SLE, Energy duality, Foliations by Weil-Petersson quasicircles

Yilin Wang (MIT)

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Introduction

Probabilistic

Deterministic

SLE_{0+}

Weil-Petersson quasicircles

SLE_{∞}

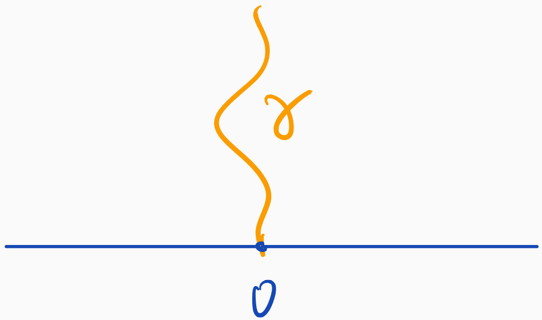
Loewner energy measures

"how round the curve is"

Foliations of Weil-Petersson quasicircles

Chordal Loewner evolution

[Loewner 1923]



a simple chord in $(\mathbb{H}, 0, \infty)$

Loewner transform

Loewner equation

$W : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $t \mapsto W_t$

driving function of γ

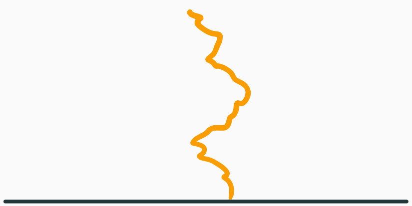
Schramm - Loewner Evolution SLE

- If $W = \sqrt{\kappa}\, B$ where B is a standard Brownian motion.

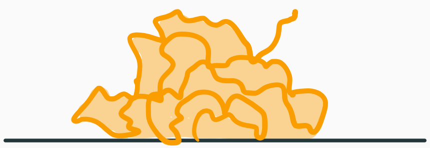
  $\gamma = \mathrm{SLE}_\kappa$ [Schramm '99]

Thm [Rohde - Schramm '05]

$0 \le \kappa \le 4$



$4 < \kappa < 8$



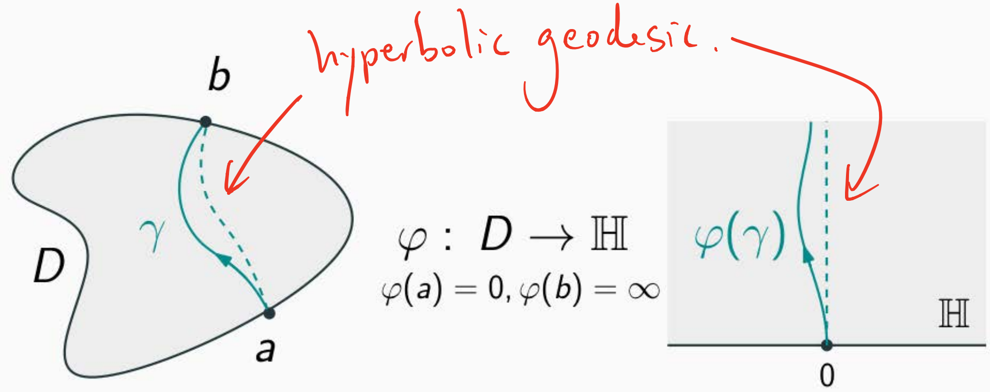
$8 \le \kappa$

For specific values of κ

SLE_κ is proved / conjectured to be scaling limit of interfaces in 2D statistical mechanics models.

Lawler. Schramm. Werner
Smirnov. Sheffield ...

Chordal Loewner energy (W. [1])



Loewner energy $I_{D,a,b}(\gamma) := I_{\mathbb{H},0,\infty}(\varphi(\gamma)) := I(W) := \frac{1}{2}\int_0^\infty W'(t)^2 dt$

if W is absolutely continuous;

$= \infty$ otherwise.

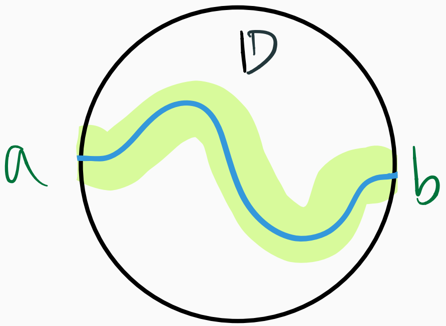
- For $c > 0$, we have $I_{\mathbb{H},0,\infty}(\gamma) = I_{\mathbb{H},0,\infty}(c\gamma)$. $\tilde{W}(t) = c\, W_{c^{-2}t}$
  $\Rightarrow$ The Loewner energy is well-defined in (D, a, b). (Same for SLE)

