IHES
Institut des Hautes Études Scientifiques

Shear coordinates of Weil-Petersson circle homeo

Yilin Wang J.w Dragomir Šarić
Catherine Wolfram

December 7, 2023

CIRM Renormalization and visualisation in Geometry, Dynamics & Number theory universal Teichmüller space

$$T(1) = \mathrm{M\ddot{o}b}(S^1) \backslash \{\text{quasisymmetric homeo} : S^1 \to S^1\}$$

$$= \{\varphi : S^1 \to S^1 \mid \text{q.s. \& fixes } \pm 1, i\}$$

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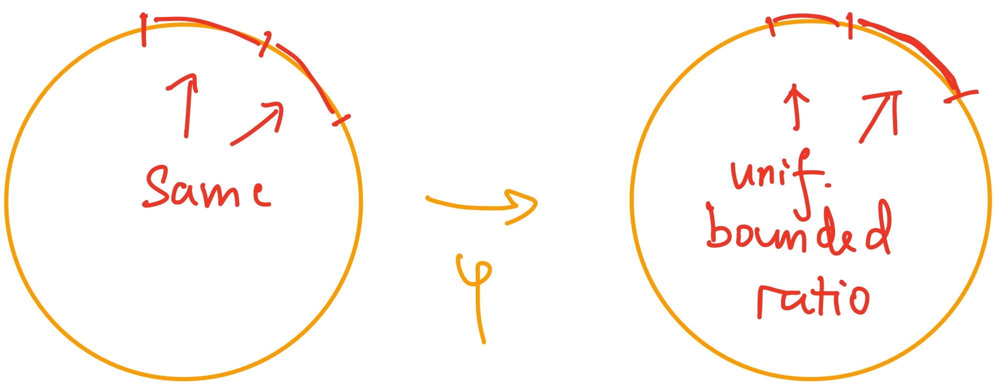

φ has q.c. extension Φ

$$\iff \mu = \frac{\bar\partial \Phi}{\partial \Phi} : \mathbb{D} \to \mathbb{C}$$

$$\|\mu\|_\infty < 1$$

Fenchel - Nielson coordinates?

Give a countable basis to parametrize universal Teichmüller space

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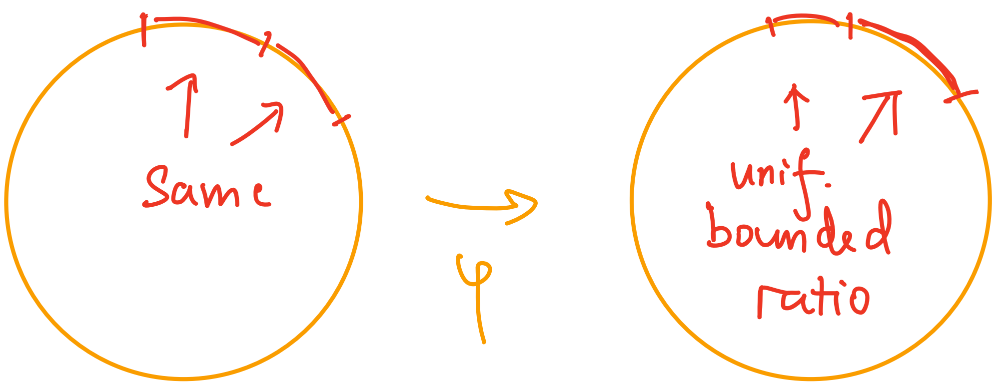

