Universal Liouville action as a renormalized volume and its gradient flow

Martin Bridgeman[*]   Kenneth Bromberg[†]   Franco Vargas Pallete[‡]   Yilin Wang[§]

April 17, 2023

Abstract

The universal Liouville action (also known as the Loewner energy) is a non-negative Kähler potential on the Weil-Petersson universal Teichmüller space which can be identified with the family of Weil-Petersson quasicircles via conformal welding. This action is invariant under Möbius transformations, our main result shows that it equals the renormalized volume of the non-compact subset of the hyperbolic 3-space bounded by the two Epstein-Poincaré surfaces associated with the quasicircle in analogy to the theory for convex co-compact hyperbolic 3-manifolds. We also study the gradient descent flow of the universal Liouville action with respect to the Weil-Petersson metric and show that the flow always converges to the origin (the circle). This provides a bound of the Weil-Petersson distance to the origin by the universal Liouville action.

Contents

[*] bridgem@bc.edu Boston College, Chestnut Hill, MA, USA

[†] bromberg@math.utah.edu University of Utah, Salt Lake City, UT, USA

[‡] franco.vargaspallete@yale.edu Yale University, New Haven, CT, USA

[§] yilin@ihes.fr Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France

1 Introduction

For a Jordan curve $\gamma \subset \hat{\mathbb{C}}$, we let Ω and Ω^* be the two connected component of $\hat{\mathbb{C}} \smallsetminus \gamma$, ρ_Ω and ρ_{Ω^*} be the Poincaré (hyperbolic) metric (with constant Gauss curvature -1) in Ω and Ω^* respectively. We consider $\hat{\mathbb{C}}$ as the conformal boundary of the hyperbolic 3-space $\mathbb{H}^3$. In [10] C. Epstein gave a natural way to associate to each conformal metric on $\hat{\mathbb{C}}$ a surface in $\mathbb{H}^3$. We will recall the basics on Epstein surfaces in Section 3. Let $\mathrm{Ep}_\Omega : \Omega \to \mathbb{H}^3$ be the Epstein-Poincaré map, namely, the Epstein map associated with the metric ρ_Ω, similarly for $\mathrm{Ep}_{\Omega^*} : \Omega^* \to \mathbb{H}^3$. The maps Ep_Ω, Ep_{Ω^*} are smooth, extend continuously to the identity map on γ, and are immersions almost everywhere. We call their images as the Epstein-Poincaré surfaces Σ_Ω and Σ_{Ω^*}. In particular, we note that the Epstein-Poincaré surfaces are non compact and have infinite area. We show the following results.

Proposition 1.1 (See Proposition 4.1)**.** *If γ is not a circle, then the two Epstein-Poincaré surfaces Σ_Ω and Σ_{Ω^*} are disjoint except at γ.*

It follows directly from the definition of Epstein-Poincaré map that if γ is a circle, then both Σ_Ω and Σ_{Ω^*} are the totally geodesic plane bounded by γ with opposite orientation (see Example 3.1).

Proposition 1.2 (See Corollary 3.13)**.** *When γ is asymptotically conformal (see Theorem 3.10), there is a neighborhood of γ in Ω on which the Epstein-Poincaré map Ep_Ω is an immersion and an embedding which fixes γ.*

Quasicircles are in natural correspondence with points in the universal Teichmüller space $T(1)$, where we identify a quasicircle with its conformal welding homeomorphism. We are interested in a special class of quasicircles, i.e. Weil-Petersson quasicircles, which corresponds to the Weil-Petersson universal Teichmüller space $T_0(1)$. This space has been studied extensively for it being the connected component of the *unique* homogeneous Kähler metric on $T(1)$ (i.e. the Weil-Petersson metric) [31], and have a big number of equivalent descriptions from very different perspectives, see, e.g., [2, 7, 12, 27, 34, 35].

Weil-Petersson quasicircles are asymptotically conformal, so Propositions 1.1 and 1.2 allow us to define the signed volume between Σ_Ω and Σ_{Ω^*}. A priori, this volume takes value in $(-\infty, \infty]$ (see Section 4.2 for more details). However, we show the following result.

Theorem 1.3. *If γ is a Weil-Petersson quasicircle, then the signed volume between the two Epstein-Poincaré surfaces, denoted as $V(\gamma)$, is finite.*

See Proposition 4.5 for the proof for smooth Jordan curves. The result for general Weil-Petersson quasicircles is obtained from an approximation argument, see Corollary 5.15.

Since $T_0(1)$ has a remarkably unique homogeneous Kähler structure, its Kähler potential is of critical importance. Takhtajan and Teo defined the *universal Liouville action* $\mathbf{S}$ on $T_0(1)$ and showed it to be such a Kähler potential [31]. In this work, we will consider the universal Liouville action as defined for Jordan curves (see Section 2.3), and denote it as $\tilde{\mathbf{S}}$ for clarity. The functional $\tilde{\mathbf{S}}(\gamma)$ can actually be defined for arbitrary Jordan curve, but it is finite if and only if γ is a Weil-Petersson quasicircle. Moreover, $\tilde{\mathbf{S}}$ is invariant under Möbius transformations of $\hat{\mathbb{C}}$ (i.e. under the $\mathsf{PSL}_2(\mathbb{C})$ action). As the $\mathsf{PSL}_2(\mathbb{C})$ action extends to orientation preserving isometries of $\mathbb{H}^3$, it is very natural to search for a characterization of the class of Weil-Petersson quasicircles and an expression of $\tilde{\mathbf{S}}$ in terms of geometric quantities in $\mathbb{H}^3$.

A pioneering work of C. Bishop [2] shows that the class of Weil-Petersson quasicircles can be characterized as Jordan curves bounding minimal surfaces in $\mathbb{H}^3$ with finite total curvature. We obtain the following similar characterization in terms of Epstein-Poincaré surfaces. In fact, the Epstein maps come with a well-defined unit normal $\vec{n}$ pointing away from Ω and from Ω^* respectively. The mean curvature $H := \mathrm{Tr}(B)/2$ is defined using the shape operator $B(v) := -\nabla_v \vec{n}$.

Theorem 1.4 (See Corollary 3.9)**.** *We have for all Jordan curves,*

$$\int_{\Sigma_\Omega} H \mathrm{d}a = \int_{\Sigma_\Omega} |\det B| \, \mathrm{d}a = \int_{\mathbb{D}} |\mathscr{S}(f)(z)|^2 \frac{(1-|z|^2)^2}{4} \mathrm{d}^2 z$$

where $f : \mathbb{D} \to \Omega$ is any conformal map, $\mathscr{S}(f) = f'''/f' - (3/2)(f''/f')^2$ is the Schwarzian derivative of f, $\mathrm{d}a$ is the area form induced from $\mathbb{H}^3$, and $\mathrm{d}^2 z$ is the Euclidean area form. In particular, Σ_Ω has finite total mean curvature (and finite total curvature) if and only if γ is a Weil-Petersson quasicircle.

However, no exact identity between the Kähler potential and geometric quantity in $\mathbb{H}^3$ was known. The main result of this work is to provide such an identity.

Definition 1.5. Let γ be a Weil-Petersson quasicircle. We define the renormalized volume (or W-volume) associated with γ as

$$V_R(\gamma) := V(\gamma) - \frac{1}{2} \int_{\Sigma_\Omega \cup \Sigma_{\Omega^*}} H \mathrm{d}a \in (-\infty, \infty).$$

The definition is reminiscent to the renormalized volume[1] for quasi-Fuchsian manifolds [16, 30]. But we emphasize again that Σ_Ω and Σ_{Ω^*} are non compact so the analysis has additional technicality.

Theorem 1.6 (See Corollary 5.10 and Theorem 5.11)**.** *If γ is a $C^{5,\alpha}$ Jordan curve with $\alpha > 0$, we have*

$$\tilde{\mathbf{S}}(\gamma) = 4V_R(\gamma). \tag{1.1}$$

If γ is a Weil-Petersson quasicircle, then we have $\tilde{\mathbf{S}}(\gamma) \geq 4V_R(\gamma)$.

Let us comment briefly on the proof of this theorem. It is easy to check that when γ is a circle, both sides of (1.1) are zero. We then show under regularity assumptions that the first variation of both sides are equal. The variation of $\tilde{\mathbf{S}}$ was proved in [31], which we recall in Theorem 2.1 (and improve in Proposition 2.4). The first variation of V_R is more laborious since the Epstein-Poincaré surfaces are not compact and are immersed only almost everywhere. After administrating appropriate truncation (where we make use of the regularity assumption), we re-derive the Schläfli formula which expresses the variation of V_R in terms of the mean curvature H, the metric I and the second fundamental form II on Epstein surfaces (Theorem 5.1), then translate the variation formula into quantities defined directly on $\Omega, \Omega^* \subset \hat{\mathbb{C}}$ (Theorem 5.7 and Corollary 5.9).

For a general Weil-Petersson quasicircle γ we use an approximation by equipotentials (they are analytic curves and the universal Liouville action increases to that of γ). We believe the identity (1.1) also holds for a general Weil-Petersson quasicircle. However, our approximation argument only implies the inequality due to the lack of tightness for the volume between the Epstein-Poincaré surfaces, see Section 5.3. We are tackling this.

The second topic of this work concerns the gradient descent flow of $\mathbf{S}$ with respect to the Weil-Petersson metric. We proceed similarly as in Bridgeman-Brock-Bromberg [4]. For $[\mu] \in T(1)$ we have a natural isomorphism $T_{[\mu]}T(1) \simeq \Omega^{-1,1}(\mathbb{D}^*)$.

Theorem 1.7 (See Theorem 6.1)**.** *The negative gradient of $\mathbf{S}$ with respect to the Weil-Petersson metric is the vector field*

$$V_{[\mu]} := -4\frac{\overline{\mathscr{S}(g_\mu)}}{\rho_{\mathbb{D}^*}} \in \Omega^{-1,1}(\mathbb{D}^*).$$

Moreover, the gradient descent flow of $\mathbf{S}$ starting from any point in $T_0(1)$ converges to the origin $[0]$ which corresponds to the round circle.

Using the gradient flow, we also obtain bounds of the Weil-Petersson distance on $T_0(1)$ in terms of the universal Liouville action.

Theorem 1.8 (See Theorem 6.3)**.** *There exist universal positive constants c and K such that for all $[\mu] \in T_0(1)$, we have $c(\mathrm{dist}_{\mathrm{WP}}([\mu], [0]) - Kc) \leq \mathbf{S}([\mu])$.*

[1]Renormalized volume of a convex co-compact hyperbolic 3-manifold is referred to the difference between the volume and half of the boundary area defined through a foliation near the ends. Our formula is similar to the definition of the *W-volume*. However, in the convex co-compact case, they only differ by a multiple of Euler characteristics of the boundary [16, Lem. 4.5].

Finally, let us make a few remarks on the motivation behind this work and additional comments on the relation with the literature.

S. Rohde and the last author introduced the *Loewner energy* for Jordan curves [24, 33] which is originally motivated from the large deviation theory of random fractal curves Schramm-Loewner evolutions (SLE) [33, 36]. In a certain sense, the Loewner energy is the *action functional* which characterizes the law of SLE. It turns out quite surprisingly that the Loewner energy equals exactly $\mathbf{S}/\pi$ as proved in [34]. Since we will not make use of Loewner theory but only the fact of $\mathbf{S}$ is a Kähler potential on $T_0(1)$, we adopt the terminology of *universal Liouville action* here. SLEs play a central role in the emerging field of two-dimensional random conformal geometry. In particular, they provide a mathematical description of the interfaces in statistical mechanics models [17, 26, 28] and also a new way of thinking about 2D conformal field theory (CFT) [1, 8, 13, 20]. On the other hand, $\mathbb{H}^3$ is the Riemannian analog of AdS_3 space. Our main result Theorem 1.6 can be interpreted as a *holographic principle* for the Loewner energy that is reminiscent of the conjectural $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence pioneered by Maldacena [18] (see also, e.g., [19, 37]). The authors are not aware of a (even conjectural) holographic principle for SLE nor for random conformal geometry in general, this work may be a first step towards this direction. We also mention [14] gives a holographic expression for determinants of discrete Dirac operator on periodic bipartite isoradial graphs.

Renormalized volume as a Liouville action has been previously studied for convex co-compact group actions in $\mathbb{H}^3$ (see work by Takhtajan-Teo [30] and Krasnov-Schlenker [16]), or equivalently, for conformally compact hyperbolic metrics. A set of applications of this study are bounds for the hyperbolic volume of mapping tori of pseudo-Anosov maps in term of their Weil-Petersson translation length (by Brock [6]) or their entropy (by Kojima-McShane [15]). This uses a bound (by Schlenker [25]) for renormalized volume in terms of Weil-Petersson distance by studying the gradient of the Liouville action, similar to our bound in Theorem 6.3. Moreover, we show in Theorem 6.1 that every flowline of the gradient converges to the absolute minimum, in analogy to the result done by the first three authors [5] for the relatively acylindrical case. This builds on work by the first two authors and Brock [4], where they used the gradient flow to find the minimum of renormalized volume for a boundary incompressible hyperbolic 3-manifold.

The paper is organized as follows: In Section 2 we collect all the basics about universal Teichmüller space, its Kähler geometry, characterizations of the Weil-Petersson universal Teichmüller space, and the universal Liouville action. In Section 3 we recall the definition of Epstein surfaces and the correspondence between geometric quantities on the surface versus on the conformal boundary. We also prove the immersion and embeddedness of the Epstein-Poincaré surfaces associated with an asymptotically conformal curve. In Section 4 we study the relation between the two Epstein-Poincaré surfaces associated with the same curve. We show that they are disjoint (except for a circle), and that if the curve is regular enough, the volume between the Epstein surfaces is finite. In Section 5, we prove the variational formula for the renormalized volume and prove the main theorem Theorem 1.6. The last section 6 is independent from Sections 3, 4 and 5 and deals with the gradient flow of the universal Liouville action.

Acknowledgments

We thank Andrea Seppi for useful discussion. We also thank Alexis Michelat for comments on an earlier version of the manuscript. M.B. is supported by NSF grant DMS-2005498. K.B. is supported by NSF grant DMS-1906095. F.V.P. is supported by NSF grant DMS-2001997. Y.W. is partially supported by NSF grant DMS-1953945.

2 Universal Weil-Petersson Teichmüller space

2.1 Universal Teichmüller space

We first briefly recall a few equivalent descriptions of the universal Teichmüller space $T(1)$. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{D} = \{z\ ,\ |z| < 1\}$, $\mathbb{D}^* = \hat{\mathbb{C}} - \overline{\mathbb{D}}$ and $\mathbb{S}^1 = \partial\mathbb{D}$. The group of orientation preserving conformal automorphism of $\hat{\mathbb{C}}$ is

$$\text{Möb}(\hat{\mathbb{C}}) = \mathsf{PSL}_2(\mathbb{C}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C},\ ad - bc = 1 \right\}_{/A\sim -A}$$

which acts on $\hat{\mathbb{C}}$ by Möbius transformations $z \mapsto \dfrac{az+b}{cz+d}$. The subgroup preserving $\mathbb{S}^1$ is

$$\text{Möb}(\mathbb{S}^1) = \mathsf{PSU}_{1,1} = \left\{ A = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1 \right\}_{/A\sim -A}$$

which is isomorphic to $\mathsf{PSL}_2(\mathbb{R})$. There are a number of equivalent descriptions that we will use.

Quasisymmetric maps: We write $\text{QS}(\mathbb{S}^1)$ for the group of sense preserving quasisymmetric homeomorphisms of $\mathbb{S}^1$. The *universal Teichmüller space* is

$$T(1) := \text{Möb}(\mathbb{S}^1)\backslash\text{QS}(\mathbb{S}^1) \simeq \{\varphi \in \text{QS}(\mathbb{S}^1),\ \varphi \text{ fixes } -1, -\mathfrak{i} \text{ and } 1\}.$$

$T(1)$ is endowed with a group operation given by the composition and the origin is the identity map $\text{Id}_{\mathbb{S}^1}$.

Beltrami Differentials: Given a Beltrami differential

$$\mu \in L_1^\infty(\mathbb{D}^*) = \{\mu \in L^\infty(\mathbb{D}^*),\ ||\mu||_\infty < 1\},$$

we extend it to $\hat{\mathbb{C}}$ by reflection, i.e. define for $z \in \mathbb{D}$,

$$\mu(z) = \overline{\mu\left(\frac{1}{\bar{z}}\right)}\frac{z^2}{\bar{z}^2}.$$

Let $w_\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the solution to the Beltrami equation $\partial_{\bar{z}} w_\mu = \mu \partial_z w_\mu$ fixing $-1, -\mathfrak{i}$ and 1. Then w_μ preserves $\mathbb{S}^1$ and $w_\mu|_{\mathbb{S}^1} \in \text{QS}(\mathbb{S}^1)$. Since every quasisymmetric circle homeomorphism can be extended to a quasiconformal self-map of $\overline{\mathbb{D}}$, we have

$$T(1) = L_1^\infty(\mathbb{D}^*)/_\sim$$

where $\mu \sim \nu$ if and only if $w_\mu|_{\mathbb{S}^1} = w_\nu|_{\mathbb{S}^1}$. We denote by $\Phi : L^\infty_1(\mathbb{D}^*) \to T(1)$ the projection $\mu \mapsto [\mu]$. Here the origin corresponds to $[0]$.

Univalent maps: If instead we extend μ by 0 on $\mathbb{D}$ and let w^μ be the unique solution to $w^\mu_{\bar z} = \mu w^\mu_z$ fixing $-1, -\mathfrak{i}$ and 1, then w^μ is conformal on $\mathbb{D}$. The map $[\mu] \mapsto w^\mu|_{\mathbb{D}}$ identifies $T(1)$ with

$$\{f : \mathbb{D} \to \hat{\mathbb{C}},\ \text{univalent fixing } -1, -\mathfrak{i} \text{ and } 1, \text{ extendable to q.c. map of } \hat{\mathbb{C}}\}, \tag{2.1}$$

since $\mu \sim \nu$ if and only if $w^\mu = w^\nu$ on $\mathbb{D}$. The origin corresponds to $\mathrm{Id}_{\mathbb{D}}$.