$\mathrm{SLE}_{0+} \longleftrightarrow$ Loewner energy

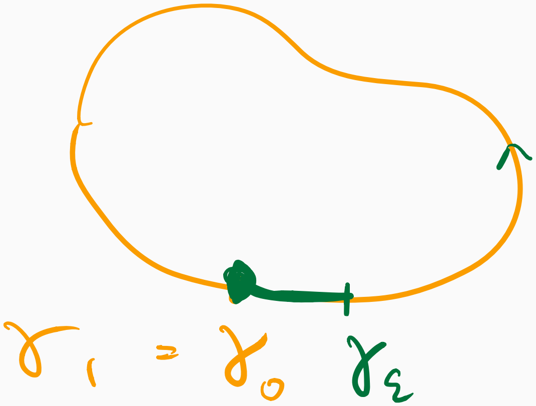
Large deviations of SLE_{0+}

Thm. W. '16 [1] Peltola. W. '20 [6]

$$\mathbb{P}(\mathrm{SLE}_\kappa \text{ is close to } \gamma) \underset{\kappa \to 0+}{\sim} \exp\left(-\frac{I_D(\gamma)}{\kappa}\right)$$



Loop energy (Rohde, W. [2])



$$\gamma : [0,1]/_{0\sim 1} \longrightarrow \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

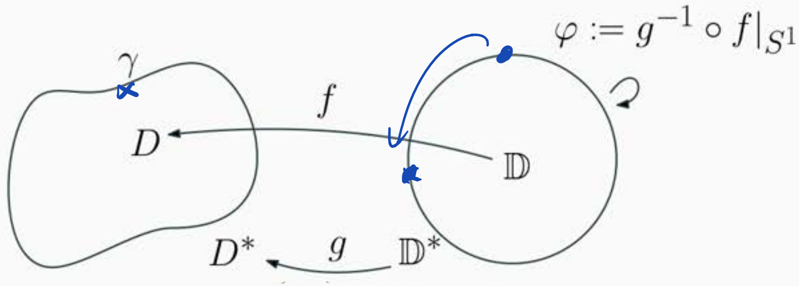
$$\boxed{I^L(\gamma, \gamma_0) := \lim_{\varepsilon \to 0} I_{\hat{\mathbb{C}} \setminus \gamma_{[0,\varepsilon]}}\left(\gamma_{[\varepsilon, 1]}\right)}$$

Thm. $I^L(\gamma, \cancel{\gamma_0})$ is independent of the root γ_0.

- $I^L(\gamma) = 0 \iff \gamma$ is a circle
- $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ conformal $\Rightarrow I^L(\varphi(\gamma)) = I^L(\gamma)$.
- $I^L(\gamma) < \infty \Rightarrow \gamma$ is a rectifiable quasicircle.

Welding homeomorphism

Idea: Associate γ with its **welding function** φ:



$I^L(\gamma) < \infty \Rightarrow \gamma$ is a quasicircle $\Leftrightarrow \varphi \in QS(S^1)$. (Beurling - Ahlfors)

$$\Updownarrow$$

$\exists k>0, \ \forall \ t, \theta \in \mathbb{R}$

$$\frac{1}{k} \leq \left| \frac{\varphi(e^{i(t+\theta)}) - \varphi(e^{it})}{\varphi(e^{it}) - \varphi(e^{i(t-\theta)})} \right| \leq k$$

Universal Teichmüller space $T(1)$

$$T(1) = \text{Möb}(S^1) \backslash QS(S^1)$$ quasicircles

$$\cup$$

$$\text{Möb}(S^1) \backslash \text{Diff}(S^1)$$ C^∞ smooth curves

has a unique homogeneous Kähler metric
Weil-Petersson metric

Universal Teichmüller space $T(1)$

$$T(1) = \text{Möb}(S^1) \backslash QS(S^1)$$ quasicircles

$$\cup$$

[Takhtajan – Teo '06]

∞-dimensional Kähler-Einstein manifold !!!

$$T_0(1) = \text{Möb}(S^1) \backslash WP(S^1)$$ Weil-Petersson quasicircles

$$\cup$$

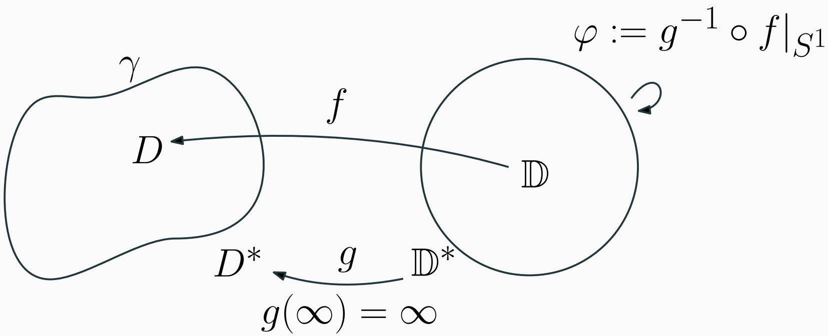
Completion

$$\text{Möb}(S^1) \backslash \text{Diff}(S^1)$$ C^∞ smooth curves

has a unique homogeneous Kähler metric

Weil-Petersson metric

Universal Liouville action



Theorem [Takhtajan & Teo '06 Memoir AMS]

The **universal Liouville action** $S_1 : T_0(1) \to \mathbb{R}$,

$$S_1([\varphi]) = \int_{\mathbb{D}} |\nabla \log |f'||^2 + \int_{\mathbb{D}^*} |\nabla \log |g'||^2 + 4\pi \log \frac{f'(0)}{g'(\infty)}$$

is a **Kähler potential** for the Weil-Petersson metric.