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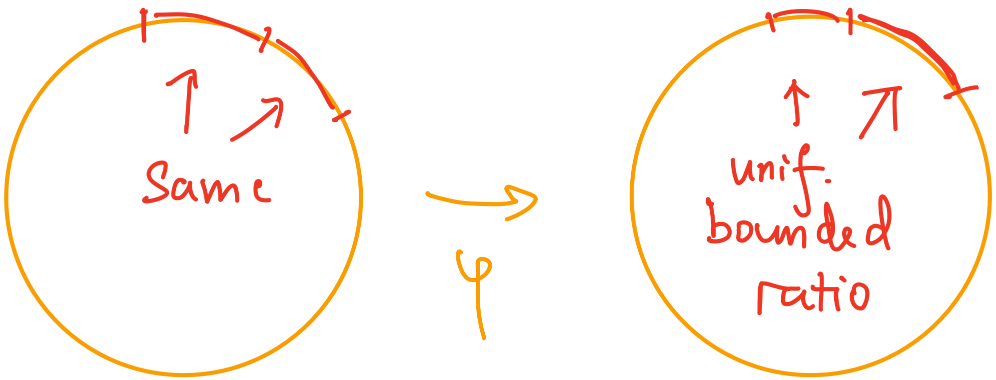

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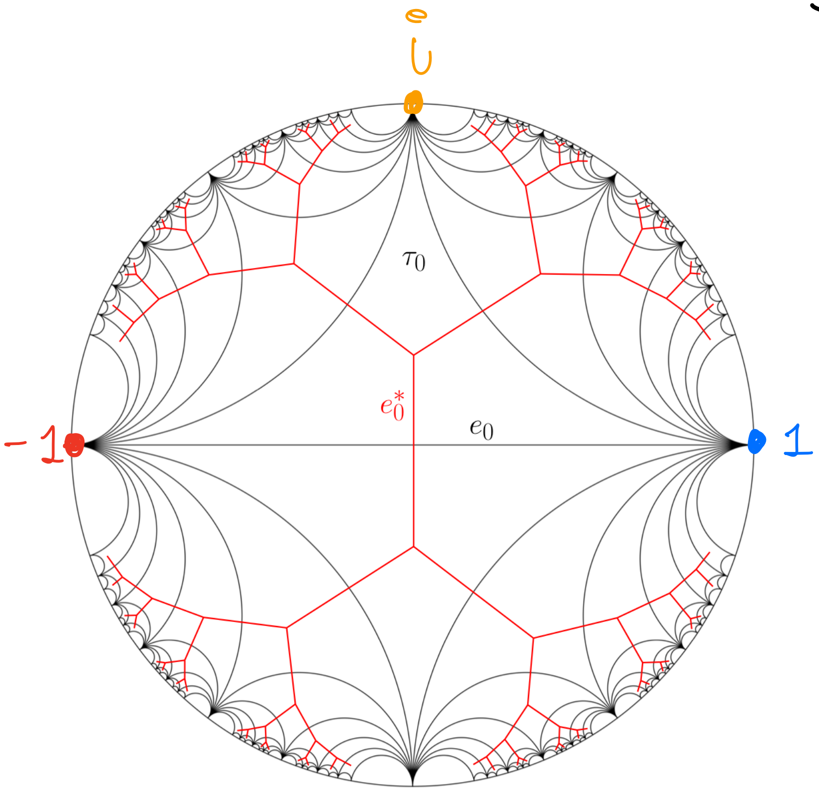