Quasicircles: By Riemann mapping theorem, the previous identification also gives

$$T(1) \simeq \{\gamma \text{ quasicircle passing through } -1, -\mathfrak{i}, \text{ and } 1\} \tag{2.2}$$

by the map $[\mu] \mapsto \gamma_\mu := w^\mu(\mathbb{S}^1)$. The origin corresponds to $\gamma_\mu = \mathbb{S}^1$. We can recover the quasisymmetric circle homeomorphism from γ_μ via conformal welding. Let Ω (resp. Ω^*) denote the connected component of $\hat{\mathbb{C}} \setminus \gamma_\mu$ where $-1, -\mathfrak{i}, 1$ are in the counterclockwise direction of $\partial\Omega$ (resp. clockwise direction of $\partial\Omega^*$). Let $f_\mu = w^\mu|_{\mathbb{D}} \colon \mathbb{D} \to \Omega$ and $g_\mu \colon \mathbb{D}^* \to \Omega^*$ be the conformal maps fixing $-1, -\mathfrak{i}, 1$. Then,

$$w_\mu|_{\mathbb{S}^1} = g_\mu^{-1} \circ f_\mu|_{\mathbb{S}^1}$$

since $g_\mu = w^\mu \circ w_\mu^{-1}|_{\mathbb{D}^*}$. We call $g_\mu^{-1} \circ f_\mu|_{\mathbb{S}^1}$ the *welding homeomorphism* of the quasicircle γ_μ passing through $-1, -\mathfrak{i}, 1$.

2.2 Kähler Structure and Weil-Petersson Teichmüller space

We first define the following spaces,

$$\begin{aligned} A_\infty(\mathbb{D}^*) &= \{\phi : \mathbb{D}^* \to \mathbb{C} \text{ holomorphic}, \ \sup_{\mathbb{D}^*} |\phi| \rho_{\mathbb{D}^*}^{-1} < \infty\}, \\ A_2(\mathbb{D}^*) &= \{\phi : \mathbb{D}^* \to \mathbb{C} \text{ holomorphic}, \ \int_{\mathbb{D}^*} |\phi|^2 \rho_{\mathbb{D}^*}^{-1} \,\mathrm{d}^2 z < \infty\} \subset A_\infty(\mathbb{D}^*), \end{aligned}$$

where $\rho_{\mathbb{D}^*}(z) = 4/(1-|z|^2)^2$ is the hyperbolic density function and $\mathrm{d}^2 z = \mathrm{d}x \wedge \mathrm{d}y$ if $z = x + \mathfrak{i}y$. The inclusion is shown in [31, Lem. I.2.1]. We define the similar spaces $A_\infty(\mathbb{D})$ and $A_2(\mathbb{D})$ (and also $A_\infty(\Omega)$ and $A_2(\Omega)$). We will also use the spaces of harmonic Beltrami differentials defined as

$$\begin{aligned} \Omega^{-1,1}(\mathbb{D}^*) &= \{\dot\nu \in L^\infty(\mathbb{D}^*), \dot\nu = \rho_{\mathbb{D}^*}^{-1} \overline{\phi}, \ \phi \in A_\infty(\mathbb{D}^*)\}; \\ H^{-1,1}(\mathbb{D}^*) &= \{\dot\nu \in L^\infty(\mathbb{D}^*), \dot\nu = \rho_{\mathbb{D}^*}^{-1} \overline{\phi}, \ \phi \in A_2(\mathbb{D}^*)\} \subset \Omega^{-1,1}(\mathbb{D}^*). \end{aligned}$$

The universal Teichmüller space $T(1)$ has a canonical complex structure such that $\Phi : L^\infty_1(\mathbb{D}^*) \to T(1)$ is a holomorphic surjection. The holomorphic tangent space at the origin is

$$T_{[0]}T(1) = L^\infty(\mathbb{D}^*)/\ker(D_0\Phi) \simeq \Omega^{-1,1}(\mathbb{D}^*)$$

where

$$\ker(D_0\Phi) = \mathfrak{N}(\mathbb{D}^*) := \{\dot\nu \in L^\infty(\mathbb{D}^*) \colon \int_{\mathbb{D}^*} \dot\nu\phi = 0, \forall\phi \text{ holomorphic and} \int_{\mathbb{D}^*} |\phi| \mathrm{d}^2 z < \infty\}$$

is the space of infinitesimally trivial Beltrami differentials.

The space $L^\infty(\mathbb{D}^*)$ has a natural group structure given by the associated quasiconformal maps. We define $\lambda = \nu \star \mu^{-1}$ if $w_\lambda = w_\nu \circ w_\mu^{-1}$. Thus

$$\lambda = \left(\frac{\nu - \mu}{1 - \overline{\mu}\nu} \frac{\partial_z w_\mu}{\overline{\partial_z w_\mu}}\right) \circ w_\mu^{-1}.$$

We define R_μ to be right multiplication by μ on $L^\infty(\mathbb{D}^*)$. This descends to give a map $R_\mu : T(1) \to T(1)$. Furthermore, the complex structure on $T(1)$ is right-invariant. Therefore, $D_0 R_{[\mu]} : T_{[0]}T(1) \to T_{[\phi]}T(1)$ is a complex linear isomorphism between holomorphic tangent spaces, and we obtain the identification of $T_{[\phi]}T(1) \simeq \Omega^{-1,1}(\mathbb{D}^*)$.

To define a Kähler metric on $T(1)$, one needs to endow $T(1)$ with a Hilbert manifold structure. It is known since [3] that on the subspace $\mathscr{M} = \text{Möb}(\mathbb{S}^1)\backslash \operatorname{Diff}(\mathbb{S}^1)$ there is a unique Kähler metric up to a scalar multiple. However, $\mathscr{M}$ is not complete under the Kähler metric. Takhtajan and Teo extend the Hilbert manifold structure on $T(1)$ by defining the Hermitian metric on the distribution $\mathscr{D}([\mu]) = D_0 R_{[\mu]}(H^{-1,1}(\mathbb{D}^*)) \subset T_{[\mu]}T(1)$ induced from $H^{-1,1}(\mathbb{D}^*)$:

$$\langle \dot\mu, \dot\nu \rangle := \int_{\mathbb{D}^*} \dot\mu \overline{\dot\nu} \rho_{\mathbb{D}^*} \mathrm{d}^2 z, \quad \forall \dot\mu, \dot\nu \in H^{-1,1}(\mathbb{D}^*).$$

They prove that this distribution is integrable and define $T_0(1)$ to be the connected component containing $[0]$ which is called the *Weil-Petersson Teichmüller space*. The Hermitian metric defined above is called the *Weil-Petersson metric*. (One may draw the similarity with the Weil-Petersson metric on Teichmüller spaces of a Fuchsian group Γ where the integral is over $\mathbb{D}^*/\Gamma$.) In terms of the four equivalent definitions of $T(1)$, the subspace $T_0(1)$ is characterized as follows:

- **Quasisymmetric maps**: Y. Shen [27] showed $\varphi \in T_0(1)$ if and only if φ is absolutely continuous with respect to the arclength measure, and $\log \varphi' \in H^{1/2}(\mathbb{S}^1)$, the fractional Sobolev space of functions u such that

$$\|u\|_{H^{1/2}}^2 := \iint_{\mathbb{S}^1 \times \mathbb{S}^1} \left| \frac{u(\zeta) - u(\xi)}{\zeta - \xi} \right|^2 \mathrm{d}\zeta \mathrm{d}\xi < \infty. \tag{2.3}$$

- **Beltrami Differentials:** It is shown in [31] that $[\mu] \in T_0(1)$ if and only if it has a representative $\mu \in L_1^\infty(\mathbb{D}^*)$ such that

$$\int_{\mathbb{D}^*} |\mu(z)|^2 \rho_{\mathbb{D}^*}(z) \mathrm{d}^2 z < \infty.$$

- **Univalent maps:** It is shown in [31, Thm. II.1.12] (see also [7]) that a univalent function $f : \mathbb{D} \to \hat{\mathbb{C}}$ fixing $-1, -\mathfrak{i}, 1$ and extendable to a quasiconformal map of $\hat{\mathbb{C}}$, corresponds to an element of $T_0(1)$ via the identification (2.1) if and only if the Schwarzian derivative

$$\mathscr{S}(f) := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

satisfies

$$\int_{\mathbb{D}} |\mathscr{S}(f)|^2 \rho_{\mathbb{D}}^{-1} \, \mathrm{d}^2 z < \infty. \tag{2.4}$$

In other words, the Bers' embedding $\beta([\mu]) := \mathscr{S}(f) \in A_2(\mathbb{D})$.

Furthermore, let $\tilde{f} = A \circ f$ where A is a Möbius map sending $\Omega = f(\mathbb{D})$ to a bounded domain (as a priori, $\overline{\Omega}$ may contain ∞). Then $f \in T_0(1)$ if and only if

$$\int_{\mathbb{D}} |\mathscr{N}(\tilde{f})|^2 \mathrm{d}^2 z < \infty \tag{2.5}$$

where $\mathscr{N}(\tilde{f}) = \tilde{f}''/\tilde{f}'$ is the *pre-Schwarzian* of $\tilde{f}$. We note that the expression in (2.4) is invariant under the transformation $f \to A \circ f \circ B$, for all $A \in \mathsf{PSL}_2(\mathbb{C})$ and $B \in \mathsf{PSU}_{1,1}$ but the expression in (2.5) is not invariant under such transformations.

- **Quasicircles:** A quasicircle passing through $-1, -\mathfrak{i}, 1$ which corresponds via (2.2) to an element of $T_0(1)$ is called a *Weil-Petersson quasicircle*. It is easy to see that if γ and $\tilde{\gamma}$ are two quasicircles passing through $-1, -\mathfrak{i}, 1$ and $\tilde{\gamma} = A(\gamma)$ for some $A \in \mathsf{PSL}_2(\mathbb{C})$, then $\tilde{\gamma}$ is Weil-Petersson if and only if γ is Weil-Petersson. Therefore, we may extend the definition to say that a Jordan curve γ is Weil-Petersson if and only if it is $\mathsf{PSL}_2(\mathbb{C})$-equivalent to a Weil-Petersson quasicircle passing through $-1, -\mathfrak{i}, 1$.

2.3 Universal Liouville action

Takhtajan and Teo introduced the *universal Liouville action* $\mathbf{S}$ on $T_0(1)$ and showed it to be a Kähler potential on $T_0(1)$. See [31, Thm. II.4.1]. We will consider it as a functional on the space of Weil-Petersson quasicircles.

Indeed, let γ be a Jordan curve which does not pass through ∞. Let D and D^* be respectively the bounded and unbounded connected component of $\hat{\mathbb{C}} \smallsetminus \gamma$, $f : \mathbb{D} \to D$ and $g : \mathbb{D}^* \to D^*$ be *any* conformal maps such that $g(\infty) = \infty$ (note that D might not be Ω, it can also be Ω^*, and f and g are different from the canonical maps f_μ and g_μ). Define

$$\tilde{\mathbf{S}}(\gamma) := \int_{\mathbb{D}} |\mathscr{N}(f)|^2 \, \mathrm{d}^2 z + \int_{\mathbb{D}^*} |\mathscr{N}(g)|^2 \, \mathrm{d}^2 z + 4\pi \log |f'(0)/g'(\infty)| \tag{2.6}$$

and is $\mathsf{PSL}_2(\mathbb{C})$-invariant (it can be seen via the identity with π times the Loewner energy of γ [34]) and finite if and only if γ is a Weil-Petersson quasicircle. The universal Liouville action $\mathbf{S}([\mu])$ for $[\mu] \in T_0(1)$ is defined as $\tilde{\mathbf{S}}(A(\gamma_\mu))$ where γ_μ is the Weil-Petersson quasicircle passing through $-1, -\mathfrak{i}, 1$ corresponding to $[\mu]$ and $A \in \mathsf{PSL}_2(\mathbb{C})$ is any Möbius transformation such that $A(\gamma_\mu)$ is bounded. The universal Liouville action $\mathbf{S}$ satisfies the following properties:

- $\mathbf{S}([\mu]) \geq 0$ for all $[\mu] \in T_0(1)$ (see, e.g., [34, Thm. 1.4]);

- $\tilde{\mathbf{S}}(\gamma) = 0$ if and only if γ is a circle, or equivalently, $[\mu] = [0]$.

The first variation formula of $\mathbf{S}$ from [31] will be a key ingredient in our proofs. We now state it for $\tilde{\mathbf{S}}$. Let γ be the Weil-Petersson quasicircle passing through $-1, -\mathfrak{i}, 1$ corresponding to an element $[\mu]$ of $T_0(1)$. Let Ω and Ω^* be the connected components of $\hat{\mathbb{C}} \smallsetminus \gamma$ as in Section 2.1. Let $f_\mu : \mathbb{D} \to \Omega$ and $g_\mu : \mathbb{D}^* \to \Omega^*$ be the conformal maps fixing $-1, -\mathfrak{i}, 1$. Let $\dot\nu \in H^{-1,1}(\mathbb{D}^*) \simeq T_{[\mu]}T_0(1)$, $t \in (-\|\dot\nu\|_\infty^{-1}, \|\dot\nu\|_\infty^{-1})$, $w_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the solution fixing $-1, -\mathfrak{i}, 1$ to the Beltrami equation

$$\partial_{\bar z} w_t(z) = \begin{cases} 0 & z \in \Omega, \\ t (g_\mu)_* \, \dot\nu(z) \, \partial_z w_t(z) & z \in \Omega^* \end{cases}$$

where

$$(g_\mu)_* \, \dot\nu(z) = \dot\nu \circ g_\mu^{-1} \, \frac{\overline{(g_\mu^{-1})'}}{(g_\mu^{-1})'}.$$

We let $\gamma_t = w_t(\gamma)$ which is a small deformation of γ.

Theorem 2.1 ([31, Cor. II.3.9])**.** *The universal Liouville action satisfies the following first variation formula. Let $\dot\nu \in H^{-1,1}(\mathbb{D}^*) \simeq T_{[\mu]}T_0(1)$,*

$$(\mathrm{d}\mathbf{S})_{[\mu]}(\dot\nu) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{\mathbf{S}}(\gamma_t) = 4 \operatorname{Re} \int_{\mathbb{D}^*} \dot\nu \mathscr{S}(g_\mu) \mathrm{d}^2 z = -4 \operatorname{Re} \int_{\Omega^*} ((g_\mu)_* \dot\nu) \, \mathscr{S}(g_\mu^{-1}) \, \mathrm{d}^2 z.$$

Remark 2.2. We note that compared to the formula in [31], we take the derivative of $\mathbf{S}$ in the real tangent space (which is canonically isomorphic to the holomorphic tangent space) while [31] takes derivative in the holomorphic tangent space and both derivatives are related by

$$(\mathrm{d}\mathbf{S})_{[\mu]}(\dot\nu) = 2 \operatorname{Re} \partial_{\dot\nu} \mathbf{S}([\mu]).$$

The last equality in Theorem 2.1 follows from a change of variable and the chain rule for Schwarzian derivatives which shows

$$\mathscr{S}(g^{-1}) = -\mathscr{S}(g) \circ g^{-1} (g^{-1\prime})^2.$$

Remark 2.3. We choose $\dot\nu$ to be *harmonic* Beltrami differential as $H^{-1,1}(\mathbb{D}^*)$ is isomorphic to $T_{[\mu]}T_0(1)$, in particular, supplementary to the infinitesimally trivial Beltrami differentials $\mathfrak{N}(\mathbb{D}^*)$. Clearly, the variational formula also holds for $\dot\nu \in H^{-1,1}(\mathbb{D}^*) + \mathfrak{N}(\mathbb{D}^*)$ if $\int |\mathscr{S}(g)| \mathrm{d}^2 z < \infty$, which is the case, e.g., whenever the curve γ is smooth.

Combining Theorem 2.1 and Remark 2.3 we obtain the following slightly modified version of the variational formula for $\tilde{\mathbf{S}}$. (We will not need the *two-sided* deformation, but it is more natural when considering the Liouville action as defined for quasicircles and there is almost no cost to add this.) We write $\mathfrak{N}(\Omega)$ (resp. $\mathfrak{N}(\Omega^*)$) for the space of infinitesimally trivial Beltrami differentials on Ω (resp. Ω^*).

Proposition 2.4. *Let $\gamma \subset \mathbb{C}$ be a smooth Jordan curve. Let Ω and Ω^* be the connected components of $\hat{\mathbb{C}} \smallsetminus \gamma$. Let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ be any conformal maps. Let*

$\dot\nu_1 \in H^{-1,1}(\Omega) + \mathfrak{N}(\Omega)$ and $\dot\nu_2 \in H^{-1,1}(\Omega^*) + \mathfrak{N}(\Omega^*)$ and w_t be any solution to the Beltrami equation

$$\frac{\partial_{\bar z} w_t}{\partial_z w_t} = \begin{cases} t\dot\nu_1, & z \in \Omega, \\ t\dot\nu_2, & z \in \Omega^*. \end{cases}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{\mathbf{S}}(w_t(\gamma)) = -4\,\mathrm{Re}\left(\int_\Omega \dot\nu_1 \mathscr{S}(f^{-1})\mathrm{d}^2 z + \int_{\Omega^*} \dot\nu_2 \mathscr{S}(g^{-1})\mathrm{d}^2 z\right).$$

The normalization of γ, w_t, f and g are not needed as the formula is invariant under other choices.