$S_1([\varphi]) < \infty \iff \gamma$ is a Weil-Petersson quasicircle.

Loewner energy $\longleftrightarrow$ Weil-Petersson quasi-circle

Thm (W. [3])

$I^L(\gamma) < \infty$ iff γ is a Weil-Petersson quasicircle.

Moreover, $I^L(\gamma) = \frac{1}{\pi} S_1(\gamma)$.

Weil-Petersson quasi-circle

__Motivated by string theory__ (smooth part $\text{Möb}(S^1)\backslash\text{Diff}(S^1)$)

Bowick - Rajeev, '87 Witten '88. Kirillov - Yurev '88

Hong, Nag, Pekonen. Sullivan '90 s

__Studied from geometric/analytic perspectives__

Cui '01, Takhtajan - Teo '06, Shen. Tang. Wu '13 - '20

Bishop '20

__Application to conformal field theory__

Radnell - Schippers - Staubach '17

Application to computer vision, KdV equations

Sharon - Mumford '06, Kushnarev '09

Schonbek - Todorov - Zubelli '99

Relation to holographic principles $\hat{\mathbb{C}} \longleftrightarrow \mathbb{H}^3$

Bishop '20 (extending [Alexakis - Mazzeo] [Anderson])

Relation to random conformal geometry

W. 16-19 Viklund - W. '19 '20

WEIL-PETERSSON CURVES, CONFORMAL ENERGIES, β-NUMBERS, AND MINIMAL SURFACES

CHRISTOPHER J. BISHOP

preprint 2020

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding midpoints
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
12	Menger curvature
13	biLipschitz involutions
14	between disjoint disks
15	thickness of convex hull
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder
21	closure of smooth curves in $T_0(1)$
22	P_φ^- is Hilbert-Schmidt
23	double hits by random lines
24	finite Loewner energy
25	large deviations of SLE(0^+)
26	Brownian loop measure

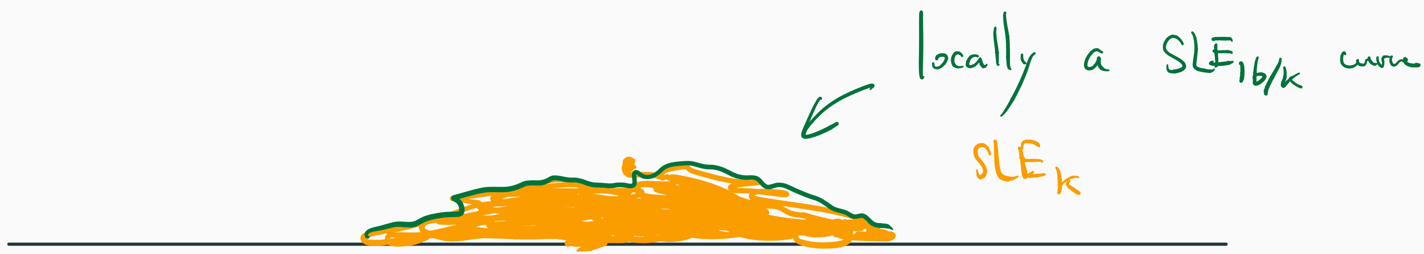
4' $\log \varphi' \in H^{1/2}(S^1)$

27 a leaf in a finite energy foliation

SLE duality

For $\kappa \geq 8$

[Dubédat] [Zhan] [Miller-Sheffield]



Asymptotic behavior of SLE_{0+} $\xleftrightarrow{\text{dual}}$ SLE_∞

Loewner energy Loewner-Kufarev energy

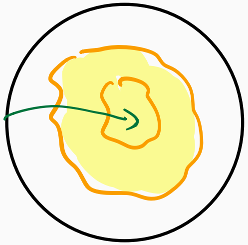
Loewner - Kufarev equation

(Measure - driven Loewner chain)

$$\mathcal{N}_+ := \left\{ (\rho_t)_{t \geq 0} \;\middle|\; \begin{array}{l} \rho_t \in \mathrm{Prob}(S^1) \\ \text{measurable in } t \end{array} \right\}$$

Loewner - Kufarev equation $\quad z \in \mathbb{D}$

$$\begin{cases} \partial_t f_t(z) = -z f_t'(z) \displaystyle\int_{S^1} \frac{\zeta + z}{\zeta - z} \, d\rho_t(\zeta) \\ f_0(z) = z \end{cases}$$



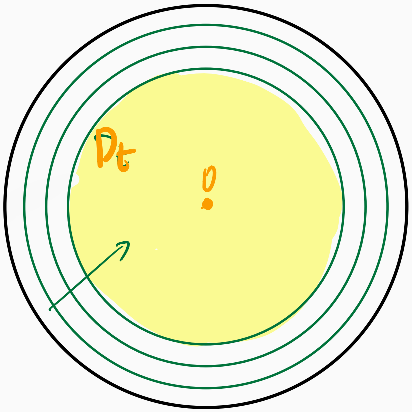
$f_t : \mathbb{D} \to D_t \subset \mathbb{D}$ with $f_t(0) = 0$, $f_t'(0) = e^{-t}$.

