2(\gamma)(\varepsilon)$ for all $\varepsilon > 0$.*

Proof. For this, we denote for $\varepsilon > 0$,

$$K_{\varepsilon,n} := \{\zeta \in \mathbb{D} \colon \xi_n \circ f_n(\zeta) \geq \varepsilon\}, \quad K_\varepsilon := \{\zeta \in \mathbb{D} \colon \xi \circ f(\zeta) \geq \varepsilon\},$$

where (Z_n, ξ_n) is the Epstein–Poincaré map on the domain $\Omega_n = f_n(\mathbb{D})$ following the notations in Section 3.3.

By Corollary 3.13

$$\frac{\operatorname{dist}(f_n(\zeta), \gamma_n)}{5} \leq |\xi_n \circ f_n(\zeta)| \leq 4 \operatorname{dist}(f_n(\zeta), \gamma_n)$$

which implies for all $\zeta \in K_{\varepsilon,n}$,

$$\operatorname{dist}(f_n(\zeta), \gamma_n) \geq \varepsilon/4.$$

Since f_n converges uniformly to f on $\overline{\mathbb{D}}$ from the explicit expression, the derivatives of f_n converges to the derivatives of f uniformly on compact sets of $\mathbb{D}$ by Cauchy's integral formula.

Hence, there exists n_0 such that for all $n \geq n_0$, we have

$$\|f_n - f\|_{\infty, \overline{\mathbb{D}}} < \varepsilon/16.$$

This implies

$$\operatorname{dist}(f(\zeta), \gamma) \geq \varepsilon/8 \quad \text{and} \quad \xi \circ f(\zeta) \geq \varepsilon/40.$$

Summarizing, we have for all $n \geq n_0$,

$$K_{\varepsilon,n} \subset K_{\varepsilon/40}.$$

Since $K_{\varepsilon/40}$ is a compact set in $\mathbb{D}$ independent of n, we have that all derivatives of f_n converge uniformly to the derivatives of f on $K_{\varepsilon/40}$. As the Epstein–Poincaré map only depends on f, f', and f'' (Theorem 3.2), $\mathrm{Ep}_{\Omega_n} \circ f_n$ converges uniformly to $\mathrm{Ep}_\Omega \circ f$ uniformly on $K_{\varepsilon/40}$. Similarly argument applies to the Epstein–Poincaré maps $\mathrm{Ep}_{\Omega_n^*} \circ g_n$. We obtain that $V_2(\gamma_n)(\varepsilon)$ converges to $V_2(\gamma)(\varepsilon)$. □

We obtain the following corollary.

Corollary 5.18. *If γ is a Weil–Petersson quasicircle, then*

$$V(\gamma) \leq \frac{1}{4}\tilde{\mathbf{S}}(\gamma) + \frac{1}{2}\int_{\Sigma_\Omega \cup \Sigma_{\Omega^*}} H \mathrm{d}a < \infty.$$

Proof. For small enough $\varepsilon > 0$,

$$\begin{aligned} V_2(\gamma)(\varepsilon) &= \lim_{n\to\infty} V_2(\gamma_n)(\varepsilon) \\ &\leq \lim_{n\to\infty} \frac{1}{4}\tilde{\mathbf{S}}(\gamma_n) + \frac{1}{2}\int_{\Sigma_{\Omega_n} \cup \Sigma_{\Omega_n^*}} H \mathrm{d}a = \frac{1}{4}\tilde{\mathbf{S}}(\gamma) + \frac{1}{2}\int_{\Sigma_\Omega \cup \Sigma_{\Omega^*}} H \mathrm{d}a \end{aligned}$$

by Theorem 5.15 and Lemma 5.16. We obtained the inequality by taking $\varepsilon \to 0$. □

Theorem 5.13 follows immediately from this corollary.

6 Gradient flow of the universal Liouville action

Following Bridgeman–Brock–Bromberg [7] and Bridgeman–Bromberg–Vargas-Pallete [10], we introduce the following flow on $T(1)$. For $[\mu] \in T(1)$, we have a natural isomorphism $T_{[\mu]}T(1) \simeq \Omega^{-1,1}(\mathbb{D}^*)$. We therefore define the vector field

$$V_{[\mu]} := -4\frac{\overline{\mathscr{S}(g_\mu)}}{\rho_{\mathbb{D}^*}} \in \Omega^{-1,1}(\mathbb{D}^*),$$

where g_μ is a conformal map defined on $\mathbb{D}^*$ associated with $[\mu]$ as defined in Section 2.1.

Theorem 6.1. *The vector field V has flowlines that exist for all time on $T(1)$. The flow restricts to a flow on $T_0(1)$ and is the (negative) Weil–Petersson gradient of the Liouville functional $\mathbf{S}$. Furthermore all flowlines on $T_0(1)$ converges to the origin $[0]$ which corresponds to the round circle.*

Proof. By the Nehari bound we have that in the Teichmüller metric on $T(1)$, $||V||_\infty \le 6$. Thus as $T(1)$ is complete in the Teichmüller metric, the flow under V exists for all time on $T(1)$. If $[\mu] \in T_0(1)$ then by the characterization (2.4) we have

$$\int_{\mathbb{D}^*} |\mathscr{S}(g_\mu)|^2 \rho_{\mathbb{D}^*}^{-1} \,\mathrm{d}^2 z < \infty.$$

Thus $V_{[\mu]} \in H^{-1,1}(\mathbb{D}^*) \simeq T_{[\mu]}T_0(1)$ and therefore by integrability the flow preserves $T_0(1)$. Furthermore if $\dot\nu \in H^{-1,1}(\mathbb{D}^*) \simeq T_{[\mu]}T_0(1)$ then by Theorem 2.1,

$$(\mathrm{d}\mathbf{S})_{[\mu]}(\dot\nu) = 4\,\mathrm{Re}\int_{\mathbb{D}^*} \dot\nu \mathscr{S}(g_\mu) = -\,\mathrm{Re}\int_{\mathbb{D}^*} \dot\nu\, \overline{V_{[\mu]}}\, \rho_{\mathbb{D}^*} = -\left\langle V_{[\mu]}, \dot\nu \right\rangle_{\mathrm{WP}}.$$