Proof. We only need to justify how the variation formula for one-sided quasiconformal deformation implies the two-sided deformation.

We consider first the two-variable family of quasiconformal maps $w_{s,t}$ whose Beltrami coefficients are $s\dot\nu_1$ in Ω and $t\dot\nu_2$ in Ω^* for $t, s \in \mathbb{R}$ small enough. We have by the composition rule of quasiconformal maps

$$w_{s,t} = u_s^t \circ w_{0,t}$$

where

$$\frac{\partial_{\bar z} w_{0,t}}{\partial_z w_{0,t}} = \begin{cases} 0, & z \in \Omega, \\ t\dot\nu_2, & z \in \Omega^*, \end{cases} \qquad \frac{\partial_{\bar z} u_s^t}{\partial_z u_s^t} = \begin{cases} s(w_{0,t})_*\dot\nu_1, & z \in w_{0,t}(\Omega), \\ 0, & z \in w_{0,t}(\Omega^*). \end{cases}$$

From the one-sided variation we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{\mathbf{S}}(w_{0,t}(\gamma)) = -4\,\mathrm{Re}\int_{\Omega^*} \dot\nu_2(z)\, \mathscr{S}(g^{-1})(z)\mathrm{d}^2 z$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \tilde{\mathbf{S}}(w_{s,t}(\gamma)) &= -4\,\mathrm{Re}\int_{w_{0,t}(\Omega)} (w_{0,t})_*\dot\nu_1(z)\, \mathscr{S}(f^{-1} \circ w_{0,t}^{-1})(z)\mathrm{d}^2 z \\ &= -4\,\mathrm{Re}\int_\Omega \dot\nu_1(z)\, \mathscr{S}(f^{-1})(z)\mathrm{d}^2 z + 4\,\mathrm{Re}\int_\Omega \dot\nu_1(z)\, \mathscr{S}(w_{0,t})(z)\mathrm{d}^2 z. \end{aligned} \tag{2.7}$$

Lemma I.2.9 in [31] shows that there exists C such that

$$\|\mathscr{S}(w_{0,t})\|_2 = \left(\int_\Omega \frac{|\mathscr{S}(w_{0,t})|^2}{\rho_\Omega}\mathrm{d}^2 z\right)^{1/2} \le C|t|\|P(\dot\nu_2)\|_2 = C|t|\left(\int_\Omega |P(\dot\nu_2)|^2 \rho_\Omega \mathrm{d}^2 z\right)^{1/2}$$

where $P : H^{-1,1}(\Omega) + \mathfrak{N}(\Omega) \to H^{-1,1}(\Omega)$ is the projection parallel to $\mathfrak{N}(\Omega)$. The second term in (2.7) converges to 0 as $t \to 0$ by Cauchy-Schwarz inequality. Therefore we can apply the chain rule and get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{\mathbf{S}}(w_t(\gamma)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{\mathbf{S}}(w_{t,t}(\gamma)) = -4\,\mathrm{Re}\left(\int_\Omega \dot\nu_1 \mathscr{S}(f^{-1})\mathrm{d}^2 z + \int_{\Omega^*} \dot\nu_2 \mathscr{S}(g^{-1})\mathrm{d}^2 z\right)$$

as claimed. $\square$

3 Epstein-Poincaré surfaces

3.1 Epstein hypersurfaces associated with conformal metrics

In [11] Epstein developed a formula for envelopes of horosphere in terms of conformal metrics in $\mathbb{S}^n = \partial_\infty \mathbb{H}^{n+1}$. Here, the hyperbolic space $\mathbb{H}^{n+1}$ is represented as the interior of the unit ball $\mathfrak{B}^{n+1}$ with the metric:

$$\mathrm{d}s^2 = \frac{4(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_{n+1}^2)}{(1-|x|^2)^2},$$

and $\mathbb{S}^n$ is represented by the unit sphere in $\mathbb{R}^{n+1}$. Let $\rho_{\mathbb{S}^n}$ denote the metric on $\mathbb{S}^n$ induced by the Euclidean metric, namely, the round metric.

Given a domain $\Omega \subseteq \mathbb{S}^n$ and a smooth function $\varphi : \Omega \to \mathbb{R}$, we can associated the conformal metric $\rho := e^\varphi \rho_{\mathbb{S}^n}$ to the parametrized surface

$$\mathrm{Ep}_\rho : z \in \Omega \mapsto \frac{|D\varphi|^2(e^{2\varphi}-1)}{|D\varphi|^2(e^\varphi+1)^2} z + \frac{2D\varphi}{|D\varphi|^2(e^\varphi+1)^2} \in \mathfrak{B}^{n+1} = \mathbb{H}^{n+1}, \tag{3.1}$$

where D denotes the gradient with respect to $\rho_{\mathbb{S}^n}$. As proved in [11, Section 2], the map Ep_ρ solves *the envelop equation* of the family of horospheres $\{H(z,\varphi)\}_{z\in\Omega}$, where

$$H(z,\varphi) := \left\{ \frac{e^{\varphi(z)}}{e^{\varphi(z)}+1} z + \frac{1}{e^{\varphi(z)}+1} Y \,\middle|\, Y \in \mathbb{S}^n \smallsetminus \{z\} \right\} \tag{3.2}$$

is a horosphere centered at z and determined by the value of $\varphi(z)$. Solving the envelop equation means for all $z \in \Omega$,

$$\mathrm{Ep}_\rho(z) \in H(z,\varphi) \quad \text{and} \quad D_z \,\mathrm{Ep}_\rho(T_z\mathbb{S}^n) \subseteq T_{\mathrm{Ep}_\rho(z)} H(z,\varphi). \tag{3.3}$$

We can expand the Epstein map Ep_ρ to the *Epstein Gauss map* $\widetilde{\mathrm{Ep}}_\rho : \Omega \to T^1\mathbb{H}^{n+1}$ by defining $\widetilde{\mathrm{Ep}}_\rho(z)$ as the outer normal vector to $H(z,\varphi)$ at $\mathrm{Ep}_\rho(z)$. The geodesic flow in the direction $-\widetilde{\mathrm{Ep}}_\rho(z)$ converges to z. We have that $\widetilde{\mathrm{Ep}}_\rho$ is always an embedding. In contrast, even though we have called Ep a parametrized surface, the map Ep_ρ need not to be an immersion. For instance, $\varphi \equiv 0$ implies that for any $z \in \mathbb{S}^n$ we have that $\mathrm{Ep}_\rho(z) = 0$ while $\widetilde{\mathrm{Ep}}_\rho(z) = (0,-z)$. Regardless, we will see in Section 5 that because we can parametrize the normal bundle by $\widetilde{\mathrm{Ep}}_\rho$ we will be able to treat the Epstein surface as a parametrized surface.

Geometrically, we can use visual metrics to describe $H(z,\varphi)$. Given $x \in \mathbb{H}^{n+1}$, we define the *visual metric* of x, denoted by ν_x, as the metric in $\mathbb{S}^n = \partial_\infty \mathbb{H}^{n+1}$ defined by the pullback $h^*(\rho_{\mathbb{S}^n})$, where h is any isometry of $\mathbb{H}^{n+1}$ so that $h(x) = 0$. The metric ν_x is well-defined, as the stabilizer of 0 in $\mathrm{Isom}_+(\mathbb{H}^{n+1})$ acts by isometries in $\mathbb{S}^n$. Moreover, as $\mathrm{Isom}_+(\mathbb{H}^{n+1})$ acts conformally in $\mathbb{S}^n$, the metric ν_x is conformal to $\rho_{\mathbb{S}^n}$, meaning that we can write $\nu_x = e^\varphi \rho_{\mathbb{S}^n}$ for some smooth function $\varphi : \mathbb{S}^n \to \mathbb{R}$ that depends only on x. Then it is a simple exercise to verify that $H(z,\varphi)$ coincides with the locus

$$\{x \in \mathbb{H}^{n+1} \,|\, \nu_x(z) = e^{\varphi(z)} \rho_{\mathbb{S}^n}\}, \tag{3.4}$$

and the inside of $H(z,\varphi)$ is the locus

$$\{x \in \mathbb{H}^{n+1} \mid \nu_x(z) > e^{\varphi(z)}\rho_{\mathbb{S}^n}\}. \tag{3.5}$$

By definition, it is easy to verify that if $x \in \mathbb{H}^{n+1}$, $h \in \mathrm{Isom}_+(\mathbb{H}^{n+1})$ we have that $h^*(v_{h(x)}) = v_x$. Hence it follows

$$\mathrm{Ep}_\rho = h \circ \mathrm{Ep}_{h^*\rho} \tag{3.6}$$

where $h^*\rho$ is the pull-back metric of ρ under h.

3.2 Explicit expression of Epstein maps in the upper-space model

Here and in the sequel, we restrict ourselves to the case $n = 3$. For the computation purpose, it is convenient to use the upper-space model of the hyperbolic 3-space. Namely,

$$\mathbb{H}^3 = \{(y,\xi) \in \mathbb{C} \times \mathbb{R}_{>0}\}$$

with the hyperbolic metric

$$\mathrm{d}s^2 = \frac{|\mathrm{d}y|^2 + \mathrm{d}\xi^2}{\xi^2}.$$

The results presented in this section were obtained in [16] and [11]. We collect them here for the readers' convenience, also because our choice of convention of Epstein map, which coincides with the horosphere envelop interpretation of the Epstein map as described in Section 3.1, is slightly different than [16]. The difference of convention results mainly in constant factors at various places. We choose to include the simple derivations or examples to verify the constant factors.

Let $e^\varphi|\mathrm{d}z|^2$ be a smooth conformal metric on an open set $U \subset \mathbb{C}$. The Epstein map $\mathrm{Ep}_\varphi := \mathrm{Ep}_{e^\varphi|\mathrm{d}z|^2} : z \in U \mapsto (y,\xi) \in \mathbb{C} \times \mathbb{R}_+ = \mathbb{H}^3$ is given explicitly by

$$\xi = \frac{2e^{-\varphi/2}}{1+|\varphi_{\bar z}|^2 e^{-\varphi}}, \qquad y = z + \frac{2\varphi_{\bar z} e^{-\varphi}}{1+|\varphi_{\bar z}|^2 e^{-\varphi}} = z + \xi \cdot \psi, \tag{3.7}$$

where

$$\psi := \varphi_{\bar z} e^{-\varphi/2}, \qquad \varphi_{\bar z} = \partial_{\bar z}\varphi.$$

The Epstein Gauss map is $\widetilde{\mathrm{Ep}}_\varphi \colon U \subset \mathbb{C} \to T_1\mathbb{H}^3$ such that the base point is Ep_φ and the vector component is $\xi\overrightarrow{\eta}$ where

$$\overrightarrow{\eta} = \left(\frac{2\varphi_{\bar z} e^{-\varphi/2}}{1+|\varphi_{\bar z}|^2 e^{-\varphi}}, \frac{1-|\varphi_{\bar z}|^2 e^{-\varphi}}{1+|\varphi_{\bar z}|^2 e^{-\varphi}}\right) = \left(\frac{2\psi}{1+|\psi|^2}, \frac{1-|\psi|^2}{1+|\psi|^2}\right) \tag{3.8}$$

is a Euclidean normal vector. It is straightforward to check that the geodesic flow $\alpha(t) \in T_1\mathbb{H}^3$ starting from $-\widetilde{\mathrm{Ep}}_\varphi(z) = (\mathrm{Ep}_\varphi(z), -\vec{n})$ satisfies

$$\alpha(t) = -\widetilde{\mathrm{Ep}}_{\varphi+2t}(z),$$

and the base point of $\alpha(t)$ tends to z as $t \to \infty$.

Example 3.1.

- If $\varphi \equiv 2t$, then for all z,
$$\mathrm{Ep}_\varphi(z) = (z, 2e^{-t}) \qquad \overrightarrow{\eta} = (0,1).$$
- If $e^\varphi = \frac{4}{(1+|z|^2)^2}$, then for all $z \in \mathbb{C}$, $(y, \xi) = (0,1)$.
- If $\varphi = \log 4 - 2\log(1-|z|^2)$, i.e., $e^\varphi |\mathrm{d}z|^2$ is the hyperbolic metric in $\mathbb{D}$, then for $z = re^{\mathrm{i}\theta} \in \mathbb{D}$,
$$\mathrm{Ep}_\varphi(re^{\mathrm{i}\theta}) = \left(\frac{2r}{1+r^2}e^{\mathrm{i}\theta}, \frac{1-r^2}{1+r^2}\right) = \overrightarrow{\eta}.$$
We see (and one of the advantage of choosing this convention is) that Ep_φ maps $\mathbb{D}$ onto the totally geodesic plane in $\mathbb{H}^3$ bounded by $\partial\mathbb{D}$.

Fix $\varphi \in C^\infty(U, \mathbb{R})$. Let Σ_t denote the Epstein surface associated with the metric $e^{\varphi+2t}|\mathrm{d}z|^2$, $\Sigma = \Sigma_0$. Let I and I_t denote the first fundamental form on Σ and on Σ_t. Let $B(v) := -\nabla_v \vec{n}$ be the shape operator on Σ, wherever Ep_φ is an immersion. We let
$$\mathrm{I\!I}(u,v) := \mathrm{I}(Bu, v) = \mathrm{I}(u, Bv), \qquad \mathrm{I\!I\!I}(u,v) := \mathrm{I}(Bu, Bv)$$
and k_-, k_+ be the two eigenvalues of B, namely the principal curvatures on Σ. Let
$$H := \frac{1}{2}\operatorname{tr}(B) = \frac{k_- + k_+}{2}$$
be the mean curvature on Σ.

Definition 3.2. We define the first, second, and third fundamental forms at infinity associated with the surface Σ as
$$\begin{aligned}
\mathrm{I}^* &:= \mathrm{I} + 2\mathrm{I\!I} + \mathrm{I\!I\!I} = \mathrm{I}((\mathrm{id} + B)\cdot, (\mathrm{id} + B)\cdot);\\
\mathrm{I\!I}^* &:= \mathrm{I} - \mathrm{I\!I\!I} = \mathrm{I}((\mathrm{id} + B)\cdot, (\mathrm{id} - B)\cdot);\\
B^* &:= (\mathrm{I}^*)^{-1}\mathrm{I\!I}^* = \frac{\mathrm{id} - B}{\mathrm{id} + B};\\
\mathrm{I\!I\!I}^* &:= \mathrm{I}^*(B^*\cdot, B^*\cdot) = \mathrm{I}((\mathrm{id} - B)\cdot, (\mathrm{id} - B)\cdot) = \mathrm{I} - 2\mathrm{I\!I} + \mathrm{I\!I\!I}
\end{aligned}$$
where id is the identity operator. We define similarly $H^* = \operatorname{tr}(B^*)/2$.

Theorem 3.3 (See [16, Lem. 5.7, Thm. 5.8, Cor. 5.11])**.** *We have*
$$I_t = \frac{1}{4}(e^{2t}\mathrm{I}^* + 2\mathrm{I\!I}^* + e^{-2t}\mathrm{I\!I\!I}^*).$$
In particular,
$$4e^{-2t} I_t \xrightarrow{t\to\infty} I^*.$$
Moreover, we have
$$\mathrm{I}^* = e^\varphi |\mathrm{d}z|^2, \qquad \mathrm{I\!I}^* = \vartheta\, \mathrm{d}z^2 + \bar{\vartheta}\, \mathrm{d}\bar{z}^2 + 2\varphi_{z\bar{z}}\, \mathrm{d}z\mathrm{d}\bar{z},$$
where $\vartheta = \varphi_{zz} - \frac{1}{2}(\varphi_z)^2$. The eigenvalues of B^* are given by
$$k_\pm^* = \frac{1-k_\pm}{1+k_\pm} = 2e^{-\varphi}\left(\varphi_{z\bar{z}} \pm \sqrt{\vartheta\bar{\vartheta}}\right) = -K^* \pm 2e^{-\varphi}\sqrt{\vartheta\bar{\vartheta}}.$$

In the last equality we used the identity

$$\varphi_{z\bar{z}} = \frac{1}{4}\Delta\varphi = -\frac{1}{2}e^{\varphi}K^* \tag{3.9}$$

where K^* denotes the Gauss curvature of the metric I^*.

Example 3.4. When $\varphi \equiv 0$, we have $\mathrm{I}_t(y,\xi) = \xi^{-2}|\mathrm{d}y|^2 = (e^{2t}/4)|\mathrm{d}y|^2$.