$(\rho_t)_{t \geq 0} \longleftrightarrow$ Evolution family $(D_t)_{t \geq 0}$

Example

- $\rho_t(d\theta) = \frac{1}{2\pi}\, d\theta \quad \forall\, t \geq 0$

$\Rightarrow \; D_t = e^{-t}\, \mathbb{D}$



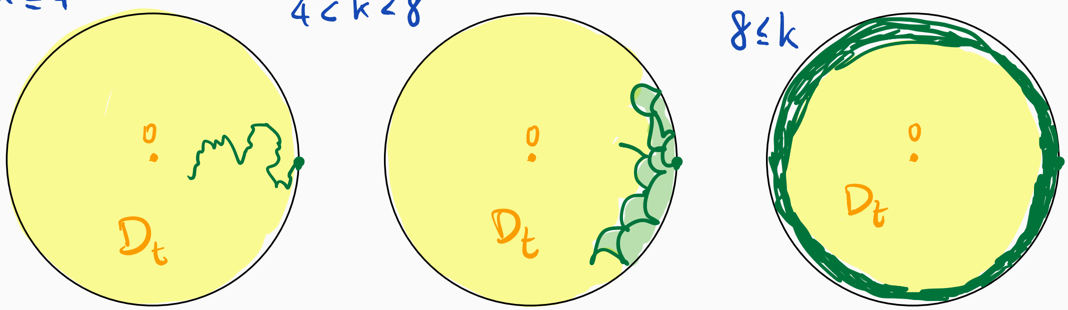
- $\rho_t(d\theta) = \delta_{e^{i\sqrt{k}\underbrace{B_t}}}$

Dirac mass

Brownian motion on $\mathbb{R}$

Radial SLE_k

$k \leq 4$ $\quad 4 < k < 8$ $\quad 8 \leq k$



LDP of SLE_∞

Thm (Ang - Park - W. [5] '20)

Radial SLE_κ process satisfies the LDP as $\kappa \to \infty$ with rate function S_+ (Loewner - Kufarev energy)

" $\mathbb{P}(SLE_\kappa \simeq (D_t)_{t\geq 0}) \simeq \exp(-\kappa S_+(\rho))$ as $\kappa \to \infty$ "

where

$$S_+(\rho) := \int_0^\infty L(\rho_t)\, dt$$

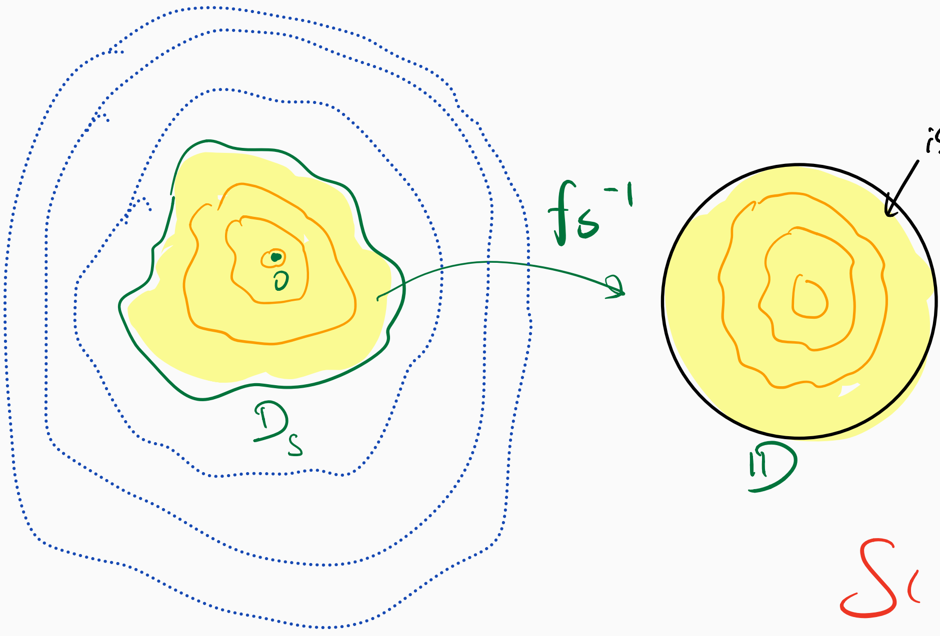
$$L(\rho_t) := \frac{1}{2}\int_{S^1} |\nu_t'(\theta)|^2\, d\theta$$

if $\rho_t = \nu_t^2(\theta)\, d\theta$ and $L(\rho_t) = \infty$ otherwise.