Therefore $\nabla_{\mathrm{WP}}\mathbf{S} = -V$ and

$$\mathrm{d}\mathbf{S}(V) = -||V||^2_{\mathrm{WP}}.$$

We consider the flowline $\mathbb{R}_+ \to T_0(1) : t \mapsto \alpha(t)$ for V starting at a point $[\mu] = \alpha(0) \in T_0(1)$. Since $\mathbf{S} \ge 0$, for all $T > 0$,

$$0 \le \int_0^T ||V(\alpha(t))||^2_{\mathrm{WP}}\,\mathrm{d}t = \mathbf{S}([\mu]) - \mathbf{S}(\alpha(T)) \le \mathbf{S}([\mu]).$$

Thus

$$\int_0^\infty ||V(\alpha(t))||^2_{\mathrm{WP}}\,\mathrm{d}t < \infty.$$

We therefore have a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} ||V(\alpha(t_n))||_{\mathrm{WP}} = 0.$$

Therefore the conformal maps $g_{\alpha(t_n)}$ satisfy $\|\mathscr{S}(g_{\alpha(t_n)})\|_2 \to 0$. From the Hilbert manifold structure of $T_0(1)$, see [39, Ch. I, Def. 2.11], this implies $\alpha(t_n)$ converges in $T_0(1)$ to the origin $[0]$. In particular, $\mathbf{S}(\alpha(t_n)) \to 0$.

To show that the flow line converges to $[0]$ (not only along a subsequence), we note first that the Liouville action of $\alpha(t)$ is decreasing which implies $\mathbf{S}(\alpha(t)) \to 0$ as $t \to \infty$. We now show that this implies the convergence of the flow line in $T_0(1)$ to the origin.

In fact, assuming the opposite, there is $\varepsilon > 0$ and a sequence $\alpha(t_k)$ such that the Weil–Petersson distance to 0 is greater than ε along the sequence. Let γ_k be the quasicircle passing through $1, -1, -\mathfrak{i}$ associated with $\alpha(t_k)$ (see Section 2), since their Liouville action is uniformly bounded, they are all K-quasicircles (see, e.g., [30, Prop. 2.9]) for some $K > 1$ (namely, image of $\mathbb{S}^1$ of a K-quasiconformal homeomorphism φ_k of $\hat{\mathbb{C}}$ fixing $1, -1, -\mathfrak{i}$). We can extract from the normal family $\{\varphi_k\}$ a subsequence $\varphi_{k(n)}$ which converges uniformly on $\mathbb{S}^1$ as $n \to \infty$. We write γ_∞ for the image of $\mathbb{S}^1$ of the limiting map $\lim_{n\to\infty} \varphi_{k(n)}$. We have $\gamma_{k(n)}$ converges uniformly to γ_∞. From [30, Lem. 2.12], we know the Liouville action is lower-semicontinuous, this implies

$$0 = \liminf_{n\to\infty} \mathbf{S}(\alpha_{t_{k(n)}}) = \liminf_{n\to\infty} \tilde{\mathbf{S}}(\gamma_{k(n)}) \geq \tilde{\mathbf{S}}(\gamma_\infty). \tag{6.1}$$

Hence $\tilde{\mathbf{S}}(\gamma_\infty) = 0$. As $\mathbb{S}^1$ is the only zero of $\tilde{\mathbf{S}}$, we have $\gamma_\infty = \mathbb{S}^1$ and the corresponding point in $T_0(1)$ is the origin 0. To see $\alpha_{t_{k(n)}}$ also converges in $T_0(1)$, consider the conformal maps $f_n : \mathbb{D} \to D_n$ fixing 0 and $g_n : \mathbb{D}^* \to D_n^*$ fixing ∞ as in the definition of $\mathbf{S}(\alpha_{t_{k(n)}})$, where D_n and D_n^* are respectively the bounded and unbounded connected component of $\mathbb{C} \smallsetminus \gamma_{k(n)}$. We have from (6.1) and Carathéodory theorem that

$$\lim_{n\to\infty} \log\left|\frac{f_n'(0)}{g_n'(\infty)}\right| = 0, \text{ which implies } \lim_{n\to\infty}\int_{\mathbb{D}} \left|\frac{f_n''(z)}{f_n'(z)}\right|^2 \mathrm{d}^2 z = 0.$$

This shows $\alpha_{t_{k(n)}}$ converges to 0 in $T_0(1)$ by [39, Cor. A.4] and contradicts with the assumption of the sequence being ε distance away from 0. ☐

Using the gradient flow we may bound the Weil–Petersson distance between $[\mu]$ and $[0]$ by the universal Liouville action. We first recall some results proved by Takhtajan and Teo that we summarize in the lemma below.

Lemma 6.2 ([39, Ch. I: Rem. 2.4, Lem. 2.5, Cor. 2.6])**.** *There exists $0 < \delta < 1$ such that for all $\mu \in \Omega^{-1,1}(\mathbb{D}^*)$ with $\|\mu\|_\infty < \delta$,*

$$\left|\frac{|\partial_z w_\mu(z)|^2}{(1-|w_\mu(z)|^2)^2} - \frac{1}{(1-|z|^2)^2}\right| < \frac{1}{(1-|z|^2)^2}.$$

Moreover, for such μ, the map $D_0(\beta \circ R_{[\mu]}) : H^{-1,1}(\mathbb{D}^*) \to A_2(\mathbb{D})$ is a bounded linear isomorphism with

$$||D_0(\beta \circ R_{[\mu]})(\nu)||_2 \leq 24||\nu||_{\mathrm{WP}} \qquad ||\nu||_{\mathrm{WP}} \leq K||D_0(\beta \circ R_{[\mu]})(\nu)||_2$$

where $K = \sqrt{2}/(1-\delta)^2$.