Corollary 3.5. *Let* $\mathrm{d}a$ *denote the area form induced by* I*,* $\mathrm{d}a^* = e^{\varphi}\,\mathrm{d}^2 z$ *the area form induced by* I^**, we have* $\mathrm{d}a^* = (1+k_+)(1+k_-)\,\mathrm{d}a$*,*

$$H\mathrm{d}a = \left(\frac{1-(K^*)^2}{4} + |\vartheta|^2 e^{-2\varphi}\right)\mathrm{d}a^*,$$

$$H^* = \frac{k_+^* + k_-^*}{2} = -K^*.$$

and

$$k_+k_-\mathrm{d}a = \frac{(1-k_+^*)(1-k_-^*)}{4}\mathrm{d}a^* = \left[\frac{(1+K^*)^2}{4} - e^{-2\varphi}|\vartheta|^2\right]\mathrm{d}a^*.$$

Proof. It follows directly from Definition 3.2 that

$$k_\pm = \frac{1-k_\pm^*}{1+k_\pm^*} \qquad \text{and} \qquad \mathrm{d}a^* = (1+k_+)(1+k_-)\,\mathrm{d}a.$$

We obtain from Theorem 3.3 and (3.9) that

$$\begin{aligned}
H\mathrm{d}a &= \frac{k_+ + k_-}{2(1+k_+)(1+k_-)}\,\mathrm{d}a^* = \frac{1}{4}(1 - k_+^* k_-^*)e^{\varphi}\,\mathrm{d}^2 z \\
&= \frac{1}{4}(1 - 4e^{-2\varphi}(\varphi_{z\bar{z}}^2 - |\vartheta|^2))e^{\varphi}\,\mathrm{d}^2 z \\
&= \frac{1}{4}(1 - 4e^{-2\varphi}((\frac{1}{2}e^{\varphi}K^*)^2 - |\vartheta|^2))e^{\varphi}\mathrm{d}^2 z \\
&= \left(\frac{1-(K^*)^2}{4} + |\vartheta|^2 e^{-2\varphi}\right)\mathrm{d}a^*.
\end{aligned}$$

Similarly,

$$H^* = \frac{k_+^* + k_-^*}{2} = 2e^{-\varphi}\varphi_{z\bar{z}} = -K^*$$

and

$$\begin{aligned}
k_+k_-\mathrm{d}a &= \frac{k_+k_-}{(1+k_+)(1+k_-)}\mathrm{d}a^* = \frac{(1-k_+^*)(1-k_-^*)}{4}\mathrm{d}a^* \\
&= \frac{1}{4}(1 + K^* - 2e^{-\varphi}\sqrt{\vartheta\bar{\vartheta}})(1 + K^* + 2e^{-\varphi}\sqrt{\vartheta\bar{\vartheta}})\mathrm{d}a^* \\
&= \left[\frac{(1+K^*)^2}{4} - e^{-2\varphi}|\vartheta|^2\right]\mathrm{d}a^*
\end{aligned}$$

as claimed. ∎

3.3 Epstein-Poincaré map on a simply connected domain

We apply the results to the special case of Epstein-Poincaré surfaces, namely, the Epstein surfaces associated with the Poincaré (or hyperbolic, namely, $K^* \equiv -1$) metric $\rho_\Omega |\mathrm{d}z|^2 = e^\varphi |\mathrm{d}z|^2$ on a simply connected domain $\Omega \subsetneq \mathbb{C}$. Let $f : \mathbb{D} \to \Omega$ be a conformal map. Then using the same notation as Theorem 3.3, we have

$$\vartheta = \mathscr{S}(f^{-1}), \qquad k^*_\pm = 1 \pm 2\|\mathscr{S}(f^{-1})\|$$

where

$$\|\mathscr{S}(f^{-1})\| := |\mathscr{S}(f^{-1})|e^{-\varphi}.$$

Remark 3.6. The fact that ϑ equals the Schwarzian derivative of a uniformization map follows from a direct computation, this holds true if and only if the metric on Ω has constant curvature. Moreover, from the Nehari bound, we have

$$\|\mathscr{S}(f^{-1})(z)\| \le 3/2, \qquad \forall z \in \Omega.$$

From the computation above and Corollary 3.5 we obtain the following result.

Theorem 3.7 (Epstein-Poincaré surface [11, Prop. 7.4])**.** *If* $\|\mathscr{S}(f^{-1})(z)\| \neq 1$, *then* Ep_Ω *is an immersion near* z *and the principal curvatures of* Σ_Ω *at* $\mathrm{Ep}_\Omega(z)$ *are given by*

$$k_\pm = \frac{-\|\mathscr{S}(f^{-1})\|}{\|\mathscr{S}(f^{-1})\| \pm 1}. \tag{3.10}$$

In particular, we have

$$H\mathrm{da} = |\vartheta|^2 e^{-2\varphi}\mathrm{da}^* = \|\mathscr{S}(f^{-1})\|^2 \rho_\Omega \,\mathrm{d}^2 z. \tag{3.11}$$

We have the total curvature

$$\int_{\Sigma_\Omega} |\det B|\,\mathrm{da} := \int_{\Sigma_\Omega} |k_- k_+|\mathrm{da} = \int_\Omega e^{-2\varphi}|\vartheta|^2 \mathrm{da}^* = \int_\Omega \|\mathscr{S}(f^{-1})\|^2 \rho_\Omega |\mathrm{d}z|^2 = \int_{\Sigma_\Omega} H\mathrm{da}.$$

We note that $\int_{\Sigma_\Omega}$ *is understood as the integral on the non-singular locus*

$$\{\mathrm{Ep}_\Omega(z) \colon \|\mathscr{S}(f^{-1})(z)\| \neq 1\}$$

which has full measure.

Example 3.8. If $\Omega = \mathbb{D}$, (3.10) shows $k_\pm \equiv 0$ which is consistent with the fact that Σ is a totally geodesic plane. See Example 3.1.

We obtain immediately the following consequence which is reminiscent to the results of Bishop obtained in [2].

Corollary 3.9. *A Jordan curve* γ *is a Weil-Petersson quasicircle if and only if*

$$0 \le \int_{\Sigma_\Omega} H\mathrm{da} = \int_{\Sigma_\Omega} |\det B|\,\mathrm{da} < \infty.$$

Proof. This follows from the characterization (2.4) of Weil-Petersson quasicircle, the identity

$$\int_{\mathbb{D}} \|\mathscr{S}(f)(\zeta)\|^2 \rho_{\mathbb{D}} \mathrm{d}^2\zeta = \int_{\Omega} \|\mathscr{S}(f^{-1})(z)\|^2 \rho_{\Omega} \mathrm{d}^2 z,$$

and Theorem 3.7. $\square$

The Epstein surface is uniquely determined and naturally associated with a simply connected domain Ω. There are two connected components of γ in $\hat{\mathbb{C}}$, we will study the relation between the two Epstein-Poincaré surfaces later in Section 4 and it will be crucial to define the renormalized volume. However, let us first record some properties of a single Epstein-Poincaré surface. We will use a few classical results from geometric function theory.

Theorem 3.10. *Let f be a conformal map from $\mathbb{D}$ onto a domain bounded by a Jordan curve γ.*

- See [22, Prop. 1.2]. *For all $\zeta \in \mathbb{D}$, we have*

$$\left| \frac{(1-|\zeta|^2)}{2} \frac{f''(\zeta)}{f'(\zeta)} - \bar{\zeta} \right| \leq 2. \tag{3.12}$$

- See [22, Cor. 1.4]. *For all $\zeta \in \mathbb{D}$, we have*

$$\frac{1}{4}(1-|\zeta|^2)\,|f'(\zeta)| \leq \operatorname{dist}(f(\zeta), \gamma) \leq (1-|\zeta|^2)\,|f'(\zeta)|\,. \tag{3.13}$$

- See [22, Thm. 11.1]. *The following are equivalent:*

(AC1) *γ is asymptotically conformal;*
(AC2) $\lim_{|\zeta|\to 1-} \frac{f''(\zeta)}{f'(\zeta)}(1-|\zeta|^2) = 0$*;*
(AC3) $\lim_{|\zeta|\to 1-} \|\mathscr{S}(f)(\zeta)\| = 0$*;*
(AC4) $\dfrac{f(\zeta)-f(x)}{(\zeta-x)f'(\zeta)} \to 1$ *as* $\zeta \to x$, $x \in \overline{\mathbb{D}}$ *and* $\dfrac{|z-x|}{1-|z|} \leq a$ *(for all* $a > 0$*).*

Example 3.11. Weil-Petersson quasicircles are asymptotically conformal. See, e.g., Corollary II.1.4 of [31].

Lemma 3.12. *The Epstein-Poincaré map $\mathrm{Ep}_\Omega : \Omega \to \Sigma_\Omega$ has the following explicit expression. For $z = f(\zeta)$, $\zeta \in \mathbb{D}$, we have*

$$\begin{aligned}
\psi(z) &= \varphi_{\overline{z}} e^{-\varphi/2} = \frac{|f'(\zeta)|}{f'(\zeta)} \left(-\frac{\overline{f''(\zeta)}}{\overline{f'(\zeta)}} \frac{(1-|\zeta|^2)}{2} + \zeta \right), \\
e^{-\varphi(z)/2} &= \frac{1}{2} |f'(\zeta)| (1-|\zeta|^2) \\
\xi(z) &= \frac{2e^{-\varphi/2}}{1+|\psi|^2} = \frac{|f'(\zeta)|(1-|\zeta|^2)}{1 + |-\frac{f''(\zeta)}{f'(\zeta)} \frac{(1-|\zeta|^2)}{2} + \bar{\zeta}|^2}, \\
y(z) &= z + \xi \cdot \psi = f(\zeta) + \frac{\left(-\frac{\overline{f''(\zeta)}}{\overline{f'(\zeta)}} \frac{(1-|\zeta|^2)}{2} + \zeta \right) f'(\zeta)(1-|\zeta|^2)}{1 + |-\frac{f''(\zeta)}{f'(\zeta)} \frac{(1-|\zeta|^2)}{2} + \bar{\zeta}|^2}.
\end{aligned}$$

Furthermore, it extends continuously to $\partial\mathbb{D}$, namely, $(y,\xi) \xrightarrow[\zeta\to e^{i\theta}]{} (f(e^{i\theta}),0)$. In particular, Ep_Ω *extends continuously to γ as the identity map.*

Proof. We express ρ_Ω as

$$\rho_\Omega = e^{\varphi(z)}|\mathrm{d}z|^2 = \frac{4|\mathrm{d}z|^2}{|f'(f^{-1}(z))|^2(1-|f^{-1}(z)|^2)^2}. \tag{3.14}$$

In ζ variable,

$$e^{-\varphi(f(\zeta))} = \left(\frac{|f'(\zeta)|(1-|\zeta|^2)}{2}\right)^2, \tag{3.15}$$

it gives

$$\varphi(f(\zeta)) = \log 4 - \log|f'(\zeta)|^2 - 2\log(1-|\zeta|^2).$$

Taking derivative in ζ we obtain

$$\varphi_z f'(\zeta) = -\frac{f''(\zeta)}{f'(\zeta)} + \frac{2\overline{\zeta}}{1-|\zeta|^2},$$

hence

$$\varphi_z = -\frac{f''(\zeta)}{(f'(\zeta))^2} + \frac{2\overline{\zeta}}{f'(\zeta)(1-|\zeta|^2)}.$$

Combining with equation (3.15) we get

$$\psi(z) = \varphi_{\overline{z}} e^{-\varphi/2} = \frac{|f'(\zeta)|}{\overline{f'(\zeta)}}\left(-\frac{\overline{f''(\zeta)}}{\overline{f'(\zeta)}}\frac{(1-|\zeta|^2)}{2} + \zeta\right). \tag{3.16}$$

The expression for ξ and y follows from their definition (3.7). We note that (3.12) implies that $|\psi| \le 2$ and (3.13) implies that $e^{-\varphi(z)/2} \to 0$ as $\zeta \to e^{\mathrm{i}\theta}$. We obtain the limit $(y(z),\xi(z)) \xrightarrow[\zeta\to e^{\mathrm{i}\theta}]{} (f(e^{\mathrm{i}\theta}),0)$. □

Lemma 3.12 and condition (AC2) show that when γ is asymptotically conformal, we have as $\zeta \to \partial\mathbb{D}$,

$$y\circ f(\zeta) - f(\zeta) \sim \frac{\zeta f'(\zeta)(1-|\zeta|^2)}{2} \simeq \frac{\zeta f'(\zeta)}{|\zeta f'(\zeta)|}\,\mathrm{dist}(f(\zeta),\gamma);$$
$$\xi\circ f(\zeta) \sim \frac{|f'(\zeta)|(1-|\zeta|^2)}{2} \simeq \mathrm{dist}(f(\zeta),\gamma).$$

Where "$\sim$" means the ratio goes to 1 and "$\simeq$" means the ratio is bounded from above and below. And for all $\zeta \in \mathbb{D}$, Lemma 3.12, inequalities (3.12), and (3.13) show that

$$\frac{\mathrm{dist}(f(\zeta),\gamma)}{5} \le \frac{|f'(\zeta)|(1-|\zeta|^2)}{5} \le |\xi\circ f(\zeta)| \le |f'(\zeta)|(1-|\zeta|^2) \le 4\,\mathrm{dist}(f(\zeta),\gamma). \tag{3.17}$$

Corollary 3.13. *If Ω is bounded by an asymptotically conformal curve γ,* $\mathrm{Ep}_\Omega \circ f$ *is an immersion and embedding in a neighborhood of $\partial\mathbb{D}$.*

Proof. Since γ is asymptotically conformal, Theorem 3.7 and (AC3) imply that $\mathrm{Ep}_\Omega \circ f$ is an immersion in a neighborhood of $\partial\mathbb{D}$.

Now we also show that $\mathrm{Ep}_\Omega \circ f$ is also an embedding near the boundary. The inequalities in (3.13) show that there exists $\delta_0 > 0$ such that if $1 - |\zeta| < \delta_0$ then

$$\frac{1}{3}\,\mathrm{dist}(f(\zeta),\gamma) \le |y(\zeta) - f(\zeta)|,\ \xi(\zeta) \le 3\,\mathrm{dist}(f(\zeta),\gamma) \tag{3.18}$$

and the principal curvatures $\lambda_\pm$ of Σ_Ω at $\mathrm{Ep}_\Omega \circ f(\zeta)$ are bounded by $1/2$. Let $A_\delta := \{\zeta \in \mathbb{D} \colon 1 - |\zeta| < \delta\}$. We now show that $\mathrm{Ep}_\Omega \circ f|_{A_\delta}$ is an embedding for small enough δ.

If it is not the case, then from (3.17) the self-intersection must occur to points with comparable distance to γ. In other words, we have for every $\delta < \delta_0$, there is $\zeta_1 = \zeta_1(\delta)$, $\zeta_2 = \zeta_2(\delta) \in A_\delta$ such that $y(\zeta_1) = y(\zeta_2)$ and $\xi(\zeta_1) = \xi(\zeta_2)$. Then we know from (3.18) that $|\zeta_1 - \zeta_2| \to 0$ as $\delta \to 0$. Let η_δ be a geodesic loop from $(y(\zeta_1), \xi(\zeta_1))$ to $(y(\zeta_2), \xi(\zeta_2))$ on the Epstein surface, and for small δ, it is contained in the image of $\mathrm{Ep}_\Omega \circ f|_{A_{\delta_0}}$. The proof of [11, Thm. 3.4] or [9, Prop. 4.15] then shows that it is not possible since the principal curvatures on $\mathrm{Ep}_\Omega \circ f|_{A_{\delta_0}}$ have modulus less than $1/2$. $\square$

4 Renormalized volume for a Jordan curve

4.1 Disjoint Epstein-Poincaré surfaces

When γ is a circle, Example 3.1 shows that both Epstein surfaces coincide with the geodesic plane bounded by γ.

Proposition 4.1. *If γ is not a circle, then Σ_Ω and Σ_{Ω^*} are disjoint in the interior.*

Proof. We will use the horosphere envelop description of the Epstein surfaces described in Section 3.1. It suffices to show that for all $\eta \in \Omega$ and $\eta' \in \Omega^*$, the horospheres at η and η' for the respective hyperbolic metric are disjoint.

By the invariance property (3.6) of Epstein map under Möbius transformation, we may assume that η is the south pole $\mathfrak{s} = (0,0,-1)$ and η' is the north pole $\mathfrak{n} = (0,0,1)$. Moreover, writing the Poincaré metrics $\rho_\Omega = e^\varphi \rho_{\mathbb{S}^2}$ and $\rho_{\Omega^*} = e^\psi \rho_{\mathbb{S}^2}$, we may also assume that $\psi(\mathfrak{n}) = 0$ after possibly applying another Möbius transformation fixing $\mathfrak{n}$ and $\mathfrak{s}$. Hence, the horospheres $H(\mathfrak{n},\psi) = H(\mathfrak{n},0)$ passes through the origin $(0,0,0)$ and is contained in the upper half-ball of $\mathfrak{B}^3$.

Let $\pi : \mathbb{S}^2 \smallsetminus \{\mathfrak{n}\} \to \mathbb{C}$ be the stereographic projection from $\mathfrak{n}$ sending $\mathfrak{s}$ to $0 \in \mathbb{C}$ and the lower half-sphere $\mathbb{S}^2_-$ onto $\mathbb{D}$. The condition $\psi(\mathfrak{n}) = 0$ is equivalent to $|\lim_{z\to\infty} g'(z)| = 1$ where g is a conformal map from $\mathbb{D}^* = \pi(\mathbb{S}^2_+)$ to $\pi(\Omega^*)$ fixing ∞.

On the other hand, we have

$$\pi_*\rho_{\mathbb{S}^2,\theta} = \frac{4}{(1+|z|^2)^2}|\mathrm{d}z|^2, \quad \pi_*\rho_{\mathbb{S}^2_-,\theta} = \frac{4}{(1-|z|^2)^2}|\mathrm{d}z|^2$$

where $z = \pi(\theta)$, and these expressions coincide when $z = \pi(\mathfrak{s}) = 0$ which shows

$$\rho_{\mathbb{S}^2,\mathfrak{s}} = \rho_{\mathbb{S}^2_-,\mathfrak{s}}.$$

Now let f be a conformal map from $\mathbb{D}$ onto $\pi(\Omega)$ fixing 0. We have

$$\pi_*\rho_{\mathbb{S}^2,\mathfrak{s}} = \pi_*\rho_{\mathbb{S}^2_-,\mathfrak{s}} = \rho_{\mathbb{D},0} = |f'(0)|^2\rho_{\pi(\Omega),0} = |f'(0)|^2\pi_*\rho_{\Omega,\mathfrak{s}},$$

where the second and the last equalities follow from the definition of push-forward, and the third equality follows from the property of the hyperbolic metric.

The following Lemma 4.2 shows that $|f'(0)| < 1$. Therefore, $\rho_{\Omega,\mathfrak{s}} > \rho_{\mathbb{S}^2,\mathfrak{s}}$ and (3.4) and (3.5) show that $H(\mathfrak{s},\varphi)$ is strictly contained in the inside of $H(\mathfrak{s},0)$. In particular, $H(\mathfrak{s},\varphi)$ is strictly contained in the lower half-ball and is disjoint from $H(\mathfrak{n},\psi)$. □

The following lemma is a special case of the Grunsky inequality, see, e.g., [21, Thm. 4.1, (21)] and [31, P. 70-71].