and study properties of QS maps in terms of the coordinates and subspaces of $T(1)$

Farey tessellation



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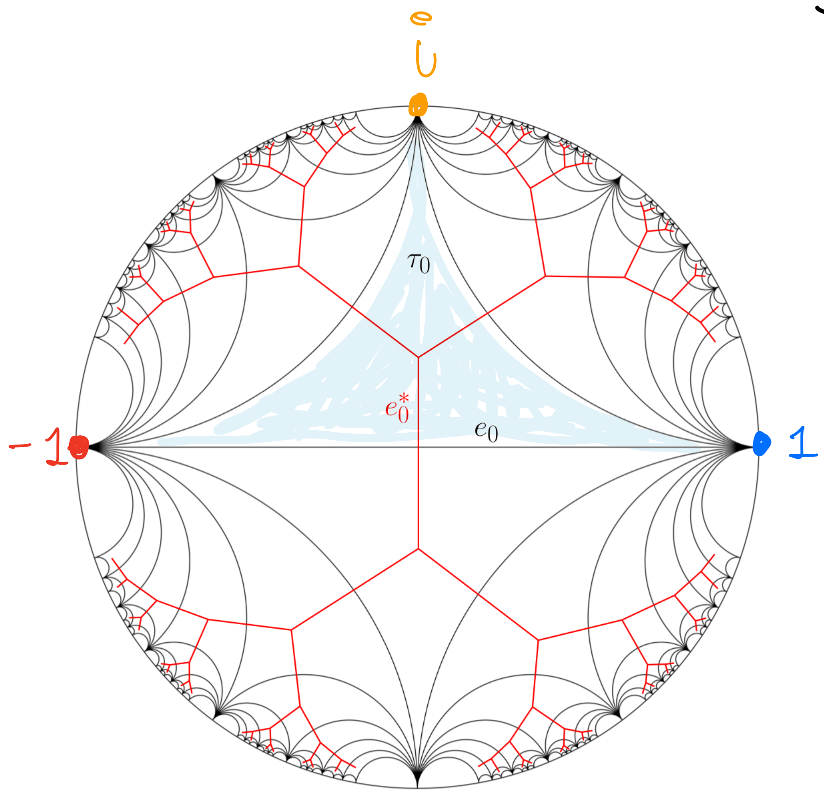