Theorem 6.3. *With the same constants δ and K as in Lemma 6.2. Let $0 < c < 2\delta\sqrt{4\pi/3}$, then for $[\mu] \in T_0(1)$, we have*

$$c(\mathrm{dist}_{\mathrm{WP}}([\mu],[0]) - Kc) \leq \mathbf{S}([\mu]). \tag{6.2}$$

Proof. We let $t \mapsto \alpha(t)$ be the gradient flow line starting at $[\mu]$. Assume first that $\|V([\mu])\| \geq c$ and let τ be the first time $\|V(\alpha(t))\|_{\mathrm{WP}} = c$. Then $\|V(\alpha(t))\|_{\mathrm{WP}} > c$ for all $t < \tau$. Thus

$$\mathbf{S}([\mu]) - \mathbf{S}(\alpha(\tau)) = \int_0^\tau \|V(\alpha(t))\|_{\mathrm{WP}}^2 \,\mathrm{d}t \geq c\int_0^\tau \|V(\alpha(t))\|_{\mathrm{WP}} \,\mathrm{d}t \geq c\, \mathrm{dist}_{\mathrm{WP}}([\mu], \alpha(\tau)).$$

We have therefore

$$\mathbf{S}([\mu]) \geq c(\mathrm{dist}_{\mathrm{WP}}([\mu],[0]) - \mathrm{dist}_{\mathrm{WP}}(\alpha(\tau),[0])).$$

By [39, Ch. I, Lem. 2.1], we have for all $\phi \in A_\infty(\mathbb{D})$,

$$||\phi||_\infty := \sup_{z\in\mathbb{D}} \|\phi(z)\|_{\mathbb{D}} \leq \sqrt{\frac{3}{4\pi}} \sqrt{\int_{\mathbb{D}} ||\phi(z)||_{\mathbb{D}}^2 \, \rho_{\mathbb{D}} \, \mathrm{d}^2 z} = \sqrt{\frac{3}{4\pi}} ||\phi||_2. \tag{6.3}$$

Hence, since $\|V(\alpha(\tau))\| = c$, we have

$$\|V(\alpha(\tau)))\|_\infty \leq c\sqrt{3/4\pi} < 2\delta.$$

Therefore

$$\|\hat{\beta}([\alpha(\tau)])\|_\infty = \|\mathscr{S}(g_{\alpha(\tau)})\|_\infty < \delta/2 < 1/2$$

where $\hat{\beta}$ is the Bers embedding $T(1) \to A_\infty(\mathbb{D}^*)$. As $\hat{\beta}(T_0(1)) = \hat{\beta}(T(1)) \cap A_2(\mathbb{D}^*)$ the linear path

$$\gamma(s) := [s\tilde{\mu}], \quad \text{where} \quad \tilde{\mu} = -\frac{2}{\bar{z}^4} \frac{\mathscr{S}(g_{\alpha(\tau)})}{\rho_{\mathbb{D}^*}} \left(\frac{1}{\bar{z}}\right) \text{ satisfies } \|\tilde{\mu}\|_{\mathbb{D},\infty} < \delta$$

for $s \in [0,1]$ from 0 to $\alpha(\tau)$ is in the ball of radius δ of $T(1)$, and also in $T_0(1)$ since by Ahlfors–Weill theorem

$$\hat{\beta}([s\tilde{\mu}]) = s\mathscr{S}(g_{\alpha(\tau)}) \in A_2(\mathbb{D}^*).$$

In $A_2(\mathbb{D}^*)$ this path has length $\|V(\alpha(\tau))\|_{\mathrm{WP}} = c$. By Lemma 6.2 we have that the preimage of this path by $\hat{\beta}$ has therefore length less than Kc and obtain (6.2).

If $\|V([\mu])\|_{\mathrm{WP}} < c$, then the above argument shows that $\mathrm{dist}_{\mathrm{WP}}([\mu], 0) \leq Kc$, so (6.2) holds trivially as $\mathbf{S} \geq 0$. □

7 Comparisons to minimal surfaces and convex core

Using Proposition 7.1 and Proposition 7.3 we will answer a question of Bishop [4] about how minimal surfaces and convex cores relate to Epstein–Poincaré maps. This section is independent from the proofs in the rest of the paper.

Let us extend the notation Σ_Ω by taking $\Sigma_\Omega(t)$ as the image of $\mathrm{Ep}_{e^{2t}\rho_\Omega}$ for $t \in \mathbb{R}$. The following proposition shows that a minimal surface in $\mathbb{H}^3$ with boundary $\gamma \subset \overline{\mathbb{C}}$ is in between appropriate equidistant images $\Sigma_\Omega(t), \Sigma_{\Omega^*}(t)$ ($\overline{\mathbb{C}} \setminus \gamma = \Omega \cup \Omega^*$). We write

$$\|\mathscr{S}(f^{-1})\|_\infty := \sup_{z\in\Omega} |\mathscr{S}(f^{-1})(z)|\rho_\Omega^{-1}(z) = \sup_{\zeta\in\mathbb{D}} |\mathscr{S}(f)(\zeta)|\rho_{\mathbb{D}}^{-1}(\zeta) = \|\mathscr{S}(f)\|_\infty.$$