Whole-plane Loewner-Kufarev Chain

(More symmetric)

$\rho = (\rho_t)_{t\in\mathbb{R}} \to (D_t)_{t\in\mathbb{R}}$ and $f_t : \mathbb{D} \to D_t$ with $f_t(0)=0$, $f_t'(0)=e^{-t}$



is the radial Loewner-Kufarev chain generated by $(\rho_{t+s})_{t\geq 0}$

$$S(\rho) := \int_{-\infty}^{\infty} L(\rho_t)\, dt$$

Foliation by Weil-Petersson quasicircles.

Thm (Viklund – W. [7])

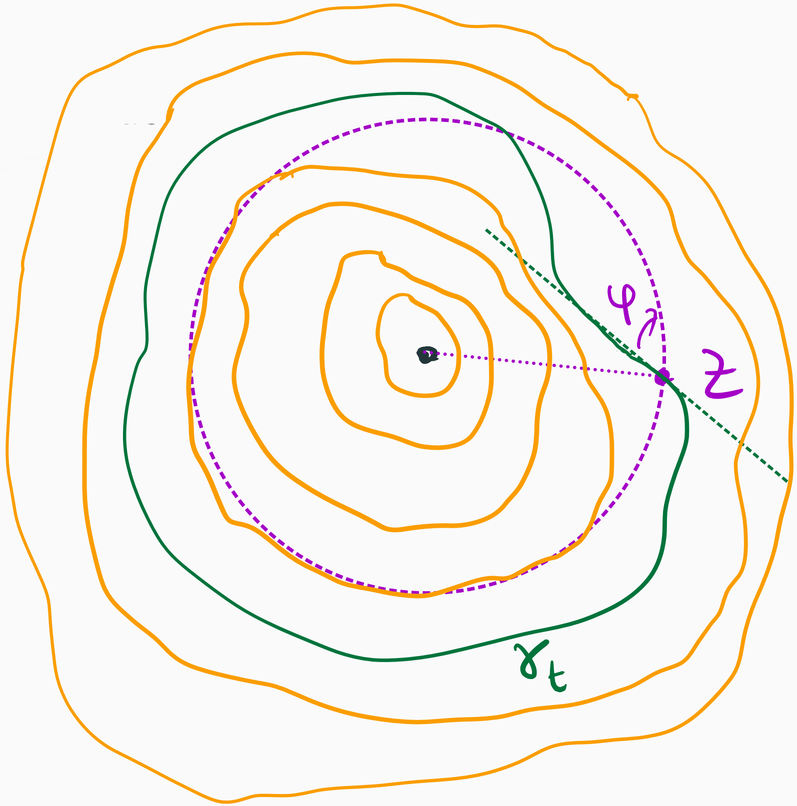
If $S(\rho) < \infty$, then

- ∂D_t is a Weil-Petersson quasicircle $\forall t \in \mathbb{R}$
- $\bigcup \partial D_t = \mathbb{C} \setminus \{0\}$.
- $t \mapsto \partial D_t$ is continuous in the sup-norm.

"Foliation" of $\mathbb{C} \setminus \{0\}$ by Weil-Petersson quasi-circles

More quantitatively?

Winding function φ



Let $g_t := f_t^{-1} : D_t \to \mathbb{D}$

$$\varphi(z) := \arg \frac{g_t'(z)\, z}{g_t(z)}$$

if $z \in \partial D_t$

Energy duality

Thm (Viklund – W.)

$$16\,S(\rho) = \mathcal{D}_{\mathbb{C}}(\varphi) \quad \left(= \frac{1}{\pi}\int_{\mathbb{C}} |\nabla\varphi|^2 \, dA(z) \right)$$