Lemma 4.2 (Grunsky inequality). *Suppose that $f : \mathbb{D} \to \mathbb{C}$ and $g : \mathbb{D}^* \to \hat{\mathbb{C}}$ are univalent functions on $\mathbb{D}$ and $\mathbb{D}^*$ such that $f(0) = 0$ and $g(\infty) = \infty$, and $f(\mathbb{D}) \cap g(\mathbb{D}^*) = \emptyset$. Then we have*

$$\int_{\mathbb{D}} \left|\frac{f'(z)}{f(z)} - \frac{1}{z}\right|^2 \mathrm{d}^2 z + \int_{\mathbb{D}^*} \left|\frac{g'(z)}{g(z)} - \frac{1}{z}\right|^2 \mathrm{d}^2 z \le 2\pi \log\left|\frac{g'(\infty)}{f'(0)}\right|.$$

Equality holds if $\mathbb{C} \setminus \{f(\mathbb{D}) \cup g(\mathbb{D}^*)\}$ has zero Lebesgue measure.

4.2 Volume between the Epstein-Poincaré surfaces

Let $\gamma \subset \hat{\mathbb{C}}$ be an asymptotically conformal Jordan curve. We now define the volume between Σ_Ω and Σ_{Ω^*}. We cautiously note that both Epstein-Poincaré surfaces are non-compact and may not be embedded. Without loss of generality, we assume that γ does not contain $\infty \in \hat{\mathbb{C}}$ and use the upper space model of $\mathbb{H}^3$. We consider an approximation of the hyperbolic volume form. For $\varepsilon > 0$, let

$$\mathrm{vol}_\varepsilon = \mathbf{1}_{\xi\ge\varepsilon}\frac{\mathrm{vol}_{eucl}}{\xi^3}$$

where vol_{eucl} is the Euclidean volume form.

Let φ_γ be continuous map $\overline{\mathbb{H}^3} \to \overline{\mathbb{H}^3}$, such that $\varphi_\gamma|_\Omega = \mathrm{Ep}_\Omega$, $\varphi_\gamma|_{\Omega^*} = \mathrm{Ep}_{\Omega^*}$, and $\varphi_\gamma|_{\mathbb{H}^3}$ is differentiable. This is possible since Ep_Ω and Ep_{Ω^*} extend to the identity map on γ. We define

$$V_2(\gamma)(\varepsilon) := \int_{\mathbb{H}^3} \varphi_\gamma^* \,\mathrm{vol}_\varepsilon\,.$$

This is the *signed volume between the Epstein surfaces* bounded by γ above level ε.

Since the boundary values of φ_γ are determined and $\varphi_\gamma(\overline{\mathbb{H}^3}) \cap \{(y,\xi) \colon \xi \ge \varepsilon\}$ is compact, we have $V_2(\gamma)(\varepsilon)$ is finite and independent from the choice of φ_γ. Since both Epstein surfaces are disjoint (unless γ is a circle) by Proposition 4.1 and embedded near the boundary by Corollary 3.13, without loss of generality, we assume further more that the Jacobian of φ_γ is positive in a neighborhood U_γ of γ in $\mathbb{H}^3$. (If γ is a circle, then we choose φ_γ such that the Jacobian is zero.) Now we define

$$\lim_{\varepsilon\to0+} V_2(\gamma)(\varepsilon) \in (-\infty, \infty]. \tag{4.1}$$

Notice that $\int_{U_\gamma} \varphi_\gamma^* \operatorname{vol}_\varepsilon$ increases as $\varepsilon \to 0+$ and $\int_{\mathbb{H}^3 \smallsetminus U_\gamma} \varphi_\gamma^* \operatorname{vol}_\varepsilon$ is constant for small enough ε. Therefore the above limit exists, and the monotonicity and (3.6) also show that the limit is invariant under actions of elements in $\mathsf{PSL}_2(\mathbb{C})$ which do not send any point of γ to $\infty \in \hat{\mathbb{C}}$.

Definition 4.3. For an asymptotically conformal Jordan curve $\gamma \subset \hat{\mathbb{C}}$, we define *the signed volume between the Epstein-Poincaré surfaces* $V(\gamma)$ to be the limit in (4.1) applied to the curve $A(\gamma)$, where A is any element in $\mathsf{PSL}_2(\mathbb{C})$ such that $A(\gamma)$ does not passes through ∞.

The above definition is clearly $\mathsf{PSL}_2(\mathbb{C})$-invariant.

4.3 Volume for smooth Jordan curves

In this subsection we will see if the Jordan curve γ is sufficiently smooth, then the map Ep_Ω extends not only continuously to γ but also osculates to the totally geodesic plane bounded by the circle osculating to the curve. This will be useful later to prove that the volume between Ep_Ω and Ep_{Ω^*} is finite, if γ is sufficiently smooth.

For a C^2 curve $\gamma(t) = x(t) + \mathrm{i}y(t)$ in $\mathbb{C}$, curvature is calculated by

$$k_\gamma(t) = \frac{x'y'' - y'x''}{((x')^2 + (y')^2)^{3/2}} = \frac{1}{|\gamma'|^3} \operatorname{Re}\left(-\mathrm{i}\overline{\gamma'}\gamma''\right)$$

If γ is $C^{2,\alpha}$ for some $0 < \alpha < 1$, Kellogg's theorem implies that the conformal map $f : \mathbb{D} \to \Omega$ extends to a $C^{2,\alpha}$ homeomorphism $\overline{\mathbb{D}} \to \overline{\Omega}$. Writing $\gamma(\theta) = f(e^{\mathrm{i}\theta})$ we have that

$$\begin{aligned} \gamma'(\theta) &= \mathrm{i}f'(e^{\mathrm{i}\theta})e^{\mathrm{i}\theta} \\ \gamma''(\theta) &= -f''(e^{\mathrm{i}\theta}) - f'(e^{\mathrm{i}\theta})e^{\mathrm{i}\theta}. \end{aligned}$$

Then it follows that

$$\begin{aligned} k_\gamma(\gamma(\theta)) &= \frac{-\operatorname{Re}\left(f''(e^{\mathrm{i}\theta})\overline{f'(e^{\mathrm{i}\theta})}e^{\mathrm{i}\theta} + |f'(e^{\mathrm{i}\theta})|^2\right)}{|f'(e^{\mathrm{i}\theta})|^3} \\ &= \frac{-\operatorname{Re}(\frac{f''(e^{\mathrm{i}\theta})}{f'(e^{\mathrm{i}\theta})}e^{\mathrm{i}\theta}) - 1}{|f'(e^{\mathrm{i}\theta})|}. \end{aligned}$$

Define then the *osculating circle*, denoted by $\mathscr{C}_\gamma(\theta)$, as the circle with center $\gamma(\theta) + \frac{\mathrm{i}\gamma'(\theta)}{k_\gamma(\theta)}$ and radius $|\frac{\mathrm{i}\gamma'(\theta)}{k_\gamma(\theta)}|$, oriented the same as γ. The circle $\mathscr{C}_\gamma(\theta)$ is then tangent to γ at $\gamma(\theta)$, and agrees with γ at this point of tangency up to second order. From this, it follows that $\mathscr{C}_\gamma(\theta)$ is invariant by parametrizations of γ.

Similarly, we can define the *osculating plane*, denoted by $\mathscr{P}_\gamma(\theta)$, as the geodesic plane in $\mathbb{H}^3$ so that the boundary of $\mathscr{P}_\gamma(\theta)$ is $\mathscr{C}_\gamma(\theta)$. Hence we will show that for γ sufficiently smooth we have that Ep_Ω and $\mathscr{P}_\gamma(\theta)$ agree up to second order.

For this, a straightforward computation using the explicit expression of the Epstein-Poincaré map (Lemma 3.12) gives the following.

Lemma 4.4. *Assume that γ is $C^{2,\alpha}$. Using the parametrization $(r,\theta)\mapsto f(re^{\mathrm{i}\theta})$ of the domain Ω, near $q=f(e^{\mathrm{i}\theta})\in\gamma$, we have*

$$\begin{aligned}
\lim_{z\to q}\psi(z) &= \frac{|f'(e^{\mathrm{i}\theta})|}{f'(e^{\mathrm{i}\theta})}e^{\mathrm{i}\theta};\\
\lim_{z\to q}\overrightarrow{\eta}(z) &= \lim_{z\to q}\left(\frac{2\psi}{1+|\psi|^2},\frac{1-|\psi|^2}{1+|\psi|^2}\right)=\left(\frac{|f'(e^{\mathrm{i}\theta})|}{f'(e^{\mathrm{i}\theta})}e^{\mathrm{i}\theta},0\right);\\
\partial_r\overrightarrow{\eta}(q) &= \left(0,-\operatorname{Re}(\frac{f''(e^{\mathrm{i}\theta})}{f'(e^{\mathrm{i}\theta})}e^{\mathrm{i}\theta})-1\right);\\
\partial_\theta\overrightarrow{\eta}(q) &= \left(\mathrm{i}\frac{e^{\mathrm{i}\theta}f'(e^{\mathrm{i}\theta})}{|f'(e^{\mathrm{i}\theta})|}\left(\operatorname{Re}(\frac{f''(e^{\mathrm{i}\theta})}{f'(e^{\mathrm{i}\theta})}e^{\mathrm{i}\theta})+1\right),0\right).
\end{aligned}$$

In particular, this implies that the Epstein-Poincaré surface, viewed as a surface embedded in $\mathbb{R}^3$ near γ, is umbilic with curvature $-(\operatorname{Re}(\frac{f''(e^{i\theta})}{f'(e^{i\theta})}e^{i\theta})+1)|f'(e^{i\theta})|^{-1}=k_\gamma(q)$ at q.

Proposition 4.5. *Let γ be a $C^{4,\alpha}$ Jordan curve in $\mathbb{C}$. Then Ep_Ω and $\mathscr{P}_\gamma(\theta)$ are tangent at $(\gamma(\theta),0)$ and agree up to order 2.*

Proof. Since γ is $C^{4,\alpha}$ then $f:\mathbb{D}\to\mathbb{C}$ extended to $\partial\mathbb{D}$ as a $C^{4,\alpha}$ map. Then the Epstein map extends by identity on γ is a $C^{2,\alpha}$ map on $\overline{\mathbb{D}}$ by Lemma 3.12. We see then from Lemma 4.4 that the continuous extension of $\overrightarrow{\eta}$ is the Euclidean unit outward orthogonal vector to γ. Hence the extension of the Epstein map at γ has to agree up to first order with the osculating plane $\mathscr{P}_q$. To verify that they actually agree up to second order, Lemma 4.4 shows that the Epstein-Poincaré surface is umbilic with the same curvature in the Euclidean space as $\mathscr{P}_q$. □

Next we show that for sufficiently regular curves γ this volume is in fact finite.

Proposition 4.6. *Let γ be a $C^{5,\alpha}$ Jordan curve in $\mathbb{C}$. Then $V(\gamma)$ is finite.*

Proof. Without loss of generality, we assume that $\gamma:S^1\to\mathbb{C}$ is parametrized by arc-length. Take φ_γ some continuous map $\overline{\mathbb{H}^3}\to\overline{\mathbb{H}^3}$ as before, meaning that $\varphi_\gamma|_\Omega=\mathrm{Ep}_\Omega$, $\varphi_\gamma|_{\Omega^*}=\mathrm{Ep}_{\Omega^*}$, and $\varphi_\gamma|_{\mathbb{H}^3}$ is differentiable. We take the following $C^{4,\alpha}$ parametrization of a neighbourhood U of γ in $\overline{\mathbb{H}^3}$, denoted $G:S^1_s\times\overline{\mathbb{H}^2}_{(a,b)}\to\overline{\mathbb{H}^3}$, by

$$G(s,a,b)=\gamma(s)+\mathrm{i}a\gamma'(s)+be_3$$

where $e_3=(0,0,1)$. It is a straightforward calculation to see that the hyperbolic metric in G-coordinates is given by

$$\frac{(1-ak(s))^2}{b^2}\mathrm{d}s^2+\frac{1}{b^2}\mathrm{d}a^2+\frac{1}{b^2}\mathrm{d}b^2,$$

where $k(s)$ is the signed curvature of γ given by $\gamma''(s)=\mathrm{i}k(s)\gamma'(s)$. Hence the volume form is given by

$$\frac{(1-ak(s))}{b^3}\,\mathrm{d}s\,\mathrm{d}a\,\mathrm{d}b.$$

If we assume that γ is $C^{5,\alpha}$, then the Epstein-Poincaré surfaces are $C^{3,\alpha}$ up to the boundary. This means that there are $C^{3,\alpha}$ functions $a_\Omega, a_{\Omega^*} : S^1_s \times [0, \varepsilon_0]_b \to \mathbb{R}$ so that the Epstein-Poincaré surfaces in the neighbourhood U of γ are given by $G(s, a_\Omega(s,b), b)$, $G(s, a_{\Omega^*}(s,b), b)$. And since by Proposition 4.5 the Epstein-Poincaré surfaces agree up to second order at γ, then there exists a constant $C > 0$ so that $|a_\Omega(s,b) - a_{\Omega^*}(s,b)| \le Cb^3$. Hence for small enough neighborhood U of γ, the integral $\int_U \varphi_\gamma^* \operatorname{vol}$ will be bounded by

$$V_1(\gamma)(\varepsilon_0) = \int_{S^1} \int_0^{\varepsilon_0} \int_{a_\Omega(s,z)}^{a_{\Omega^*}(s,z)} \frac{(1 - ak(s))}{b^3} \, \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}s.$$

This integral is well-defined and convergent since

$$\left| \int_{a_\Omega(s,b)}^{a_{\Omega^*}(s,b)} \frac{(1 - ak(s))}{b^3} \mathrm{d}a \right| = \frac{1}{b^3} \left| a_{\Omega^*}(s,b) - a_\Omega(s,b) \right| . \left| \left(\frac{a_\Omega(s,b) + a_{\Omega^*}(s,b)}{2} \right) k(s) - 1 \right|$$

is bounded by a constant independent of (s,b). Hence $V(\gamma) = \lim_{\varepsilon \to 0+} V_2(\gamma)(\varepsilon)$ is a finite real value. $\square$

Definition 4.7. Let γ be a Weil-Petersson quasicircle in $\mathbb{C}$. Then we define $V_R(\gamma)$, the renormalized volume of γ, as

$$\begin{aligned} V_R(\gamma) &:= V(\gamma) - \frac{1}{2} \int_{\Sigma_\Omega \cup \Sigma_{\Omega^*}} H \mathrm{d}a \\ &= V(\gamma) - \frac{1}{2} \int_\Omega \|\mathscr{S}(f^{-1})(z)\|^2 \rho_\Omega \mathrm{d}^2 z - \frac{1}{2} \int_{\Omega^*} \|\mathscr{S}(g^{-1})(z)\|^2 \rho_{\Omega^*} \mathrm{d}^2 z. \end{aligned} \tag{4.2}$$

Remark 4.8. The second identity follows from Theorem 3.7. A priori, $V_R(\gamma) \in (-\infty, \infty]$ as $V(\gamma) \in (-\infty, \infty]$ and the integrals of mean curvature are finite by Corollary 3.9. Proposition 4.6 shows that if γ is $C^{5,\alpha}$, then $V_R(\gamma) < \infty$. From the $\mathsf{PSL}_2(\mathbb{C})$-invariance of each summand in ((4.2)) we can easily see that V_R is $\mathsf{PSL}_2(\mathbb{C})$-invariant.

5 Universal Liouville action as renormalized volume

Our objective in this Section is to prove that the renormalized volume in Definition 4.7 agrees up to a constant with the Loewner energy for $C^{5,\alpha}$ curves.

5.1 Variation of the volume

For this subsection we consider a 1-parameter family $(\gamma_t)_{t \in (-1,1)}$ of $C^{5,\alpha}$ Jordan curves ($\alpha > 0$). We will define a parametrization of the Epstein surfaces that allows us to compute the derivative $\frac{\partial}{\partial t}|_{t=0} V(\gamma_t)$. Since scalar multiplications are isometries of $\mathbb{H}^3$, we can assume without loss of generality that all curves γ_t have Euclidean arclength 2π. Furthermore, for any sufficiently small ε we have that $V(\gamma_t) = V_1(\gamma_t)(\varepsilon) + V_2(\gamma_t)(\varepsilon)$. Moreover, we assume that $V_2(\gamma_t)(\varepsilon) \xrightarrow{\varepsilon \to 0} V(\gamma_t)$ converges uniformly in t.