Proposition 7.1. *Let $M \subset \mathbb{H}^3$ be a minimal surface so that $\partial_\infty M = \gamma$. Denote by M_Ω the closure of the component of $\mathbb{H}^3 \setminus M$ with conformal boundary Ω. Given conformal map $f : \mathbb{D} \to \Omega$, denote by $t_0 = \frac{1}{2}\log(\max\{1, 2\|\mathscr{S}(f)\|_\infty - 1\})$. Then for any $t \geq t_0$ we have that $\Sigma_\Omega(t) \subset M_\Omega$.*

Proof. Recall that by the discussion at the start of Subsection 3.2, the principal curvatures at infinity of ρ_Ω are bounded below by $1 - 2\|\mathscr{S}(f)\|_\infty$. By taking any $\Omega' \subset \Omega$ bounded by an equipotential, we have that the same lower bound $1 - 2\|\mathscr{S}(f)\|_\infty$ holds for $\rho_{\Omega'}$. We will start by showing that $\Sigma_{\Omega'}(t) \subset M_\Omega$ for any $t \geq t_0$.

Fixing $\Omega' \subset \Omega$, define $t' = \inf\{t \in \mathbb{R} \mid \Sigma_{\Omega'}(t) \subset M_\Omega\}$. As we have $\partial\Omega' \subset \Omega$, then $t' < +\infty$ and $\Sigma_{\Omega'}(t') \cap M$ is a non-empty compact subset of $\mathbb{H}^3$. If we assume by contradiction that $t' > t_0$ then $e^{2t'}\rho_{\Omega'}$ has principal curvatures at infinity bounded strictly below by -1. As we have that the mean curvature at infinity $\hat{H}$ is the opposite of Gaussian curvature of the metric (see Corollary 3.4), then the principal curvatures at infinity $\hat{k}_{1,2}$ of $e^{2t'}\rho$ satisfy $\frac{\hat{k}_1+\hat{k}_2}{2} = e^{-2t'} < 1$ at every point. This implies that the mean curvature vector of $\Sigma_{\Omega'}(t')$, given by $\frac{1-\hat{k}_1\hat{k}_2}{(1+\hat{k}_1)(1+\hat{k}_2)} > 0$ times the outer normal to the associated horosphere, is hence positively parallel to the outer normal to the horosphere. This leads to a contradiction as at the tangent point between $\Sigma_{\Omega'}(t')$ and M we will have that the mean curvature vector points in the opposite direction by the relative position of $\Sigma_{\Omega'}(t')$ and M with respect to the outer normal to the horosphere.

As we can obtain $\Sigma_\Omega(t)$ as limits of $\Sigma_{\Omega'}(t)$, the conclusion follows for Ω. □

Remark 7.2. From Proposition 7.1 we have that if $\|\mathscr{S}(f)\|_\infty, \|\mathscr{S}(g)\|_\infty < 1$ (where f, g are uniformization maps for Ω, Ω^*) then the minimal surface M lies in between the Epstein–Poincaré maps from Ω, Ω^*. This is an alternate proof of Proposition 4.2 under the assumption $\|\mathscr{S}(f)\|_\infty, \|\mathscr{S}(g)\|_\infty < 1$.

We can impose instead conditions on the curvatures of the minimal surface to obtain the same conclusion as in Remark 7.2.

Proposition 7.3. *Let $M \subset \mathbb{H}^3$ be a minimal surface so that $\partial_\infty M = \gamma$. Denote by M_Ω the closure of the component of $\mathbb{H}^3 \smallsetminus M$ with conformal boundaries Ω. Assume that any point of M the principal curvatures are strictly between -1 and 1. Then for any $t \geq 0$ we have that $\Sigma_\Omega(t) \subset M_\Omega$.*

Proof. For $z \in \Omega$ define ν_M as the visual metric of M, given by the value of the visual metric at z of the first horosphere H_z at z that intersects M (or equivalently, $H_z = \partial B_z$, where B_z is the largest open horosphere based at z disjoint from M). As $z \notin \partial_\infty M$ we have that ν_M is well-defined, and by the condition on mean curvatures we have that H_z is tangent to M at a unique point. Indeed, $H_z \cap M$ is an isolated set, and if it is not a singleton then we would find an intrinsic geodesic segment of M that on its interior belong to $\mathbb{H}^3 \smallsetminus \overline{B_z}$ but its endpoints lie in H_z. Such curve will have an interior point of geodesic curvature greater than 1, but as a geodesic segment of M its geodesic curvature is always less than 1, which is a contradiction.

Using that the point of tangency of H_z and M is unique, one has that the Gaussian curvature of ν_M at z is given by $-\frac{1+\lambda^2}{1-\lambda^2} \leq -1$, where $\pm\lambda$ are the principal curvatures of M at the point of tangency. By the Ahlfors-Schwarz lemma we have that $\nu_M \leq e^{2t}\rho_\Omega$ for any $t \geq 0$, which in particular implies that $\Sigma_\Omega(t) \subset M_\Omega$. □

Finally, we observe that the Epstein–Poincaré map has image inside a neighbourhood of the convex core.

Proposition 7.4. *Define* $\epsilon = \log(\max\{1, 2\|\mathscr{S}(f)\|_\infty - 1, 2\|\mathscr{S}(g)\|_\infty - 1\})$*. Then* Σ_Ω*,* Σ_{Ω^*} *belong to* $C_\epsilon(\gamma)$*, the* ϵ *neighbourhood of the convex core* $C(\gamma)$*.*

Proof. We start by proving the following claim.

Claim: Take $t > \frac{\epsilon}{2}$. For any round disk $\overline{D} \subset \Omega$ we will show that $\Sigma_\Omega(t)$ lies in the component of $\mathbb{H}^3 \setminus \Sigma_D(t)$ whose boundary at infinity contains γ.