Dynamic   static

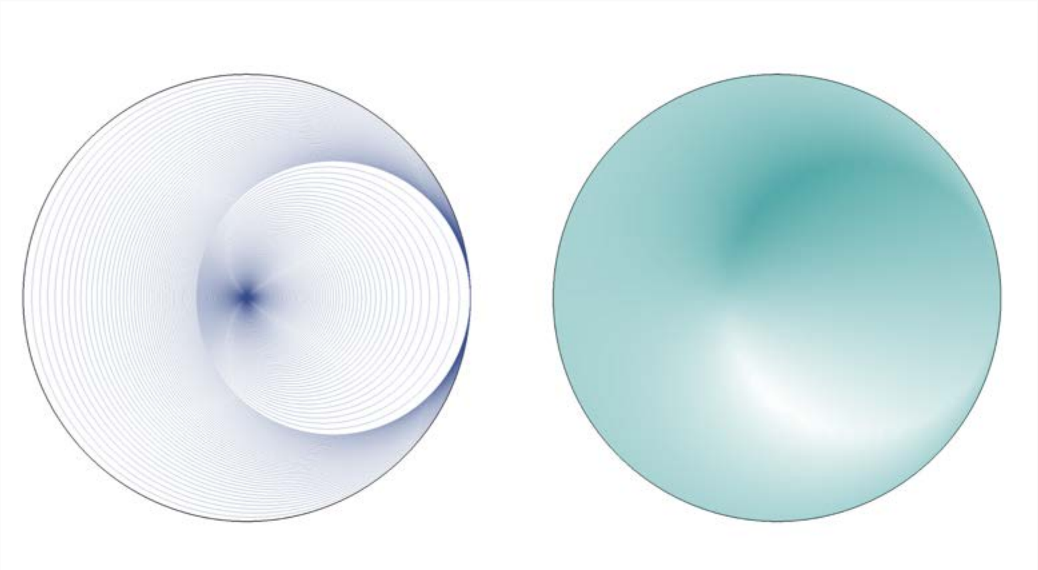
One is finite iff the other is finite

- "16" is consistent with SLE duality

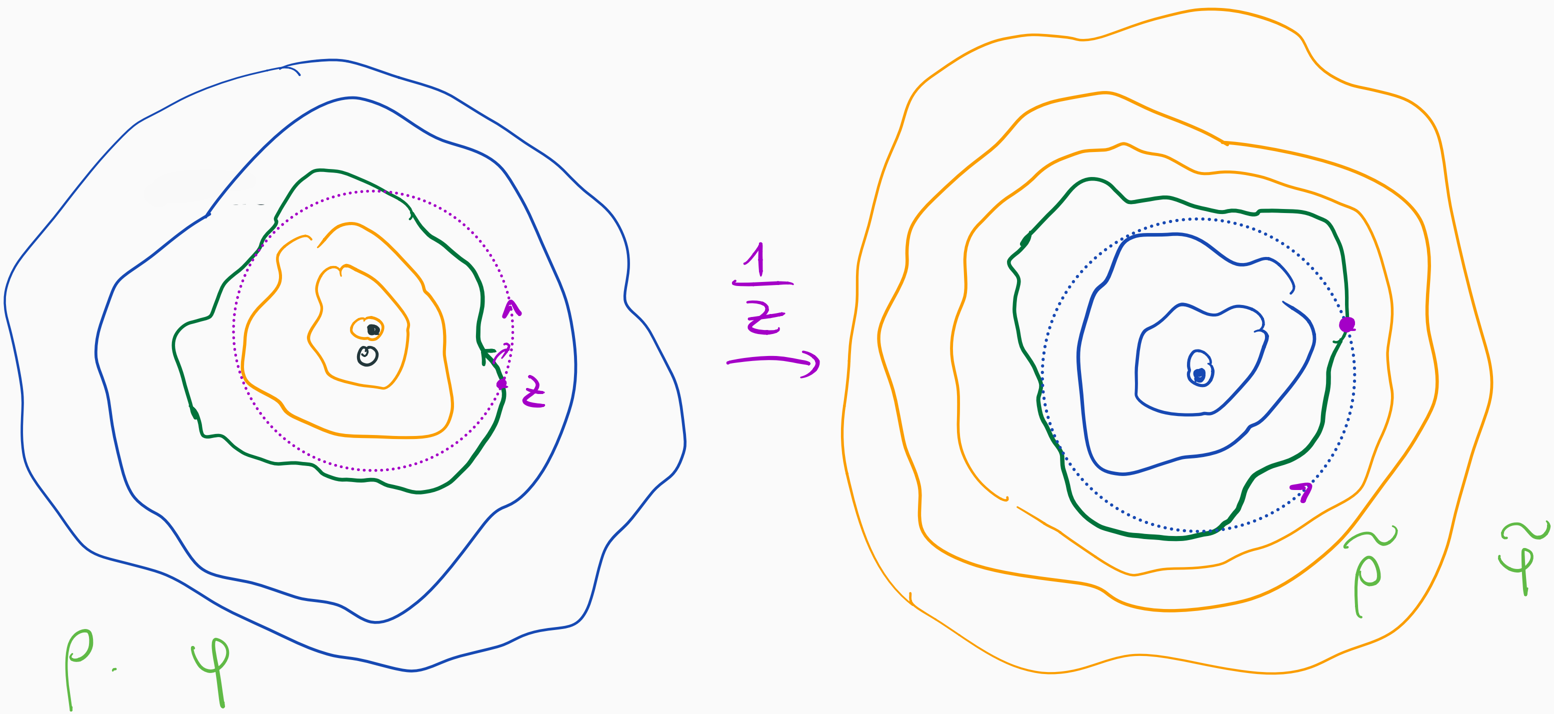
$$\kappa \longleftrightarrow \frac{16}{\kappa}$$

Example

$$\rho_t(d\theta) = \begin{cases} \frac{1}{\pi}\sin^2\left(\frac{\theta}{2}\right)d\theta & \text{for } t\in[0,1]; \\ \frac{1}{2\pi}d\theta & \text{otherwise.} \end{cases}$$