Let $f_t : \overline{\mathbb{D}} \to \Omega_t$, $g_t : \overline{\mathbb{D}^*} \to \Omega_t^*$ be univalent functions. As the 1-parameter family γ_t is $C^{5,\alpha}$, we can take the 1-parameter family of maps f_t, g_t to be $C^{5,\alpha}$ on $\overline{\mathbb{D}}$ and $\overline{\mathbb{D}^*}$

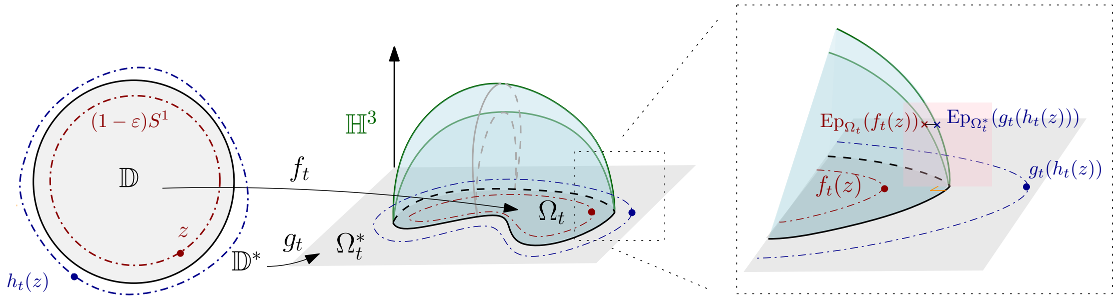

Figure 1: Illustration of the two Epstein-Poincaré surfaces associated with the two connected component of $\hat{\mathbb{C}} \setminus \gamma_t$ and the map h_t.

respectively and in t parameters. Consider ε sufficiently small so that for $z \in \overline{\mathbb{D}}$ with $|z| > 1-\varepsilon$ we have that $\mathrm{Ep}_{\Omega_t}(f_t(z))$ belongs to the parametrized neighbourhood U_{γ_t} from Proposition 4.6. Take the horizontal line $L_{t,z}$ (horocycle centered at $\infty \in \hat{\mathbb{C}}$) obtained by varying the second G-coordinate in U_{γ_t} starting from $\mathrm{Ep}_{\Omega_t}(f_t(z))$, and define $h_t(z) \in \overline{\mathbb{D}^*}$ to be the point such that $\mathrm{Ep}_{\Omega_t^*}(g_t(h_t(z)))$ is the first point of intersection of the horizontal line with $\mathrm{Ep}_{\Omega_t^*}$. See Figure 1 for an illustration.

Clearly along $\partial\mathbb{D}$ the map h_t agrees with $g_t^{-1} \circ f_t$, and from the regularity of f_t, g_t and the G-coordinates of U_{γ_t} we can see that the 1-parameter family of functions h_t is $C^{3,\alpha}$ in $1-\varepsilon < |z| \leq 1$ and t-parameters. Moreover, we can make ε sufficiently small so that h_t is a diffeomorphism with its image.

For r sufficiently close to 1, define the cylindrical neighbourhood $A(r)$ of γ_0 as $f_0(\{r \leq |z| \leq 1\}) \cup g_0(h_0(\{r \leq |z| \leq 1\}))$, which we parametrize by $S^1 \times [r, 1/r]$, sending (p,s) to $f_0(sp)$ if $s \leq 1$ and sending (p,s) to $g_0(h_0(\frac{p}{s}))$ if $s \geq 1$. These cylindrical neighbourhoods are nested as r grows, and their intersection as $r \to 1^-$ is γ_0. Define as well $\Omega(r), \Omega^*(r)$ the components of $\mathbb{C} \setminus A(r)$ in Ω_0 and Ω_0^*, respectively.

Define a 1-parameter family of homeomorphisms $F_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ so that for $z \in \overline{\Omega_0}$ we define $F_t(z) := f_t(f_0^{-1}(z))$, for $z \in g_0(h_t(\{1-\varepsilon < |z| \leq 1\}))$ we define $F_t(z) := g_t(h_t^{-1}(g_0^{-1}(z)))$, and we extend F_t to the rest of Ω_0^* as a $C^{3,\alpha}$ map in both Ω_0^* and t parameters. Let us also fix F_0 to be the identity. It follows then that $F_t|_{\Omega_0}$ is a conformal map between Ω_0 and Ω_t, and $F_t|_{\gamma_0}$ parametrizes γ_t. Given a parameter r so that $1-\varepsilon < r < 1$, we construct the family of piecewise smooth maps $E_{r,t} : S^2 \to \mathbb{H}^3$ satisfying the following properties:

- (C1) In $\Omega(r), \Omega^*(r)$ the map $E_{r,t}$ is defined as the composition of F_t with the Epstein-Poincaré maps $\mathrm{Ep}_{\Omega_t}, \mathrm{Ep}_{\Omega_t^*}$ of γ_t.
- (C2) Considering the parametrization of $A(r)$, for each $p \in S^1$, $E_{r,t}(\{p\} \times [r, 1/r])$ is the straight $\mathbb{R}^3$ segment joining $\mathrm{Ep}_{\Omega_t}(f_t(z))$ and $\mathrm{Ep}_{\Omega_t^*}(g_t(h_t(z)))$.
- (C3) The curve $E_{r,t}(\gamma \times \{r\})$ is given by the image of a curve $\gamma_{r,t}$ in $\Omega^*(\gamma_t)$ under the Epstein-Poincaré map.

Given that under our conditions the Epstein-Poincaré maps agree up to order 2 at γ_t, conditions (C1), (C2) and (C3) can be all satisfied for r sufficiently close to 1. For such fixed r the map $E_{r,t}$ is piecewise smooth, and it is $C^{3,\alpha}$ while restricted to $\Omega(r), A(r), \Omega^*(r)$

on both those parameters and t.

We prove the following main theorem in this section.

Theorem 5.1. *Let γ_t be a 1-parameter family of $C^{5,\alpha}$ Jordan curves ($\alpha > 0$). Then the first derivative of the volume $V(\gamma_t)$ is computed by*

$$\left.\frac{\partial}{\partial t}\right|_{t=0} V(\gamma_t) = \int_{\Omega} \mathrm{Ep}_{\Omega}^{*} \left(\delta H + \frac{1}{4}\langle \delta \mathrm{I}, \mathrm{I\!I}\rangle \mathrm{d}a\right) + \int_{\Omega^*} \mathrm{Ep}_{\Omega^*}^{*} \left(\delta H + \frac{1}{4}\langle \delta \mathrm{I}, \mathrm{I\!I}\rangle \mathrm{d}a\right)$$

where $\mathrm{Ep}_{\Omega}, \mathrm{Ep}_{\Omega^*}$ are the Epstein-Poincaré maps of Ω, Ω^* (respectively); $\mathrm{I}, \mathrm{I\!I}, H, \mathrm{d}a$ are the metric, second fundamental form, mean curvature and area form of the images of $\mathrm{Ep}_{\Omega}, \mathrm{Ep}_{\Omega^*}$; and δ denotes first order variation.

As in the convex cocompact case (see for instance [23, 29]) the main idea is to prove the analogous to the Schläfli formula for domains with piecewise smooth boundary by using Stokes theorem. While in our case the region bounded by Ep_{Ω} and Ep_{Ω^*} is non-compact and hence a new difficulty has been introduced, we have already taken the first step to deal with this by approximating with the (compact) region bounded by $E_{r,t}$. Even with this approximation, the map $E_{r,t}$ could fail to be a piecewise immersion, as the Epstein-Poincaré maps define branched surfaces in general. This happens when the principal curvatures at infinity are equal to ± 1 at a given point, and hence the term $\delta H + \frac{1}{4}\langle \delta \mathrm{I}, \mathrm{I\!I}\rangle \mathrm{d}a$ is not well-defined. Regardless, we will see (Lemma 5.2) that we have a well-defined normal and a well-defined (parametrized) shape operator. This will allow us to still establish a geometric identity (Proposition 5.4) that will express the variation of volume as the integral of a well defined 2-form $\mathrm{Tr}(\langle \nabla_{\xi}(B\cdot), DE_p \cdot\rangle)$ plus an exact form (see (5.4)). The notation $\mathrm{Tr}(\langle \nabla_{\xi}(B\cdot), DE_p \cdot\rangle)$ will be explained by the discussion following (5.3), from where we will see that $\mathrm{Tr}(\langle \nabla_{\xi}(B\cdot), DE_p \cdot\rangle)$ is the continuous extension of $\mathrm{Ep}_{\Omega,\Omega^*}^{*}\left(\delta H + \frac{1}{4}\langle \delta \mathrm{I}, \mathrm{I\!I}\rangle \mathrm{d}a\right)$ to the non-immersed points. Then after using Stokes theorem and making $r \to 1^-$, we will obtain the identity of Theorem 5.1 by verifying that all other integrals (both from the approximation by $E_{r,t}$ and Stokes theorem) go to 0.

We first address the definition of the normal and (parametrized) shape operator for $E_{r,t}$.

Lemma 5.2. *Along each $\Omega(r), A(r), \Omega^*(r)$, on the image of $E_{r,t}(p)$ there is a well-defined vector $\vec{n}$ that is normal to the image of $DE_{r,t}$. Such normal vector $\vec{n}$ varies piecewise $C^{3,\alpha}$ on r, t, p, and more precisely it is $C^{3,\alpha}$ while restricting p to either $\Omega(r), A(r), \Omega^*(r)$. Moreover, there is a piecewise $C^{2,\alpha}$ family of linear maps $B_{r,t}(p) : \mathbb{R}^2 \to \vec{n}^{\perp}(E_{r,t}(p))$ so that at any point where $E_{r,t}$ is an immersion, $B_{r,t}$ agrees with the pullback by $E_{r,t}$ of the shape operator of the image of $E_{r,t}$.*

Proof. For $\Omega(r), \Omega^*(r)$ the existence of $\vec{n}$ follows from the construction of the Epstein-Poincaré map, see (3.8), and from the map $F_{r,t}$ being piecewise $C^{3,\alpha}$. For $A(r)$, each curve $E_{r,t}(\gamma \times \{r\})$ is embedded for r sufficiently close to 1, as it converges to $F_t(S_1)$ as $r \to 0$. Since the segment $E_{r,t}(\{p\} \times [-r, r])$ belongs to a perpendicular of γ_t that varies smoothly on the data, we define $\vec{n}$ as the orthogonal vector to this line and $E_{r,t}(\gamma \times \{r\})$, taken so that the third coordinate of $\vec{n}$ is positive. This makes $\vec{n}$ well-defined for r sufficiently close to 1.

For $\Omega(r), \Omega^*(r)$, $B_{r,t}(p)$ is clearly defined as the shape operator in $E_{r,t}$ coordinates at points where $E_{r,t}$ is an immersion. This is not well-defined at points of $\Omega(r), \Omega^*(r)$ where the curvatures at infinity are ± 1, since by the duality the metric $I(X,Y) = \frac{1}{4}I^*(X + B^*X, Y + B^*Y)$ will vanish precisely at directions X at infinity whenever $B^*X = -X$. In particular X is an eigenvalue of B^*, which remains true if we rescale the metric by a constant factor. Rescale then the conformal metric by a factor e^{ε}, so now the map $E_{r,t}$ becomes an embedding and $|BX| = (1 + e^{-\varepsilon})|X|^*$. Sending $\varepsilon \to 0$ we see that we can extend $B_{r,t}X$ as a vector of norm 2 orthogonal to $\vec{n}$ for $|X|^* = 1$.

For the region $A(r)$ we can define $B_{r,t}$ by observing that the map $E_{r,t}$ is the composition of a smooth map into the horizontal lines described in step (C2). The union of these lines are immersed for r sufficiently close to 1, and hence have a well-defined shape operator. Hence we define $B_{r,t}$ as the pullback of such shape operator by $E_{r,t}$.

It is clear form the definitions that $\vec{n}$ and $B_{r,t}$ are piecewise $C^{3,\alpha}, C^{2,\alpha}$ respectively. □

Remark 5.3. While $E_{r,t}$ may fail in general to be a piecewise immersion, it is an immersion while restricted to the edge locus $\partial A(r)$. Moreover, from the definition of the normal vector $\vec{n}$ we have that the dihedral angles are well-define and vary $C^{2,\alpha}$ along t and the base point. When appropriate, we will simplify notation by dropping r, t sub-indices.

The following proposition generalizes the key formula to prove the differential Schläfli-formula (see [29, Proposition 5]). Let $\frac{\partial}{\partial t}|_{t=0} E_{r,t} = \xi$ be the piecewise defined vector field by the first order variation on t, and let ∇ denote the Levi-Civita connection of $\mathbb{H}^3$.

Proposition 5.4. *For any $p \in \hat{\mathbb{C}} \setminus \gamma_t$ and $u, v \in \mathbb{R}^2$ we have*

$$\langle \nabla_\xi (Bu), DE_p v\rangle = -\langle \nabla_{DE_p v} \nabla_\xi \vec{n}, DE_p u\rangle + \langle R(\xi, DE_p u)\vec{n}, DE_p v\rangle \tag{5.1}$$

where we follow the convention $R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$.

Proof. Let us verify first that we have the equality

$$\langle B_{r,t} u, DE_{r,t} v\rangle = -\langle \nabla_{DE_{r,t} u} \vec{n}, DE_{r,t} v\rangle$$

Where $E_{r,t}$ is an immersion, this follows from the relation between the shape operator and the second fundamental form. In directions when $DE_{r,t}$ fails to be injective both sides vanish. Taking then derivative in t we have

$$\begin{aligned} \langle \nabla_\xi Bu, DEv\rangle + \langle Bu, \nabla_\xi v\rangle &= -\langle \nabla_\xi \nabla_{DE_{r,t} u} \vec{n}, DE_{r,t} v\rangle - \langle \nabla_{DE_{r,t} u} \vec{n}, \nabla_\xi DE_{r,t} v\rangle \\ = -\langle \nabla_{DE_{r,t} u} \nabla_\xi \vec{n}, DE_{r,t} v\rangle &+ \langle R(\xi, DE_p u)\vec{n}, DE_p v\rangle - \langle \nabla_{DE_{r,t} u} \vec{n}, \nabla_\xi DE_{r,t} v\rangle \end{aligned} \tag{5.2}$$

If E is an immersion we have that $\langle Bu, \nabla_\xi v\rangle = -\langle \nabla_{DE_{r,t} u} \vec{n}, \nabla_\xi DE_{r,t} v\rangle$. Since we can extend the equality by continuity, we have then

$$\langle \nabla_\xi Bu, DEv\rangle = -\langle \nabla_{DEu} \nabla_\xi \vec{n}, DEv\rangle + \langle R(\xi, DE_p u)\vec{n}, DE_p v\rangle$$

as claimed. □

Remark 5.5. At points where E is an immersion, we can write $\langle \nabla_\xi Bu, DEv\rangle$ as

$$\langle \nabla_\xi Bu, DEv\rangle = \langle B'(DEu), DEv\rangle + \langle \nabla_{Bu}\xi, DEv\rangle$$

which is the formula appearing in [29, Proposition 5], where B' is the derivative of the shape operator in the immersed surface image.

The next step involves tracing the formula (5.1) with respect to the metric in the image of $E_{r,t}$ and multiply it by its area form (both induced from $\mathbb{H}^3$). Note that the trace of $\langle R(\xi, DE_p\cdot)\vec{n}, DE_p\cdot\rangle$ is $-2\langle \xi, \vec{n}\rangle E^* da$, which is a multiple of the 2-form that appears in the variational formula for the volume enclosed by the maps $E_{r,t}$. The remaining terms in (5.1) lead to the Schläfli formula we are interested in. Hence our next concern is how to perform this trace when $E_{r,t}$ is not am immersion. Let us address first $E_{r,t}$ in $\Omega(r), \Omega^*(r)$.

For $U \subseteq \mathbb{C}$, $\varphi \in C^\infty(U)$ denote by $E_\varphi, \widetilde{E_\varphi}$ the Epstein map and Epstein Gauss map for $e^\varphi |dz|^2$, respectively. We say that a 2-tensor $T : C^\infty(U) \to \Lambda^{2,0}(U)$ is *compatible* if T is a differentiable map so that for any $\varphi \in C^\infty(U)$, $x \in U, v \in \mathbb{R}^2$ so that $\langle (DE_\varphi)_x\cdot, (DE_\varphi)_x v\rangle \equiv 0$, we have that $T(\varphi)_x(\cdot, v) \equiv 0$. Examples of compatible maps are each summand in formula (5.1).

For any compatible T we define $\mathrm{Tr}(T) \in \Omega^2(U)$ as

$$\mathrm{Tr}(T(\varphi))_x := \lim_{\varepsilon\to 0} E^*_{\varphi+\varepsilon}(tr((E^{-1}_{\varphi+\varepsilon})^* T(\varphi+\varepsilon)_x)\mathrm{d}a_\varepsilon), \tag{5.3}$$

where $tr, \mathrm{d}a_{\varphi+\varepsilon}$ are respectively the trace and area form on the orthogonal complement of $\widetilde{E_{\varphi+\varepsilon}}$. To see that this limit is well defined, observe that for $\varepsilon \neq 0$ sufficiently small we have that $E_{\varphi+\varepsilon}$ is an immersion at $x \in U$. In particular, the limit agrees with $E^*_\varphi(tr((E^{-1}_\varphi)^* T)da)$ if E_φ is an immersion at $x \in U$. As an application of Theorem 3.3 we can take orthonormal $u, v \in \mathbb{R}^2$ so that $u_\varepsilon := (DE_{\varphi+\varepsilon})_x u, v_\varepsilon := (DE_{\varphi+\varepsilon})_x v \in T^1\mathbb{H}^3$ are orthogonal for all ε. Hence

$$\begin{aligned} &E^*_{\varphi+\varepsilon}(tr((E^{-1}_{\varphi+\varepsilon})^* T(\varphi+\varepsilon)_x)\mathrm{d}a_\varepsilon) \\ &\quad = \left(\frac{1}{|u_\varepsilon|^2} T(\varphi+\varepsilon)(u,u) + \frac{1}{|v_\varepsilon|^2} T(\varphi+\varepsilon)(v,v)\right) |u_\varepsilon|.|v_\varepsilon|\, \mathrm{d}x\mathrm{d}y \end{aligned}$$

and the limit (5.3) exists even if either or both $|u_\varepsilon|, |v_\varepsilon|$ go to 0 linearly with ε, since in that case we have that the respective $T(\varphi)(u,u), T(\varphi)(v,v)$ vanishes and the corresponding $\frac{1}{|u_\varepsilon|} T(\varphi+\varepsilon)(u,u)$, $\frac{1}{|v_\varepsilon|} T(\varphi+\varepsilon)(v,v)$ converges to a derivative of T.