Observe first that by Theorem 3.3 the curvatures at infinity of $\Sigma_\Omega(t)$ are given by $e^{-2t}(1 \pm 2\|\mathscr{S}(f)\|_\infty)$. Hence as $t > \frac{\epsilon}{2}$ we have that the curvatures at infinity of $\Sigma_\Omega(t)$ are greater than -1 and therefore $\Sigma_\Omega(t)$ is an immersed surface. Moreover as the principal curvatures at infinity add up to $e^{-2t} < 1$ we have that the surface $\Sigma_\Omega(t)$ is mean-convex with respect to the normal vector field given by $\widetilde{\mathrm{Ep}_\Omega}$.

Define the (signed) distance function $d : \mathbb{H}^3 \to \mathbb{R}$ to $\Sigma_D(t)$ so that $d^{-1}\{s\} = \Sigma_D(s+t)$. Considering $d \circ \Sigma_\Omega$, we define $d_0 = \sup\{d(\Sigma_\Omega(t)(z)) \mid z \in \Omega\}$. The claim is equivalent to show that $d_0 \leq 0$, so let us argue by contradiction and assume $d_0 > 0$.

As γ lies in the exterior of D we have that for $\lim_{z\in\Omega,\, z\to\gamma} d(\Sigma_\Omega(t)(z)) = -\infty$, from which it follows that d_0 is realized at a point $z_0 \in \Omega$. Hence the normal vector to $\Sigma_\Omega(t)(z_0)$ must be perpendicular to $\Sigma_\Omega(d_0 + t)$, since otherwise we can produce z_1 close to z_0 so that $d(z_1) > d(z_0)$. And as $\Sigma_D(d_0 + t)$ is mean-convex with respect to $\widetilde{\mathrm{Ep}_\Omega}(t)(z_0)$ and $d_0 > 0$, we have that $\widetilde{\mathrm{Ep}_\Omega}(t)(z_0)$ extends to a geodesic orthogonal to $\Sigma_D(d_0 + t)$ with z_0 as it backwards endpoint. But then at $\Sigma_\Omega(t)(z_0) = \Sigma_D(d_0 + t)(z_0)$ (with respect to $\widetilde{\mathrm{Ep}_\Omega}(t)(z_0)$) the surface $\Sigma_\Omega(t)$ has a principal curvature $\leq \tanh(t)$ while the surface $\Sigma_D(d_0 + t)$ has principal curvatures $> \tanh(t)$, which contradicts the definition of d_0. Hence the claim is proven.

The claim combined with Proposition 4.2 show that if $t > \frac{\epsilon}{2}$ then $\Sigma_\Omega(t)$ is contained in $C_t(\gamma)$. As $\Sigma_\Omega(t)$ is obtained from Σ_Ω by flowing distance t along the normal flow and $C_t(\gamma)$ is the t-neighbourhood of $C(\gamma)$, then from $\Sigma_\Omega(t) \subseteq C_t(\gamma)$ it follows that Σ_Ω lies in $C_{2t}(\gamma)$. The result now follows by taking $t \to (\epsilon/2)^+$. $\square$

Remark 7.5. As with Remark 7.2, we have that Proposition 7.4 shows that if we have that $\|\mathscr{S}(f)\|_\infty, \|\mathscr{S}(g)\|_\infty < 1$ then the Epstein–Poincaré map has image inside the convex core. This is in contrast with the analogous result for convex co-compact hyperbolic 3-manifolds, where the condition $\|\mathscr{S}(f)\|_\infty, \|\mathscr{S}(g)\|_\infty < 1$ is not required. This is because in the convex co-compact case we can take a point at infinity where the conformal factor between the Poincaré and the Thurston metric is maximized (by the co-compact action in the boundary) and show that such maximum is bounded by 1, rather than argue with tangencies of surfaces as in the proof of Proposition 7.4.

Remark 7.6. While in Propositions 7.1, 7.3, 7.4 the restrictions on the norm of the Schwarzian or the curvature are not necessarily sharp, some restriction is necessary. This can be seen for instance in the following example. Denote by Ω_0 the complement of the real segment $[0,1]$ in $\hat{\mathbb{C}}$ and by Ω_n a sequence of domains bounded by equipotentials of Ω_0 so that $\Omega_n \xrightarrow{n\to\infty} \Omega_0$. Since that the convex core of Ω_0 is given by the half-plane defined by the circular arc, it is easy to see that Ep_{Ω_0} pierces through the convex core. As Ω_n bound equipotentials of Ω_0 and $\Omega_n \xrightarrow{n\to\infty} \Omega_0$, one can verify that the stronger conclusions from Propositions 7.1, 7.3, 7.4 do not follow for n sufficiently large.

8 Extending variational formula to non-immersed case

The goal of this section is to extend the Schläfli formula to the case when the Poincaré–Epstein surfaces are not immersions and prove Theorem 5.2. In order to do so, we will need to generalize various parts of the proof of Schläfli found in [36].

Let us recall the setup in Section 5.2. Let $(\gamma_t)_{t\in[0,1]}$ be a $C^{5,\alpha}$ family of Jordan curves ($\alpha > 0$). Let $\Omega(r), A(r), \Omega^*(r)$ be the piecewise decomposition of $\hat{\mathbb{C}}$ where we defined the family of piecewise smooth maps $E_{r,t} : \hat{\mathbb{C}} \to \mathbb{H}^3$. We define the unit normal vector field $\vec{n}$ using $\widetilde{\mathrm{Ep}}_{\Omega_t}$ and $\widetilde{\mathrm{Ep}}_{\Omega_t^*}$, and on $A(r)$ we choose $\vec{n}$ to have positive vertical component as in Lemma 5.5.

We first extends the notion of the shape operator.