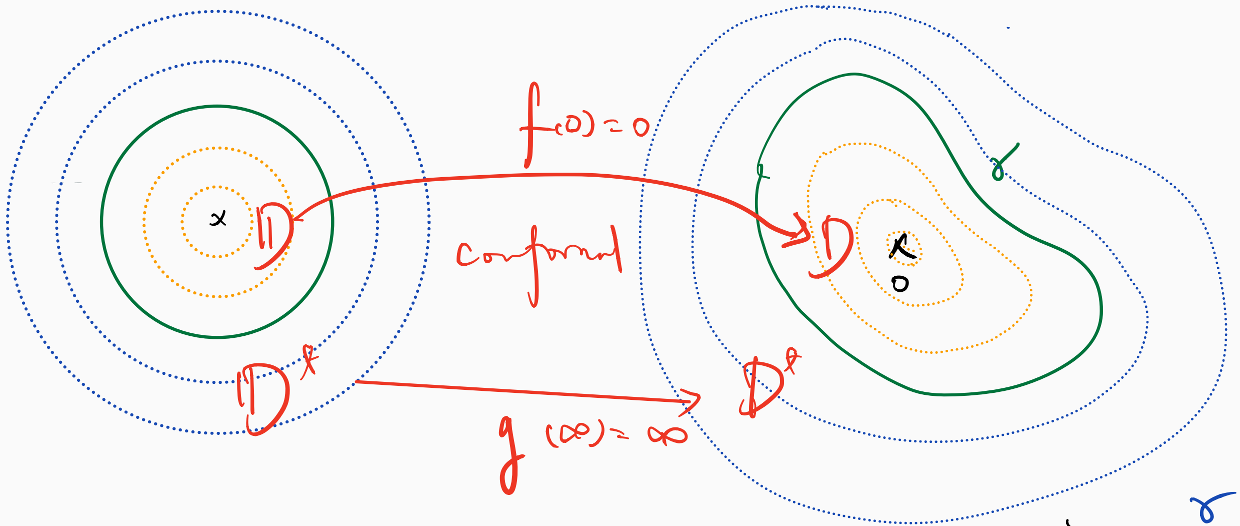
Energy Reversibility



<u>Cor (V.W.)</u> $S(\rho) = S(\tilde{\rho})$.

Proof: $\tilde{\varphi}(z) = \varphi(\frac{1}{z}) \Rightarrow \mathcal{D}(\tilde{\varphi}) = \mathcal{D}(\varphi)$. ▨

$S \leftrightarrow I^L$



$\hookrightarrow \rho^\gamma \in \mathcal{N}$

Cor (V.W.)

$$16\, S(\rho^\gamma) = I^L(\gamma) + 2\log\left|\frac{g'(\infty)}{f'(0)}\right| \geq I^L(\gamma)$$

φ is harmonic in $\mathbb{C} \setminus \gamma$.

Cor. (V.W.) Definition 27

A Jordan curve γ separating 0 and ∞ is Weil-Petersson

$\iff$ γ can be realized as a leaf in the foliation generated by a measure with $S(\rho) < \infty$.

And $I^L(\gamma) \leq 16\, S(\rho^\gamma) \leq 16\, S(\rho)$.

Key step in the proof of $16\, S(\rho) = \mathcal{D}(\varphi)$

- Loewner chain $\rightsquigarrow$ evolution family of conformal mappings
  $\rightsquigarrow$ powerful tool in studying conformally invariant quantities / systems

- Dirichlet energy is conformally invariant

Assume ρ generates a foliation. $\mathcal{D}(\phi) < \infty$

$$\phi = \underbrace{\phi^{h,t}}_{\text{harmonic in } D_t} + \underbrace{\phi^{0,t}}_{\in W_0^{1,2}(D_t)}$$

Disintegration isometry

$\rho := \rho_t(d\theta)\,dt$ measure on $S^1 \times \mathbb{R}_+$

Thm (V. W.)

$\left(W_0^{1,2}(\mathbb{D}), \mathcal{D}^{1/2}\right) \longrightarrow L^2(S^1 \times \mathbb{R}_+, 2\rho)$

$\partial_n \phi^{0,t} \circ f_t$

$$\iota : \quad \phi \mapsto \frac{1}{2\pi} \int_{\mathbb{D}} \Delta(\phi \circ f_t)(z) P_{\mathbb{D}}(z, e^{i\theta})\, dA(z).$$

is a bijective isometry with inverse operator

$$\varkappa[u](w) = 2\pi \int_0^{\tau(w)} P_{\mathbb{D}}[u_t \rho_t](g_t(w))\, dt, \qquad u_t(\cdot) := u(\cdot, t).$$

Gaussian free field ⇝ White noise decomp.

generalizing [Hedenmalm - Nieminen]

Proof cont'd.