For the terms in (5.1) we can make this computation explicit for $-\langle \nabla_{DE_p v}\nabla_\xi \vec{n}, DE_p u\rangle$ and $\langle R(\xi, DE_p u)\vec{n}, DE_p v\rangle$. Observe that at points where E is an immersion we have that $\mathrm{Tr}(-\langle \nabla_{DE_p v}\nabla_\xi \vec{n}, DE_p u\rangle)$ is equal to $-E^*(div(\nabla_\xi \vec{n})da) = E^*(d(\langle ., \nabla_\xi \vec{n}\rangle)) = -d(i_{\nabla_\xi \vec{n}})$, where $i_{\nabla_\xi \vec{n}}$ is the 1-form defined by $u \mapsto \langle DE_p u, \nabla_\xi \vec{n}\rangle$. Hence for all points we get $\mathrm{Tr}(-\langle \nabla_{DE_p v}\nabla_\xi \vec{n}, DE_p u\rangle) = -d(i_{\nabla_\xi \vec{n}})$.

For $\langle R(\xi, DE_p u)\vec{n}, DE_p v\rangle = -\langle \xi, \vec{n}\rangle\langle DE_p u, DE_p v\rangle$, we can see that this symmetric tensor is the pullback of a symmetric tensor in $\vec{n}^\perp$. Then $\mathrm{Tr}(\langle R(\xi, DE_p u)\vec{n}, DE_p v\rangle = -2\langle \xi, \vec{n}\rangle E^* da$, which vanishes if E fails to be an immersion.

Lemma 5.6 (See [29]). *At points where E is an immersion, $\frac{1}{2}\operatorname{Tr}(\langle \nabla_\xi(B\cdot), DE_p\cdot\rangle)$ agrees with the pullback by E of the form $(\delta H + \frac{1}{4}\langle \delta \mathrm{I}, \mathrm{I\!I}\rangle)\,\mathrm{d}a$.*

We now relate the variation of the volume with $\operatorname{Tr}(\langle \nabla_\xi(B\cdot), DE_p\cdot\rangle)$. Following (5.3) and (5.1) we obtain

$$\operatorname{Tr}(\langle \nabla_\xi(B\cdot), DE_p\cdot\rangle) = -d(i_{\nabla_\xi \vec{n}}) - 2\langle \xi, \vec{n}\rangle E^*\mathrm{d}a. \tag{5.4}$$

For $E_{r,t}$ in $A(r)$, we can establish and trace (5.1) in the embedded surface that contains the image of $E_{r,t}$ (for r sufficiently close to 1) and then take the pullback by $E_{r.t}$.
Let $V_2(r,t)$ be defined as the volume bounded by $E_{r,t}$. Namely, extend $E_{r,t} : S^2 = \hat{\mathbb{C}} \to \mathbb{H}^3$ as a map from the closed ball B^3 and define

$$V_2(r,t) := \int_{B^3} E^*_{r,t}(\mathrm{vol}_{\mathbb{H}^3})$$

By Stokes, this definition does not depend on the specific extension of $E_{r,t}$ to B^3. Since $E_{r,t}$ vary $C^{3,\alpha}$ as piecewise maps from $\Omega(r), \Omega^*(r), A(r)$, we can take the extension to vary $C^{3,\alpha}$ on t and check that $\partial_t V_2(r,t)$ is given by

$$\partial_t V_2 = \left(\int_{\Omega(r)} + \int_{\Omega^*(r)} + \int_{A(r)} -\langle \xi, \vec{n}\rangle E^*\mathrm{d}a \right)$$

where $\xi = \partial_t E_{r,0}$, $\vec{n}$ is the normal vector described in Lemma 5.2 and da is the area form of the orthogonal plane to $\vec{n}$. The negative sign is due to the fact that we are taking normal vector $\vec{n}$ pointing *inward* the region bounded by $E_{r,t}$.
Applying (5.4) we have then

$$\partial_t V_2 = \left(\int_{\Omega(r)} + \int_{\Omega^*(r)} + \int_{A(r)} \frac{1}{2}\operatorname{Tr}(\langle \nabla_\xi(Bu), DE_p v\rangle) + \frac{1}{2}\mathrm{d}(i_{\nabla_\xi \vec{n}}) \right)$$

Applying Stokes theorem for $\frac{1}{2}d(i_{\nabla_\xi \vec{n}})$ yields the integral of $\frac{1}{2}i_{\nabla_\xi \vec{n}}$ over each boundary component. Since $E^{r,t}$ is embedded along $\partial A(r)$, then as in [29] we have that along $\partial A(r)$, we have

$$\begin{aligned} i_{\nabla_\xi(\vec{n}^{\Omega(r)})} + i_{\nabla_\xi(\vec{n}^{A(r)})} &= \frac{\partial \theta^+}{\partial t} E^*\mathrm{d}\ell \\ i_{\nabla_\xi(\vec{n}^{\Omega^*(r)})} + i_{\nabla_\xi(\vec{n}^{A(r)})} &= \frac{\partial \theta^-}{\partial t} E^*\mathrm{d}\ell \end{aligned}$$

where $\theta^+(x)$ (respectively $\theta^-(x)$) is the exterior dihedral angle of the planes orthogonal to $\vec{n}^{\Omega(r)}, \vec{n}^{A(r)}$ at $E(x)$ (respectively $\vec{n}^{\Omega(r)^*}, \vec{n}^{A(r)}$ at $E(x)$), and dℓ is the length form in $\mathbb{H}^3$.
Applying then Stokes for $\partial_t V_2$ we get

$$\begin{aligned} \partial_t V_2 =& \left(\int_{\Omega(r)} + \int_{\Omega^*(r)} + \int_{A(r)} \frac{1}{2}\operatorname{Tr}(\langle \nabla_\xi(B\cdot), DE_p\cdot\rangle) \right) \\ &+ \frac{1}{2}\left(\int_{\partial\Omega(r)} \frac{\partial \theta^+}{\partial t} E^*\mathrm{d}\ell + \int_{\partial\Omega^*(r)} \frac{\partial \theta^-}{\partial t} E^*\mathrm{d}\ell \right). \end{aligned} \tag{5.5}$$

Proof of Theorem 5.1. Following Lemma 5.6 and Equation (5.5), we only need to prove that

$$\lim_{r\to 0}\int_{A(r)}\frac{1}{2}\operatorname{Tr}(\langle\nabla_\xi(B\cdot),DE_p\cdot\rangle)=0 \qquad \text{and}$$

$$\lim_{r\to 0}\frac{1}{2}\left(\int_{\partial\Omega(r)}\frac{\partial\theta^+}{\partial t}E^*\mathrm{d}\ell+\int_{\partial\Omega^*(r)}\frac{\partial\theta^-}{\partial t}E^*\mathrm{d}\ell\right)=0.$$

For the first term, observe that $A(r)$ belongs to the surface described in (C2). These families of surfaces can be described by

$$(r,s)\to(x(r,s,t),y(r,s,t),z(r,s,t)),$$

where r,s parametrize the surface as in (C2) for γ_t. This parametrization extends smoothly for $r=1$ towards the boundary of $\mathbb{H}^3$ by making $z(1,s,t)\equiv 0$. Moreover, given (C1) and Lemma 3.12 we have that $z(r,s,t)=O((1-r))$.

Hence the first and second fundamental form (as well as their first order variations) and the inverse of the first fundamental form are of order at most $(1-r)^{-2}$, from which the terms $H,\delta H,\langle\delta I,\mathbb{I}\rangle$ are uniformly bounded. As the area of $A(r)$ decays at least of the order of $1-r$, we have that

$$\lim_{r\to 1^-}\int_{A(r)}\frac{1}{2}\operatorname{Tr}(\langle\nabla_\xi(B\cdot),DE_p\cdot\rangle)=0$$

Likewise, the functions $\theta^\pm$ that take each (r,s,t) to the angle between $\Omega(r),\Omega^*(r)$ and $A(r)$ at $(x(r,s,t),y(r,s,t),z(r,s,t))$, extend smoothly to $r=1$ as right angles. Hence in particular $\frac{\partial\theta^\pm}{\partial t}=O((1-r))$. This is not enough for the desired limit, as the curves $\partial\Omega(r),\Omega^*(r)$ have length comparable to $(1-r)^{-1}$. What we can rather do is use again that the Epstein-Poincaré surfaces agree up to second order to use parametrizations $\gamma_\varepsilon^\pm(s)$ satisfying ($\varepsilon=(1-r)$)

$$\begin{aligned}\left\|\frac{d\gamma_\varepsilon^+}{ds}(s)-\frac{d\gamma_\varepsilon^-}{ds}(s)\right\|&\le C\varepsilon^2\\ |\theta'(\gamma_\varepsilon^+(s))+\theta'(\gamma_\varepsilon^-(s))|&\le C\varepsilon^2\end{aligned}\tag{5.6}$$

for some uniform constant $C>0$. Then since the last coordinate of $\gamma_\varepsilon^\pm$ is $O(\varepsilon)$, we have that for some uniform constant $C>0$

$$\begin{aligned}&\left|\int_{\partial\Omega(r)}\frac{\partial\theta^+}{\partial t}E^*\mathrm{d}\ell+\int_{\partial\Omega^*(r)}\frac{\partial\theta^-}{\partial t}E^*\mathrm{d}\ell\right|\\ &\le C\int_{S^1}\left|\frac{\theta'(\gamma_\varepsilon^+(s))}{\varepsilon}\right|.\left\|\frac{d\gamma_\varepsilon^+}{ds}\right\|+\left|\frac{\theta'(\gamma_\varepsilon^-(s))}{\varepsilon}\right|.\left\|\frac{d\gamma_\varepsilon^-}{ds}\right\|ds\\ &\le\frac{1}{\varepsilon}\int_{S^1}|\theta'(\gamma_\varepsilon^+(s))|.\left\|\frac{d\gamma_\varepsilon^+}{ds}(s)-\frac{d\gamma_\varepsilon^-}{ds}(s)\right\|+|\theta'(\gamma_\varepsilon^+(s))+\theta'(\gamma_\varepsilon^-(s))|.\left\|\frac{d\gamma_\varepsilon^-}{ds}\right\|ds\end{aligned}\tag{5.7}$$

goes to 0 as $\varepsilon\to 0$ uniformly in t.

We define then $V_2(r,t)$ using the parameters of Proposition 4.6, so that $V(\gamma_t)=V_1(r,t)+V_2(r,t)$ for any r sufficiently small. For the parametrized region in $V_2(r,t)$ we can see that the t derivatives of the functions f, g in the proof of Proposition 4.6 agree as well up to order 2 (in z variable), so by the same argument we have that $\lim_{r\to 0} \partial_t V_1(r, 0) = 0$. As for any r small we have that $\partial_t V(\gamma_t) = \partial_t V_1(r, t) + \partial_t V_2(r, t)$, we send r to 0 on the right hand side to obtain

$$\frac{\partial}{\partial t}\bigg|_{t=0} V(\gamma_t) = \int_{\Omega} + \int_{\Omega^*} \frac{1}{2} \operatorname{Tr}(\langle \nabla_\xi (B\cdot), DE_p \cdot\rangle)$$

which completes the proof by Lemma 5.6. □

5.2 Variation of mean curvature and Schläfli formula

The following result is proved by Krasnov-Schlenker see [16, Cor. 6.2] for the renormalized volume of convex co-compact manifolds. We adapt it to the renormalized volume associated with a smooth Jordan curve.

Theorem 5.7. *We have the first order variation of the V_R*

$$\delta V_R(\gamma) = -\frac{1}{4} \int_{\Omega \cup \Omega^*} \delta H^* + \frac{1}{2} \langle \delta \mathrm{I}^*, \mathrm{I\!I}_0^* \rangle \, \mathrm{d}a^*$$

where $\mathrm{I\!I}_0^* = \vartheta \, \mathrm{d}z^2 + \bar\vartheta \, \mathrm{d}\bar z^2$ is the traceless part of $\mathrm{I\!I}^*$, $\langle A, B\rangle$ stands for $\operatorname{tr}[(I^*)^{-1} A (I^*)^{-1} B]$.

Proof. By Definition 4.7 and Remark 4.8, we can express δV_R as the integral of smooth 2-forms in Ω, Ω^*, so that at points where the respective Epstein-Poincaré maps are immersions these forms are given by the pullback of the form

$$(\delta H + \frac{1}{4} \langle \delta \mathrm{I}, \mathrm{I\!I}\rangle) \, \mathrm{d}a - \frac{1}{2} (\delta H \mathrm{d}a - H \delta(\mathrm{d}a))$$

by the respective Epstein-Poincare map. Following [16, Section 6] this pullback is expressed precisely as $-\frac{1}{4}(\delta H^* + \frac{1}{2} \langle \delta \mathrm{I}^*, \mathrm{I\!I}_0^* \rangle) \, \mathrm{d}a^*$. As points where the Epstein-Poincaré maps are immersions are dense in Ω, Ω^* and all forms discussed are continuous, the result follows. □

More explicitly, we can write the variation of V_R in terms of the Beltrami coefficients. We consider a $C^{5,\alpha}$ family of Jordan curves (γ_t) as in the previous section and let F_t be the corresponding homeomorphism of $\hat{\mathbb{C}}$ which maps Ω_0 conformally onto Ω_t and a diffeomorphism from Ω_0^* to Ω_t^*. For $z \notin \gamma_0$, let

$$\mu_t := \frac{\partial_{\bar z} F_t}{\partial_z F_t} = t\dot\nu + O(t^2).$$

We have in particular, $\dot F := \frac{\mathrm{d}}{\mathrm{d}t} F_t|_{t=0}$ satisfies

$$\partial_{\bar z} \dot F = \dot \nu, \qquad F_t(z) = z + t\dot F(z) + O(t^2).$$

Lemma 5.8. *We have $\|\dot\nu\|_\infty < \infty$. Moreover, $\dot\nu|_{\Omega^*} \in H^{-1,1}(\Omega^*) + \mathfrak{N}(\Omega^*)$.*

Proof. On Ω we have that the 1-parameter family F_t is conformal, while in $\overline{\Omega^*}$ we can write F_t as the composition $g_t \circ H_t \circ g_0$, where H_t is a $C^{3,\alpha}$ family of maps that agree with h_t^{-1} near $\partial\mathbb{D}$. As g_t, H_t, g_0 extend to the boundary of their respective domains and are $C^{3,\alpha}$ as a family of maps, the L^∞ bound of $\dot\nu$ follows from the compactness of the domains.

For the second claim, as (γ_t) corresponds to a differentiable path in $T_0(1)$, the projection of $\dot\nu$ onto harmonic Beltrami differentials $\Omega^{-1,1}(\Omega^*)$ parallel to $\mathfrak{N}(\Omega^*)$ lies in $H^{-1,1}(\Omega^*)$. This completes the proof. ☐

Corollary 5.9. *The first variation of the renormalized volume associated with the family of deformed Jordan curves* $(\gamma_t := F_t(\gamma_0))$ *is given by*

$$\delta V_R(\gamma) = -\operatorname{Re}\int_{\Omega^*} \dot\nu \mathscr{S}[g^{-1}]\mathrm{d}^2 z. \tag{5.8}$$

where we recall $g : \mathbb{D}^* \to \Omega^*$ *is any conformal map.*

Proof. As $\dot\nu \in L^\infty$ and $\mathscr{S}[g^{-1}]$ is continuous functions up to the boundary. The integrals in (5.8) are absolutely convergent. We only need to check the pointwise identity

$$\left(\frac{1}{4}\delta H^* + \frac{1}{8}\left\langle \delta \mathrm{I}^*, \mathrm{I\!I}_0^*\right\rangle\right)\mathrm{d}a^* = \dot\nu \mathscr{S}[g^{-1}]\,\mathrm{d}^2 z$$

on Ω^*. We have

$$\mathrm{d}F_t(z) = \mathrm{d}z + t\partial_z \dot F\,\mathrm{d}z + t\partial_{\bar z}\dot F\,\mathrm{d}\bar z + O(t^2) = \mathrm{d}z + t\partial_z \dot F\,\mathrm{d}z + t\dot\nu\,\mathrm{d}\bar z + O(t^2)$$

and in the $\mathrm{d}z, \mathrm{d}\bar z$ coordinates

$$\mathrm{d}F_t(z)\mathrm{d}\overline{F_t(z)} = \begin{pmatrix} t\bar{\dot\nu} & \frac{1}{2}(1+2t\operatorname{Re}(\partial_z \dot F)) \\ \frac{1}{2}(1+2t\operatorname{Re}(\partial_z \dot F)) & t\dot\nu \end{pmatrix} + O(t^2).$$

Therefore, the hyperbolic metric in Ω_t is

$$e^{\varphi}(1+2ts+O(t^2))\,\mathrm{d}F_t(z)\mathrm{d}\overline{F_t(z)} = \mathrm{I}^* + te^{\varphi}\begin{pmatrix} \bar{\dot\nu} & \operatorname{Re}(\partial_z \dot F)+s \\ \operatorname{Re}(\partial_z \dot F)+s & \dot\nu \end{pmatrix} + O(t^2).$$

where s is some smooth function on Ω and

$$\mathrm{I}^* = e^{\varphi}\mathrm{d}z\mathrm{d}\bar z = \frac{1}{2}\begin{pmatrix} 0 & e^{\varphi} \\ e^{\varphi} & 0 \end{pmatrix}.$$

We obtain

$$\delta \mathrm{I}^* = e^{\varphi}\begin{pmatrix} \bar{\dot\nu} & \operatorname{Re}(\partial_z \dot F)+s \\ \operatorname{Re}(\partial_z \dot F)+s & \dot\nu \end{pmatrix}.$$

Recall that

$$\mathrm{I\!I}_0^* = \begin{pmatrix} \vartheta & 0 \\ 0 & \bar\vartheta \end{pmatrix} = \begin{pmatrix} \mathscr{S}(g^{-1}) & 0 \\ 0 & \overline{\mathscr{S}(g^{-1})} \end{pmatrix},$$

we have (using the complexified inner product $\langle A, B\rangle = \operatorname{Re}\operatorname{Tr}\left[\overline{(I^*)^{-1}A}(I^*)^{-1}B\right]$)

$$\langle \delta I^*, \mathrm{I\!I}_0^*\rangle = 8e^{-\varphi}\operatorname{Re}(\dot\nu\mathscr{S}[g^{-1}]).$$

We obtain the claimed variation formula from Corollary 3.5 which shows $H^* = -K^* \equiv 1$ and which implies $\delta H^* \equiv 0$. ☐

Corollary 5.10. *We have for all $C^{5,\alpha}$ Jordan curves γ, we have*

$$\tilde{\mathbf{S}}(\gamma) = 4V_R(\gamma).$$

Proof. When γ is a circle, we have $\tilde{\mathbf{S}}(\gamma) = 0$ and $V_R(\gamma) = 0$ since both Epstein surfaces are the geodesic plane bounded by γ. Given a smooth Jordan curve γ. The variational formula Proposition 2.4 and Corollary 5.9 show that

$$\tilde{\mathbf{S}}(\gamma) = 4V_R(\gamma)$$

by taking a smooth deformation from γ to a circle. □

5.3 Approximation of general WP curve

The goal of the section is to prove the following theorem.