Lemma 8.1. *There is a piecewise $C^{2,\alpha}$ family of linear maps $B_{r,t}(p) : \mathbb{R}^2 \to \vec{n}^{\perp}(E_{r,t}(p))$ for $\{(r,t,p) \,|\, r \in (1-\varepsilon, 1), t \in [0,1], p \in \Omega(r)$ or $A(r)$ or $\Omega^*(r)\}$, so that at any point where $E_{r,t}$ is an immersion, $B_{r,t}(p)v$ agrees with $B(D_pE_{r,t}(v))$, where B is the shape operator of the image of $E_{r,t}$.*

Proof. For $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_{\Omega} \neq 1\} \subseteq \Omega$ (and analogously for Ω^*) and $v \in \mathbb{R}^2$, it follows from elementary differential geometry that $B_{r,t}(p)v$ satisfies

$$B_{r,t}(p)v = -\left(D^i(p)\vec{n}^j(p)\Gamma^k_{i,j}(p) + \frac{\partial \vec{n}^k}{\partial v}(p)\right) e_k(p), \tag{8.1}$$

where we are using Einstein's notation, $e_1(p), e_2(p), e_3(p)$ is the canonical base for $T\mathbb{H}^3$ at $\mathrm{Ep}_{\Omega}(p)$, $\Gamma^k_{i,j}$ its Christoffel symbols, $\vec{n} = \vec{n}^i e_i$ are the coordinates of the normal vector $\vec{n} = \widetilde{\mathrm{Ep}}_{\Omega}$ and $D^i e_i$ the coordinates of $D\,\mathrm{Ep}_{\Omega}(v)$. As the right-hand side of (8.1) is well-defined along $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_{\Omega} = 1\}$, we use (8.1) to define $B_{r,t}(p)v$. Usually the right-hand side of (8.1) is denoted by $-\frac{D\vec{n}}{dv}$, where $\frac{D\vec{n}}{dv}$ is the *covariant derivative* of the vector field $\vec{n}$ along the parametrization Ep_{Ω}. By abuse of notation we will still denote $\frac{D\vec{n}}{dv}$ as $\nabla_{D_pEv}\vec{n}$, even though this only holds if E is an immersion at p.

For the region $A(r)$ we can define $B_{r,t}$ by observing that the map $E_{r,t}$ is the composition of a smooth map into the horizontal lines described in step (C2). The union of these lines are immersed for r sufficiently close to 1, and hence have a well-defined shape operator. Hence we define $B_{r,t}$ as the pullback of such shape operator by $E_{r,t}$. As $B_{r,t}$ is defined as a pullback of a shape operator, it satisfies the analogous identity to (8.1).

It follows then by Lemma 5.5 that $B_{r,t}$ is piecewise $C^{2,\alpha}$ in $\{(r,t,p) \,|\, r \in (1-\varepsilon, 1), t \in [0,1], p \in \Omega(r)$ or $A(r)$ or $\Omega^*(r)\}$, and agrees with the pullback by $E_{r,t}$ of the shape operator of the image of $E_{r,t}$ whenever $E_{r,t}$ is an immersion. ☐

Remark 8.2. Observe that given that we are defining $B_{r,t}$ as a pullback along A_r, we have that whenever $E_{r,t}$ is not immersed at a point of A_r, $B_{r,t}$ will be the 0 vector along the non-immersed direction. This is quite different from the behavior at $\Omega(r), \Omega^*(r)$. $E_{r,t}$ is not immersed at points of $\Omega(r), \Omega^*(r)$ where the curvatures at infinity are -1, since the metric

$$\mathrm{I}(X,Y) = \frac{1}{4}\hat{\mathrm{I}}\left((\mathrm{id} + \hat{B})X, (\mathrm{id} + \hat{B})Y\right)$$

will vanish precisely at directions X (at infinity) whenever $\hat{B}X = -X$. We note that whenever $\hat{B}$ does not have eigenvalue -1 then $(\mathrm{id} + B)(\mathrm{id} + \hat{B}) = 2\,\mathrm{id}$. Therefore

$$\mathrm{I}(BX, BY) = \frac{1}{4}\hat{\mathrm{I}}\left((\mathrm{id} + \hat{B})BX, (\mathrm{id} + \hat{B})BY\right) = \frac{1}{4}\hat{\mathrm{I}}\left((\mathrm{id} - \hat{B})X, (\mathrm{id} - \hat{B})Y\right).$$

Hence for a given eigenvector X ($\|X\|_{\rho_\Omega} = 1$) with principal curvature $k \neq -1$ we see that $\|BX\|_{\mathbb{H}^3} = \frac{|1-k|}{2}$, where this norm is given by the hyperbolic metric in $\mathbb{H}^3$. Hence for $k = -1$ we have that $\|BX\|_{\mathbb{H}^3} = 1$ is a unit vector.

Remark 8.3. When there is no ambiguity we will drop the subscripts r, t in $E_{r,t}$ to simplify the notation.

The following generalizes the key formula to prove the differential Schläfli formula (see [36, Proposition 5]). Let $\frac{\partial}{\partial t}|_{t=0} E_{r,t} = \xi$ be the piecewisely defined vector field by the first order variation on t, and let $\frac{D}{dt}$ denote the covariant derivative of a vector field along a curve in $\mathbb{H}^3$.

Proposition 8.4. *For any $p \in \hat{\mathbb{C}} = \Omega(r) \cup A(r) \cup \Omega^*(r)$ and $u, v \in \mathbb{R}^2$ we have*

$$\left\langle \frac{D}{d\xi}(B(p)u), D_pE(v) \right\rangle = -\left\langle \frac{D}{du}\frac{D}{d\xi}\vec{n}, D_pE(v) \right\rangle + \langle R(\xi, D_pE(u))\vec{n}, D_pE(v)\rangle \tag{8.2}$$

where we follow the convention $R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z$, and $\langle\cdot,\cdot\rangle$ is the hyperbolic metric tensor in $\mathbb{H}^3$.