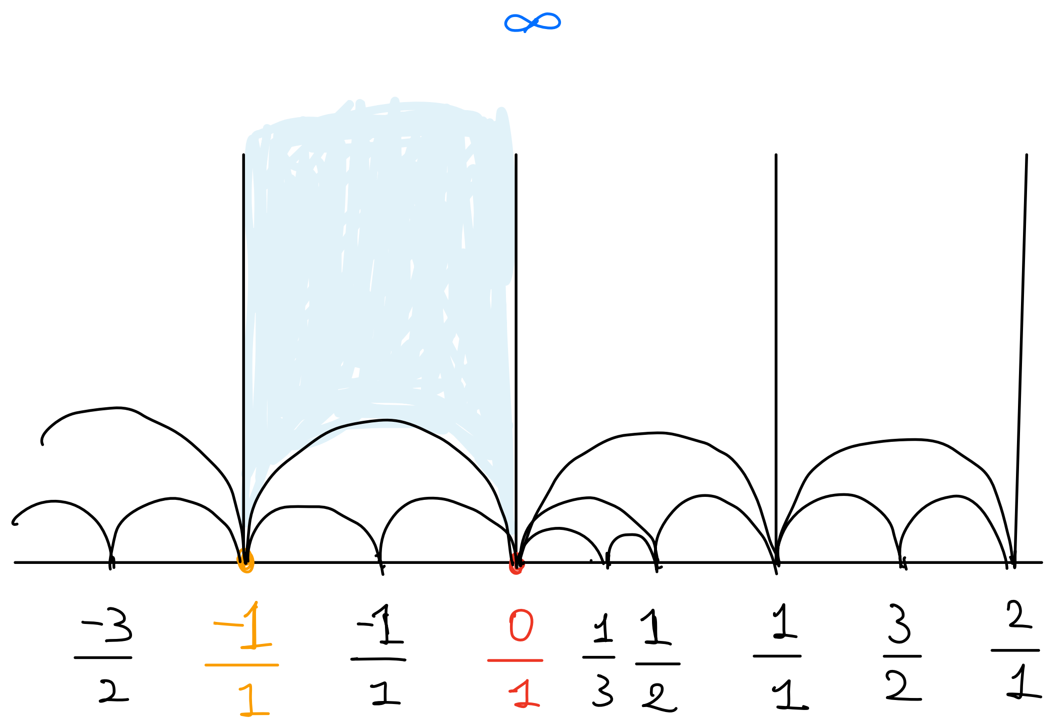

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