Theorem 5.11. *We have for all Weil-Petersson quasicircle γ,*

$$\tilde{\mathbf{S}}(\gamma) \geq 4V_R(\gamma).$$

We have already proved the equality when γ is $C^{5,\alpha}$. We also believe the equality holds for arbitrary Weil-Petersson quasicircle but are only able to prove the inequality.

For the inequality, we will use the approximation using equipotentials. Let γ be a Weil-Petersson quasicircle, $f : \mathbb{D} \to \Omega$ be a conformal map as before. Up to post-composing f by a Möbius map, we may assume that $f(0) = 0$, $f'(0) = 1$ and $f''(0) = 0$. The family of equipotentials

$$\gamma_n = f_n(S^1), \quad \text{where } f_n(z) := \frac{n}{n-1} f\left(\frac{n-1}{n} z\right)$$

is a family of analytic Jordan curves. The map f_n satisfies the same normalization as f at 0. We let $\Omega_n^* := \hat{\mathbb{C}} \smallsetminus \overline{f_n(\mathbb{D})}$ (resp. $\Omega^* := \hat{\mathbb{C}} \smallsetminus \overline{f(\mathbb{D})}$) and g_n (resp. g) be an arbitrary conformal map $\mathbb{D}^* \to \Omega_n^*$ (resp, $\mathbb{D}^* \to \Omega^*$). Apart from the analyticity, the family of equipotentials is particularly nice because of the following theorem.

Theorem 5.12 (See [32, Cor. 1.5])**.** *Along the family of equipotentials the universal Liouville action converges and is non-decreasing. We have*

$$\lim_{n\to\infty} \uparrow \tilde{\mathbf{S}}(\gamma_n) = \tilde{\mathbf{S}}(\gamma).$$

If γ is not a circle, then $\tilde{\mathbf{S}}(\gamma_{n+1}) > \tilde{\mathbf{S}}(\gamma_n)$.

Lemma 5.13. *We have*

$$\int_{\Sigma_{\Omega_n} \cup \Sigma_{\Omega_n^*}} H \mathrm{da} \xrightarrow{n\to\infty} \int_{\Sigma_\Omega \cup \Sigma_{\Omega^*}} H \mathrm{da}. \tag{5.9}$$

Proof. It follows from [31, Cor. A.4., Cor. A.6] that the element $[\mu_n]$ in $T_0(1)$ associated with γ_n converges to $[\mu]$ which is associated with γ. In particular, [31, Chap. I, Thm. 2.13, Thm. 3.1] imply that

$$\int_{\mathbb{D}} \|\mathscr{S}(f_n)\|^2 \rho_{\mathbb{D}} \,\mathrm{d}^2 z = \int_{\mathbb{D}} |\mathscr{S}(f_n)|^2 \rho_{\mathbb{D}}^{-1} \,\mathrm{d}^2 z \xrightarrow{n\to\infty} \int_{\mathbb{D}} \|\mathscr{S}(f)\|^2 \rho_{\mathbb{D}} \,\mathrm{d}^2 z.$$

As $T_0(1)$ is a topological group, we have $[\mu_n]^{-1}$ converges to $[\mu]^{-1}$ which implies

$$\int_{\mathbb{D}^*} \|\mathscr{S}(g_n)\|^2 \rho_{\mathbb{D}^*} \, \mathrm{d}^2 z = \int_{\mathbb{D}^*} |\mathscr{S}(g_n)|^2 \rho_{\mathbb{D}^*}^{-1} \, \mathrm{d}^2 z \xrightarrow{n\to\infty} \int_{\mathbb{D}^*} \|\mathscr{S}(g)\|^2 \rho_{\mathbb{D}^*} \, \mathrm{d}^2 z.$$

Using (3.11) the proof is completed. □

Lemma 5.14. *Let $V_{2,\varepsilon}(\gamma)$ denote the signed volume between* Ep_Ω *and* Ep_{Ω^*} *above the level ε. We have $V_{2,\varepsilon}(\gamma_n)$ converges to $V_{2,\varepsilon}(\gamma)$ for all $\varepsilon \in (0,1)$.*

Proof. For this, we denote for $\varepsilon \in (0,1)$,

$$K_{\varepsilon,n} := \{\zeta \in \mathbb{D} \colon \xi_n \circ f_n(\zeta) \geq \varepsilon\}, \quad K_\varepsilon := \{\zeta \in \mathbb{D} \colon \xi \circ f(\zeta) \geq \varepsilon\},$$

where (y_n, ξ_n) is the Epstein-Poincaré map on the domain $\Omega_n = f_n(\mathbb{D})$. By (3.17), we have for all n,

$$\frac{\operatorname{dist}(f_n(\zeta), \gamma_n)}{5} \leq |\xi_n \circ f_n(\zeta)| \leq 4 \operatorname{dist}(f_n(\zeta), \gamma_n)$$

which implies for all $\zeta \in K_{\varepsilon,n}$,

$$\operatorname{dist}(f_n(\zeta), \gamma_n) \geq \varepsilon/4.$$

It is not hard to see that f_n converges uniformly to f on $\overline{\mathbb{D}}$ from the explicit expression. However, it holds more generally for any sequence of normalized conformal maps representing converging sequence in $T_0(1)$. In fact, we extend f_n to a K-quasiconformal map of $\hat{\mathbb{C}}$, where K is independent of n since a converging sequence in $T_0(1)$ is also bounded in $T(1)$. The family of K-quasiconformal maps, normalized as f_n, is a normal family and converges uniformly along subsequences on all compact sets of $\mathbb{C}$. As the limit on $\overline{\mathbb{D}}$ is f, the convergence is thus along the whole sequence when restricted to $\overline{\mathbb{D}}$. Moreover, the derivatives of f_n converges to the derivatives of f uniformly on compact sets of $\mathbb{D}$ by Cauchy's integral formula.

Hence, there exists n_0 such that for all $n \geq n_0$, we have

$$\|f_n - f\|_{\infty, \overline{\mathbb{D}}} < \varepsilon/16.$$

This implies

$$\operatorname{dist}(f(\zeta), \gamma) \geq \varepsilon/8 \quad \text{and} \quad \xi \circ f(\zeta) \geq \varepsilon/40.$$

Summarizing, we have for all $n \geq n_0$,

$$K_{\varepsilon,n} \subset K_{\varepsilon/40}.$$

Since $K_{\varepsilon/40}$ is a compact set in $\mathbb{D}$ independent of n, we have that all derivatives of f_n converge uniformly to the derivatives of f on $K_{\varepsilon/40}$. As the Epstein-Poincaré map only depends on f, f', and f'', $\mathrm{Ep}_{\Omega_n} \circ f_n$ converges uniformly to $\mathrm{Ep}_\Omega \circ f$ uniformly on $K_{\varepsilon/40}$. Similarly argument applies to the Epstein-Poincaré maps $\mathrm{Ep}_{\Omega_n^*} \circ g_n$. We obtain that $V_{2,\varepsilon}(\gamma_n)$ converges to $V_{2,\varepsilon}(\gamma)$. □

We obtain the following corollary.

Corollary 5.15. *If γ is a Weil-Petersson curve, then*

$$V(\gamma) \leq \frac{1}{4}\left(\tilde{\mathbf{S}}(\gamma) - \frac{1}{2}\int_{\Sigma_\Omega \cup \Sigma_{\Omega^*}} H \mathrm{da}\right) < \infty.$$

Proof. For small enough $\varepsilon > 0$,

$$V_{2,\varepsilon}(\gamma) = \lim_{n\to\infty} V_{2,\varepsilon}(\gamma_n) \leq \lim_{n\to\infty} \frac{1}{4}\left(\tilde{\mathbf{S}}(\gamma_n) - \frac{1}{2}\int_{\Sigma_{\Omega_n} \cup \Sigma_{\Omega_n^*}} H \mathrm{da}\right) = \frac{1}{4}\left(\tilde{\mathbf{S}}(\gamma) - \frac{1}{2}\int_{\Sigma_\Omega \cup \Sigma_{\Omega^*}} H \mathrm{da}\right)$$

by Theorem 5.12. We obtained the inequality by taking $\varepsilon \to 0$. □

Theorem 5.11 follows immediately from this corollary.

6 Gradient flow of the universal Liouville action

Following Bridgeman-Brock-Bromberg [4] and Bridgeman-Bromberg-Vargas Pallete [5], we introduce the following flow on $T(1)$. For $[\mu] \in T(1)$ we have a natural isomorphism $T_{[\mu]}T(1) \simeq \Omega^{-1,1}(\mathbb{D}^*)$. We therefore define the vector field

$$V_{[\mu]} := -4\frac{\overline{\mathscr{S}(g_\mu)}}{\rho_{\mathbb{D}^*}} \in \Omega^{-1,1}(\mathbb{D}^*).$$

Theorem 6.1. *The vector field V has flowlines that exist for all time on $T(1)$. The flow restricts to a flow on $T_0(1)$ and is the (negative) Weil-Petersson gradient of the Liouville functional $\mathbf{S}$. Furthermore all flowlines on $T_0(1)$ limit to the origin $[0]$ which corresponds to the round circle.*

Proof. By the Nehari bound we have that in the Teichmüller metric on $T(1)$, $||V||_\infty \leq 6$. Thus as $T(1)$ is complete in the Teichmüller metric, the flow under V exists for all time on $T(1)$. If $[\mu] \in T_0(1)$ then by the characterization (2.4) we have

$$\int_{\mathbb{D}^*} |\mathscr{S}(g_\mu)|^2 \rho_{\mathbb{D}^*}^{-1} < \infty.$$

Thus $V_{[\mu]} \in H^{-1,1}(\mathbb{D}^*) \simeq T_{[\mu]}T_0(1)$ and therefore by integrability the flow preserves $T_0(1)$. Furthermore if $\dot{\nu} \in H^{-1,1}(\mathbb{D}^*) \simeq T_{[\mu]}T_0(1)$ then by Theorem 2.1

$$(\mathrm{d}\mathbf{S})_{[\mu]}(\dot{\nu}) = 4 \operatorname{Re} \int_{\mathbb{D}^*} \dot{\nu} \mathscr{S}(g_\mu) = -\operatorname{Re} \int_{\mathbb{D}^*} \dot{\nu}\, \overline{V_{[\mu]}}\, \rho_{\mathbb{D}^*} = -\left\langle V_{[\mu]}, \dot{\nu} \right\rangle_{\mathrm{WP}}.$$

Therefore $\nabla_{\mathrm{WP}}\mathbf{S} = -V$ and

$$\mathrm{d}\mathbf{S}(V) = -||V||_{\mathrm{WP}}^2.$$

We consider the flowline $\mathbb{R}_+ \to T_0(1) : t \mapsto \alpha(t)$ for V starting at a point $[\mu] = \alpha(0) \in T_0(1)$. Since $\mathbf{S} \geq 0$, for all $T > 0$,

$$0 \leq \int_0^T ||V(\alpha(t))||^2 \,\mathrm{d}t = \mathbf{S}([\mu]) - \mathbf{S}(\alpha(T)) \leq \mathbf{S}([\mu]).$$

Thus

$$\int_0^\infty ||V(\alpha(t))||_{\mathrm{WP}}^2 \, \mathrm{d}t < \infty.$$

We therefore have a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} ||V(\alpha(t_n))||_{\mathrm{WP}} = 0.$$

By [31, Ch. I, Lem. 2.1], we have if $\phi \in A_\infty(\mathbb{D})$ then

$$||\phi||_\infty \leq \sqrt{\frac{3}{4\pi}} ||\phi||_2. \tag{6.1}$$

Therefore

$$\lim_{n\to\infty} ||V(\alpha(t_n))||_\infty = 0.$$

Thus the conformal maps $g_{\alpha(t_n)}$ have Schwarzian $\mathscr{S}(g_{\alpha(t_n)}) \to 0$. By normalcy, we obtain a subsequence $g_{\alpha(t_{n_i})}$ converging uniformly on compact sets to a Möbius map preserving $-1, 1, \mathfrak{i}$. Therefore $\lim_{i\to} \alpha(t_{n_i}) = [0]$, the origin of $T_0(1)$.

To show that the flow line converges to [0], we observe that [0] is the unique global minimum for $\mathbf{S}$ on $T_0(1)$. Therefore there is a neighborhood of $[0] \in T_0(1)$ which is an attractor. By the above, α enters this neighborhood and therefore it converges to [0]. □

Using the gradient flow we may bound the Weil-Petersson distance between $[\mu]$ and [0] by the universal Liouville action. We first recall some results proved by Takhtajan and Teo that we summarize in the lemma below.

Lemma 6.2 ([31, Ch.I, Lem. 2.5, Rem. 2.4, Cor. 2.6])**.** *There exists $0 < \delta < 1$ such that for all $\mu \in \Omega^{-1,1}(\mathbb{D}^*)$ with $\|\mu\|_\infty < \delta$,*

$$\left| \frac{|\partial_z w_\mu(z)|^2}{(1 - |w_\mu(z)|^2)^2} - \frac{1}{(1-|z|^2)^2} \right| < \frac{1}{(1-|z|^2)^2}.$$

Moreover, for such μ, the map $D_0(\beta \circ R_{[\mu]}) : H^{-1,1}(\mathbb{D}^*) \to A_2(\mathbb{D})$ is a bounded linear isomorphism with

$$||D_0(\beta \circ R_{[\mu]})(\nu)||_2 \leq 24||\nu||_2 \qquad ||\nu||_2 \leq K||D_0(\beta \circ R_{[\mu]})(\nu)||_2$$

where $K = \sqrt{2}/(1-\delta)^2$.

Theorem 6.3. *With the same constants δ and K as in Lemma 6.2. Let $c < 2\delta\sqrt{4\pi/3}$ then for $[\mu] \in T_0(1)$, we have*

$$c(\mathrm{dist}_{\mathrm{WP}}([\mu],[0]) - Kc) \leq \mathbf{S}([\mu]).$$

Proof. We let $t \mapsto \alpha(t)$ be the gradient flow line starting at $[\mu]$ and τ be the first time $\|V(\alpha(t))\|_{\mathrm{WP}} = c$. Then $\|V(\alpha(t))\|_{\mathrm{WP}} > c$ for all $t < \tau$. Thus

$$\mathbf{S}([\mu]) - \mathbf{S}(\alpha(\tau)) = \int_0^\tau \|V(\alpha(t))\|_{\mathrm{WP}}^2 \, \mathrm{d}t \geq c \int_0^\tau \|V(\alpha(t))\|_{\mathrm{WP}} \, \mathrm{d}t \geq c \; \mathrm{dist}_{\mathrm{WP}}([\mu], \alpha(\tau)).$$

We have therefore

$$\mathbf{S}([\mu]) \geq c(\mathrm{dist}_{\mathrm{WP}}([\mu],[0]) - \mathrm{dist}_{\mathrm{WP}}(\alpha(\tau),[0])).$$

As $\|V(\alpha(\tau))\|_{\mathrm{WP}} = c$ then by (6.1), $\|V(\alpha(\tau))\|_\infty \leq \sqrt{3/4\pi}c < 2\delta$. Therefore

$$\|\hat{\beta}([\alpha(\tau)])\|_\infty = \|\mathscr{S}(g_{\alpha(\tau)})\|_\infty < \delta/2$$

where $\hat{\beta}$ is the Bers embedding $T(1) \to A_\infty(\mathbb{D}^*)$. As $\hat{\beta}(T_0(1)) = \hat{\beta}(T(1)) \cap A_2(\mathbb{D}^*)$ the linear path

$$\gamma(s) := [s\tilde{\mu}], \quad \text{where} \quad \tilde{\mu} = -\frac{2}{\bar{z}^4}\frac{\mathscr{S}(g_{\alpha(\tau)})}{\rho_{\mathbb{D}^*}}\left(\frac{1}{\bar{z}}\right) \text{ satisfies } \|\tilde{\mu}\|_{\mathbb{D},\infty} < \delta$$

for $s \in [0,1]$ from 0 to $\alpha(\tau)$ is in the ball of radius δ of $T(1)$, and also in $T_0(1)$ since by Ahlfors-Weill theorem

$$\hat{\beta}([s\tilde{\mu}]) = s\mathscr{S}(g_{\alpha(\tau)}) \in A_2(\mathbb{D}^*).$$

In the L_2 metric on $A_2(\mathbb{D}^*)$ this path has length $\|V(\alpha(\tau))\|_{\mathrm{WP}} \leq c$. By Lemma 6.2 we have that the preimage of $\hat{\beta}$ has therefore length less than Kc. □

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