Proof. Recall that we have the equality (all evaluated at p and using the notation in (8.1))

$$Bu = -\frac{D}{du}\vec{n}$$

Differentiating along ξ and taking inner product with $DE(v)$

$$\begin{aligned} \left\langle \frac{D}{d\xi}Bu, DE(v) \right\rangle &= -\left\langle \frac{D}{d\xi}\frac{D}{du}\vec{n}, DE(v) \right\rangle \\ &= -\left\langle \frac{D}{du}\frac{D}{d\xi}\vec{n}, DE(v) \right\rangle + \langle R(\xi, DE(u))\vec{n}, DE(v)\rangle, \end{aligned} \tag{8.3}$$

where we are using the curvature tensor to exchange the order of derivations. ∎

Remark 8.5. At points where E is an immersion, we can write $\langle \nabla_\xi(Bu), DE(v)\rangle$ as

$$\langle \nabla_\xi(Bu), DE(v)\rangle = \langle B'(DE(u)), DE(v)\rangle + \langle \nabla_{Bu}\xi, DE(v)\rangle$$

which is the formula appearing in [36, Prop. 5], where B' is the covariant derivative with respect to t of B in the immersed surface image. At points where E is an immersion (8.2) can be written as

$$\langle \nabla_\xi(B(p)u), D_pE(v)\rangle = -\left\langle \nabla_{D_pE(v)}\nabla_\xi\vec{n}, D_pE(u) \right\rangle + \langle R(\xi, D_pE(u))\vec{n}, D_pE(v)\rangle$$

given the identification between covariant derivatives and connections.

We recall that da is the area form on the Epstein surface. We next extend the form $\left\langle \frac{D}{d\xi}(B\cdot), DE(\cdot) \right\rangle$ da to the non-immersed case.
Note that the trace of $\langle R(\xi, DE(\cdot))\vec{n}, DE(\cdot)\rangle$ is $-2\langle \xi, \vec{n}\rangle$ at immersion points, while the derivative of volume $V_2(r,t)$ bounded by $E_{r,t}$ is given by

$$\left.\frac{\partial}{\partial t}\right|_{t=0} V_2(r,t) = \int_{\hat{\mathbb{C}}} -\langle \xi, \vec{n}\rangle E^*(\mathrm{d}a), \tag{8.4}$$

where the minus sign is due to the fact that the normal vector points inwards.
Our goal is to show that if we take the trace in the remaining terms $\langle \nabla_\xi(B(p)\cdot), D_pE(v\cdot)\rangle$, $\left\langle \nabla_{D_pE(\cdot)}\nabla_\xi \vec{n}, D_pE(\cdot)\right\rangle$ in (8.2), integrate them against $E^*(\mathrm{d}a)$ over p and make $r \to 1^-$ we obtain the right-hand side of the equation in Theorem 5.2. As $E_{r,t}$ is not a piecewise immersion, our main concern is how to perform this trace for a non-immersion. The answer is that even though the trace is not well defined (as the metric degenerates), the trace times the area form $E^*(\mathrm{d}a)$ extends to a piecewise differential for in $\hat{\mathbb{C}}$. Let us address first this procedure in $\Omega(r), \Omega^*(r)$.

Lemma 8.6. *The* 2*-form* $\operatorname{tr}\langle R(\xi, DE(\cdot))\vec{n}, DE(\cdot)\rangle E^*\mathrm{d}a$ *defined on the set of immersion points* $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_\Omega \neq 1\}$ *extends as* $C^{2,\alpha}$ *differential* 2*-form to the locus* $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_\Omega = 1\}$ *as* $-2\langle \xi, \vec{n}\rangle E^*\mathrm{d}a$.
Similarly the 2*-form* $\operatorname{tr}\left\langle \nabla_{D_pE(\cdot)}\nabla_\xi \vec{n}, D_pE(\cdot)\right\rangle E^*(\mathrm{d}a)$ *extends as* $\mathrm{d}(i_{\nabla_\xi \vec{n}})$*, where* $i_{\nabla_\xi \vec{n}}$ *is the* 1*-form defined by* $u \mapsto \langle D_pEu, \nabla_\xi \vec{n}\rangle$ *and* d *denotes the exterior derivative.*

Proof. As per the discussion before this Lemma, in the set $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_\Omega \neq 1\}$ the 2-form $\operatorname{tr}\langle R(\xi, DE(\cdot))\vec{n}, DE(\cdot)\rangle E^*(\mathrm{d}a)$ agrees with $-2\langle \xi, \vec{n}\rangle E^*\mathrm{d}a$. As $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_\Omega \neq 1\}$ is an open dense set of Ω, then we can extend uniquely $\operatorname{tr}\langle R(\xi, DE(\cdot))\vec{n}, DE(\cdot)\rangle E^*(\mathrm{d}a)$ as $-2\langle \xi, \vec{n}\rangle E^*\mathrm{d}a$ to all of Ω, as $-2\langle \xi, \vec{n}\rangle E^*\mathrm{d}a$ is a well-defined $C^{2,\alpha}$ 2-form in Ω.
Similarly, on the set $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_\Omega \neq 1\}$ we have that

$$\operatorname{tr}\left\langle \nabla_{D_pE(\cdot)}\nabla_\xi \vec{n}, D_pE(\cdot)\right\rangle E^*(\mathrm{d}a) = E^*(\operatorname{div}(\nabla_\xi \vec{n})\mathrm{d}a) = E^*(\mathrm{d}\langle \cdot, \nabla_\xi \vec{n}\rangle) = \mathrm{d}(i_{\nabla_\xi \vec{n}}).$$