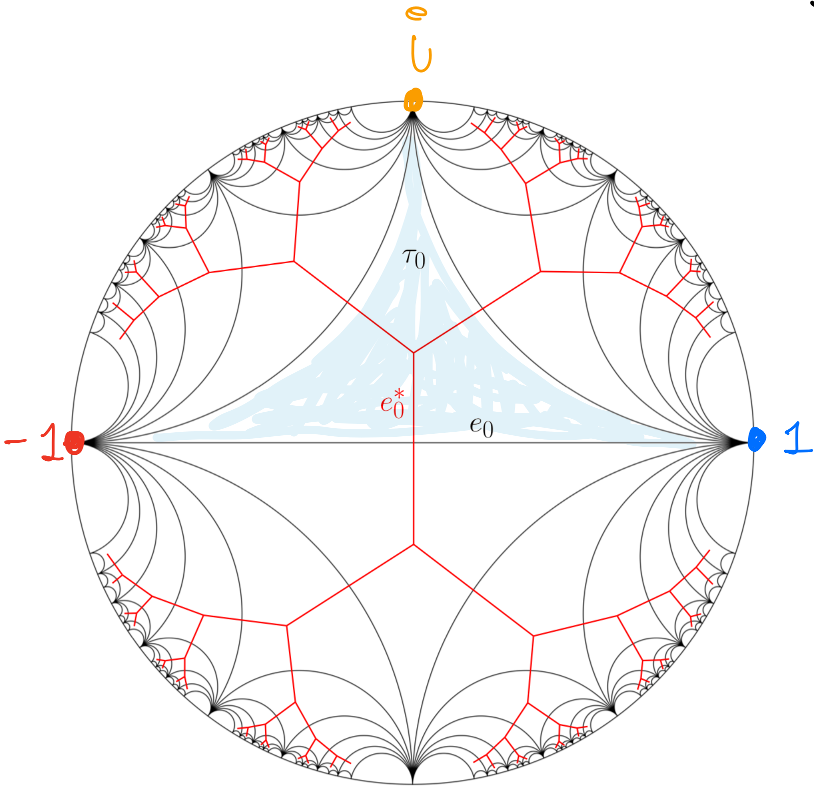
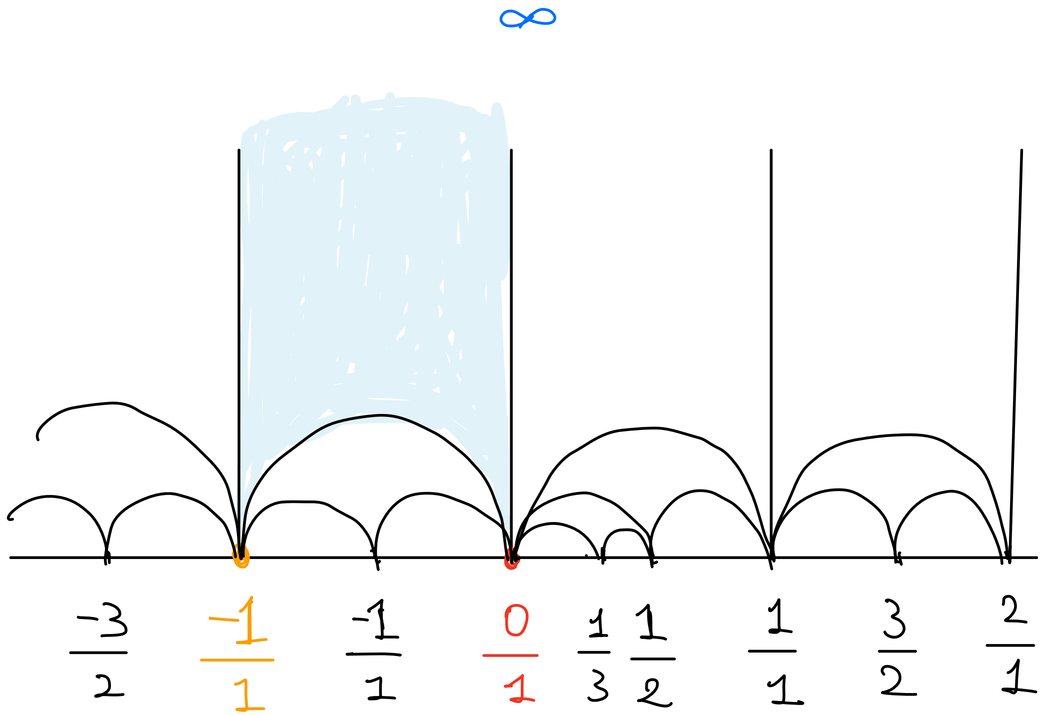