φ = winding function

$\rho = \nu_t^2(\theta)\, d\theta\, dt$

Show $\iota[\varphi](\theta, t) = \dfrac{-2\nu_t'(\theta)}{\nu_t(\theta)}$

$$\Rightarrow \quad \|\iota[\varphi]\|^2_{L^2(\rho)} = \int_0^\infty \int_{S^1} \left[\frac{2\nu_t'(\theta)}{\nu_t(\theta)}\right]^2 2\nu_t^2(\theta)\, d\theta\, dt$$

$\|$

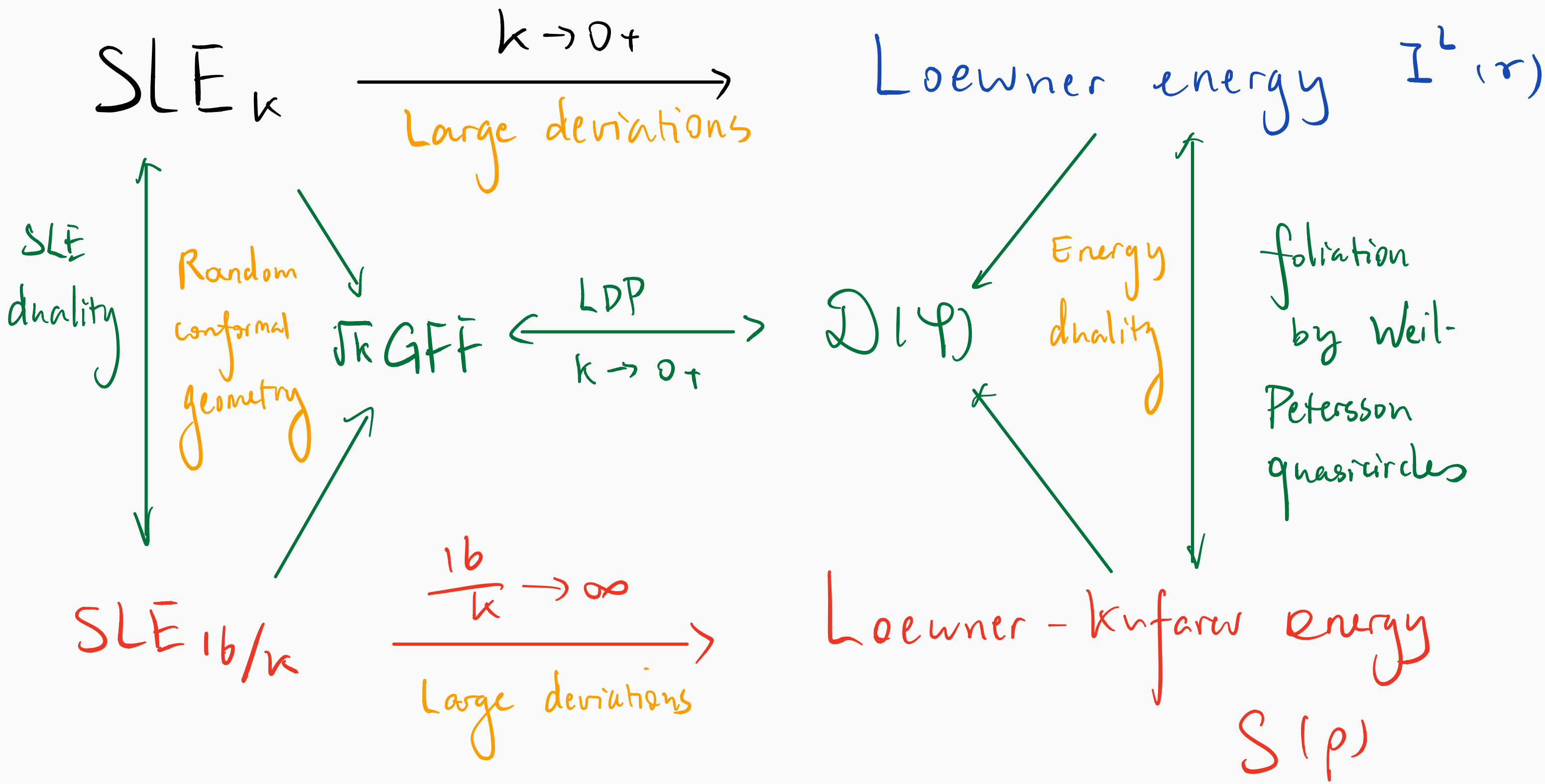
$\mathcal{D}(\varphi)$

$$= 8 \int_0^\infty \int_{S^1} \left[\nu_t'(\theta)\right]^2 d\theta\, dt$$

$$= 16 \int_0^\infty L(\rho_t)\, dt$$

$\blacksquare$

Conclusion



SLE_κ
$\kappa \to 0+$
Large deviations
Loewner energy $I^L(\gamma)$
SLE duality
Random conformal geometry
$\sqrt{\kappa}$ GFF
LDP
$\kappa \to 0+$
$\mathcal{D}(\varphi)$
Energy duality
foliation by Weil-Petersson quasicircles
$SLE_{16/\kappa}$
$\frac{16}{\kappa} \to \infty$
Large deviations
Loewner-Kufarev energy
$S(\rho)$

References

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yilwang.weebly.com