As $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_\Omega \neq 1\}$ is an open dense set in Ω and $-\mathrm{d}(i_{\nabla_\xi \vec{n}})$ is a well-defined $C^{2,\alpha}$ form in Ω, we have that $-\operatorname{tr}\left\langle \nabla_{D_pE(\cdot)}\nabla_\xi \vec{n}, D_pE(\cdot)\right\rangle E^*(\mathrm{d}a)$ extends uniquely as a $C^{2,\alpha}$ 2-form. □

Observe that by Proposition 8.4 and Lemma 8.6, on $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_\Omega \neq 1\}$ we have that

$$\operatorname{tr}\langle \nabla_\xi(B\cdot), D_pE\cdot\rangle E^*(\mathrm{d}a) = -\mathrm{d}(i_{\nabla_\xi \vec{n}}) - 2\langle \xi, \vec{n}\rangle E^*\mathrm{d}a, \tag{8.5}$$

which by Lemma 5.4 yields that in particular on $\{z \in \Omega \mid \|\mathscr{S}(f^{-1})(z)\|_\Omega \neq 1\}$ of Ω we have that

$$2E^*(\delta H + \frac{1}{4}\langle \delta \mathrm{I}, \mathrm{I\!I}\rangle\ \mathrm{d}a) = -\mathrm{d}(i_{\nabla_\xi \vec{n}}) - 2\langle \xi, \vec{n}\rangle E^*\mathrm{d}a. \tag{8.6}$$

Proof of Theorem 5.2. We proceed as in the proof of Theorem 5.7 by taking r close to 1 and defining $V_1(r,t), V_2(r,t)$ as before, so that we have $V(\gamma_t) = V_1(r,t) + V_2(r,t)$. In particular, $V_2(r,t)$ is defined as the volume bounded by $E_{r,t}$. Namely, extend $E_{r,t} : \hat{\mathbb{C}} \to \mathbb{H}^3$ as a map from the closed ball B^3 so that

$$V_2(r,t) = \int_{B^3} E^*_{r,t}(\mathrm{vol}_{\mathbb{H}^3}).$$

For $E_{r,t}$ in $A(r)$, we can establish and trace (8.2) in the embedded surface that contains the image of $E_{r,t}$ (for r sufficiently close to 1) and then take the pullback by $E_{r.t}$.

By Stokes, this definition does not depend on the specific extension of $E_{r,t}$ to B^3. Since $E_{r,t}$ vary $C^{3,\alpha}$ as piecewisely defined map from $\Omega(r), \Omega^*(r), A(r)$, we can take the extension to vary $C^{3,\alpha}$ on t and check that $\partial_t V_2(r,t)$ is given by

$$\partial_t V_2 = \left(\int_{\Omega(r)} + \int_{\Omega^*(r)} + \int_{A(r)} - \langle \xi, \vec{n} \rangle \, E^* \mathrm{d}a \right)$$

where $\xi = \partial_t E_{r,0}$ and $\mathrm{d}a$ is the area form of the orthogonal plane to $\vec{n}$. The negative sign is due to the fact that we are taking normal vector $\vec{n}$ pointing *inward* the region bounded by $E_{r,t}$.

Applying (8.5) we have then

$$\partial_t V_2 = \left(\int_{\Omega(r)} + \int_{\Omega^*(r)} + \int_{A(r)} \frac{1}{2} \operatorname{tr} \langle \nabla_\xi (B\cdot), D_p E \cdot \rangle \, E^*(\mathrm{d}a) + \frac{1}{2} \mathrm{d}(i_{\nabla_\xi \vec{n}}) \right).$$

Applying Stokes theorem for $\frac{1}{2}\mathrm{d}(i_{\nabla_\xi \vec{n}})$ yields the integral of $\frac{1}{2} i_{\nabla_\xi \vec{n}}$ over each boundary component. Since $E^{r,t}$ is embedded along $\partial A(r)$, then as in [36] we have that along $\partial A(r)$, we have

$$\begin{aligned} i_{\nabla_\xi (\vec{n}^{\Omega(r)})} + i_{\nabla_\xi (\vec{n}^{A(r)})} &= \frac{\partial \theta^+}{\partial t} E^* \mathrm{d}\ell \\ i_{\nabla_\xi (\vec{n}^{\Omega^*(r)})} + i_{\nabla_\xi (\vec{n}^{A(r)})} &= \frac{\partial \theta^-}{\partial t} E^* \mathrm{d}\ell \end{aligned}$$

where $\theta^+(x)$ (respectively $\theta^-(x)$) is the exterior dihedral angle of the planes orthogonal to $\vec{n}^{\Omega(r)}, \vec{n}^{A(r)}$ at $E(x)$ (respectively $\vec{n}^{\Omega(r)^*}, \vec{n}^{A(r)}$ at $E(x)$), and $\mathrm{d}\ell$ is the length form in $\mathbb{H}^3$.

Applying then Stokes for $\partial_t V_2$ we get

$$\begin{aligned} \partial_t V_2 = &\left(\int_{\Omega(r)} + \int_{\Omega^*(r)} + \int_{A(r)} \frac{1}{2} \operatorname{tr} \langle \nabla_\xi (B\cdot), D_p E \cdot \rangle \, E^*(\mathrm{d}a) \right) \\ &+ \frac{1}{2} \left(\int_{\partial\Omega(r)} \frac{\partial \theta^+}{\partial t} E^* \mathrm{d}\ell + \int_{\partial\Omega^*(r)} \frac{\partial \theta^-}{\partial t} E^* \mathrm{d}\ell \right). \end{aligned} \tag{8.7}$$

This proves the analog of Theorem 5.3 for the non-immersed case. Then finally by applying (8.6) on the open dense set where $\mathrm{Ep}_\Omega, \mathrm{Ep}_{\Omega^*}$ are immersions, taking $r \to 1^-$ and proceeding as in the proof of Theorem 5.7, we have the desired formula. $\square$

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