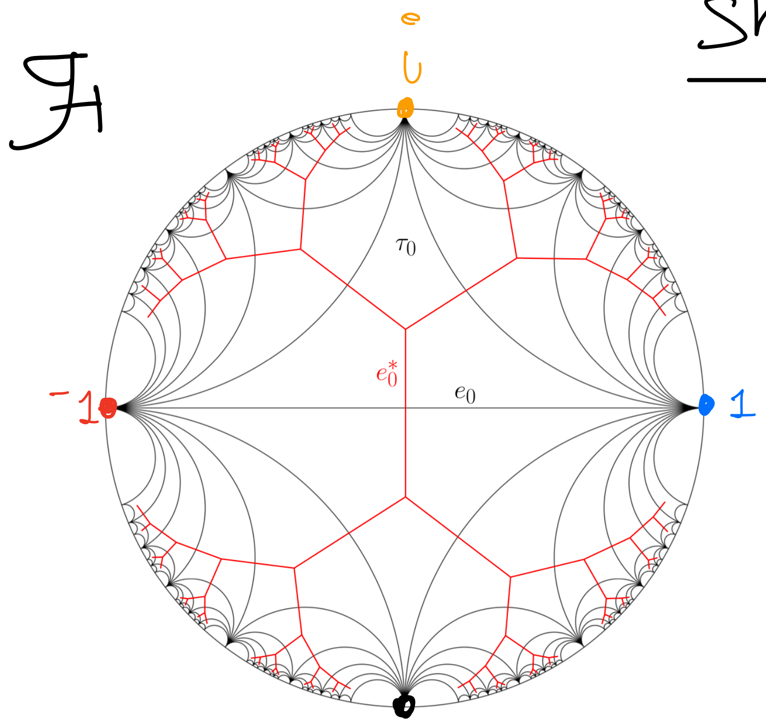


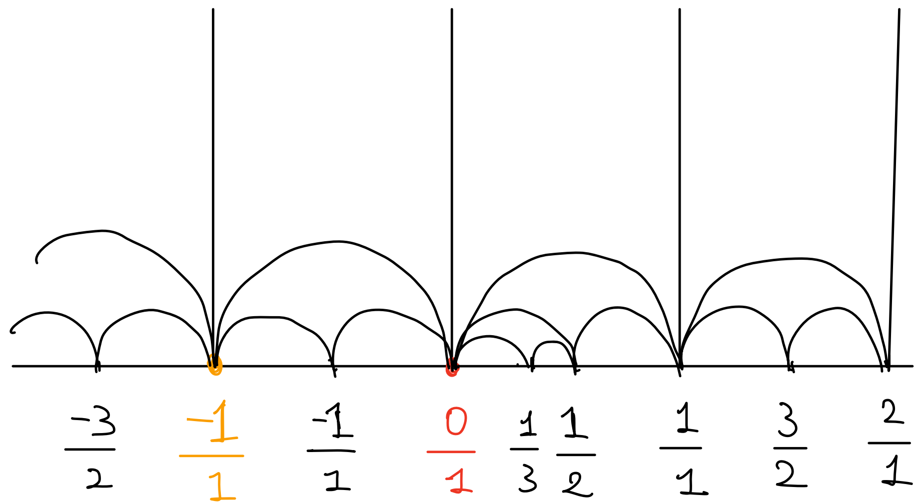
↺

$PSL(2,\mathbb{Z})$

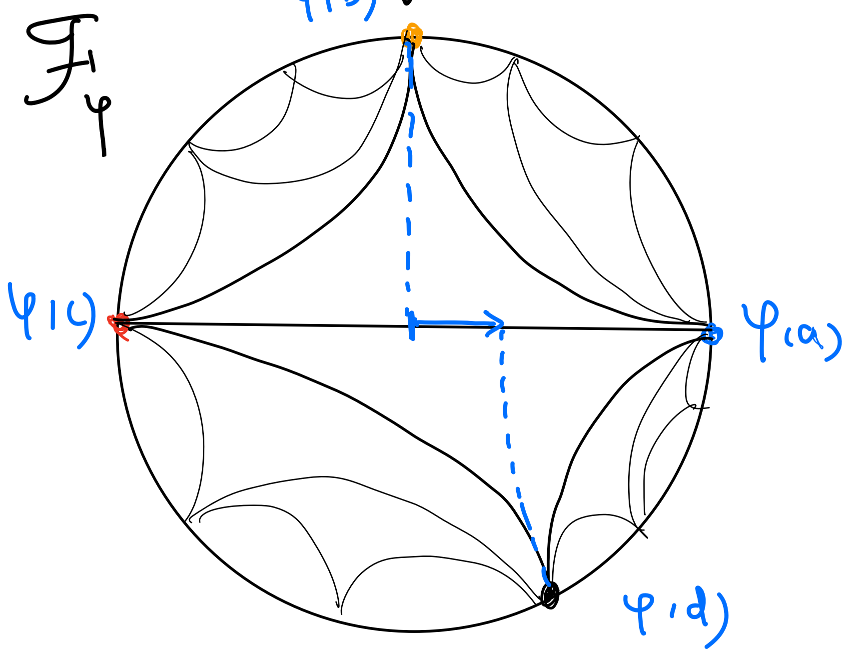
$$\mathcal{F} = (\underset{\mathbb{Q}\cup\{\infty\}}{\underset{\shortparallel}{V}}, \quad \underset{\left\{\left(\frac{p}{q}, \frac{r}{s}\right) \; : \; \left|\det\begin{pmatrix} p & r \\ q & s \end{pmatrix}\right| = 1\right\}}{\underset{\shortparallel}{E}})$$

Shear coordinates





$\downarrow \varphi$



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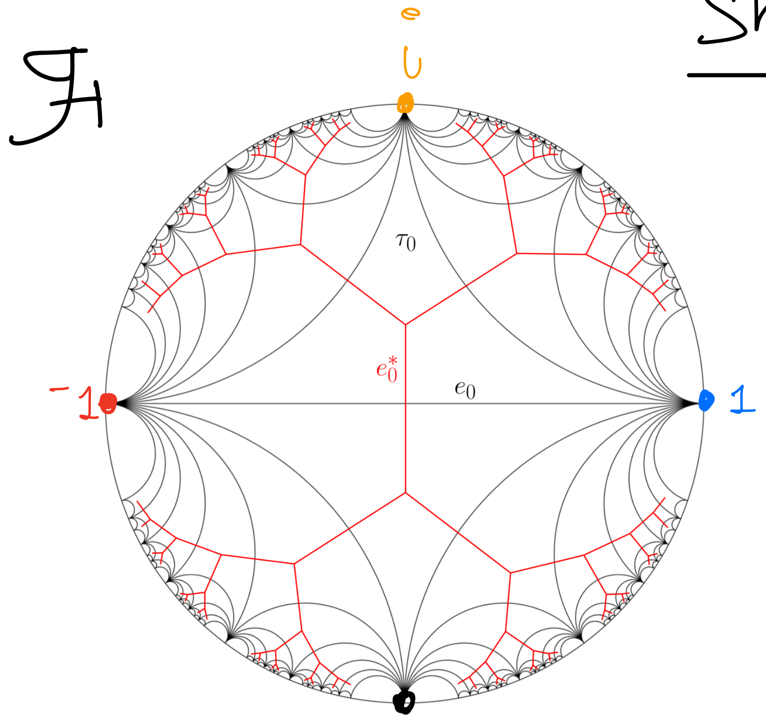
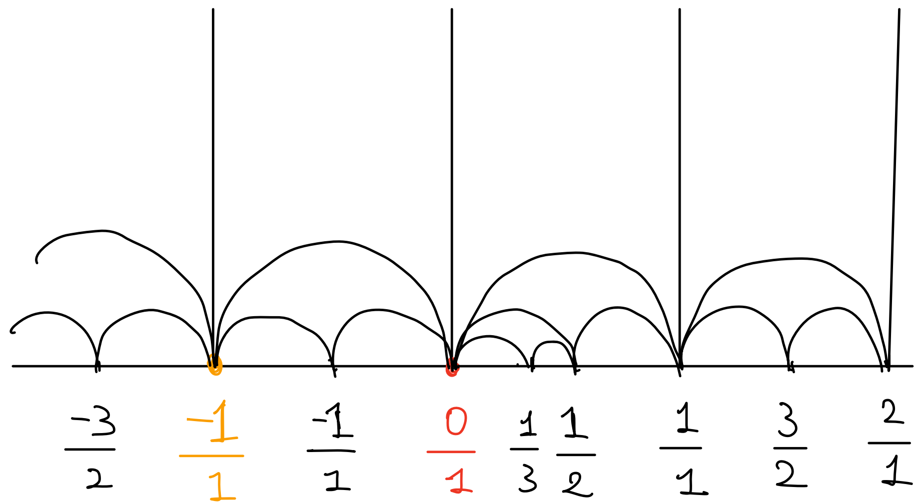
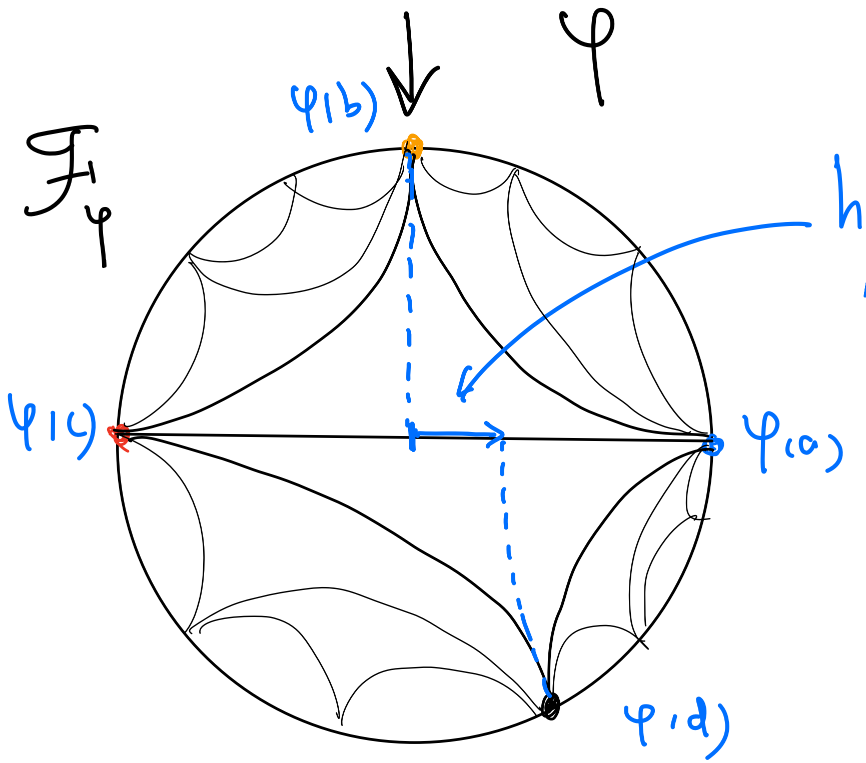




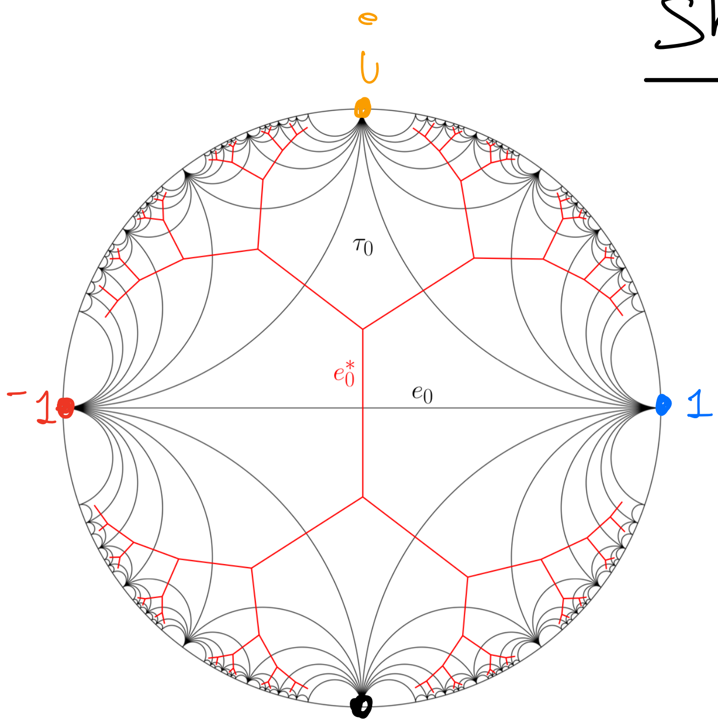


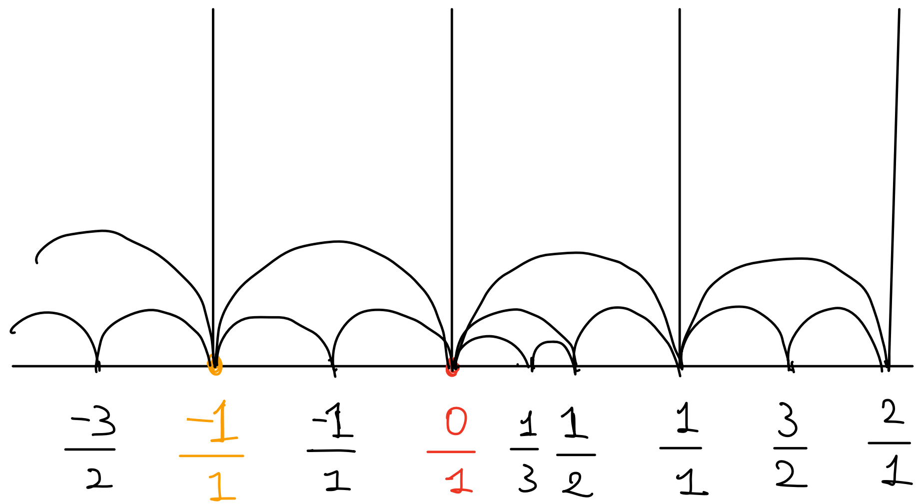
hyperbolic length $S_\varphi(e_0) \in \mathbb{R}$

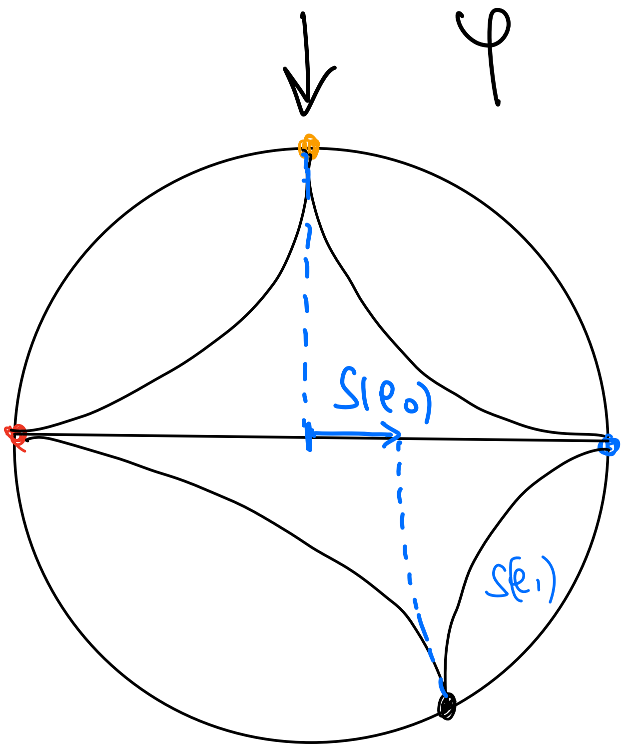
$S_\varphi(e) := \log CR(\varphi(a), \varphi(b), \varphi(c), \varphi(d))$

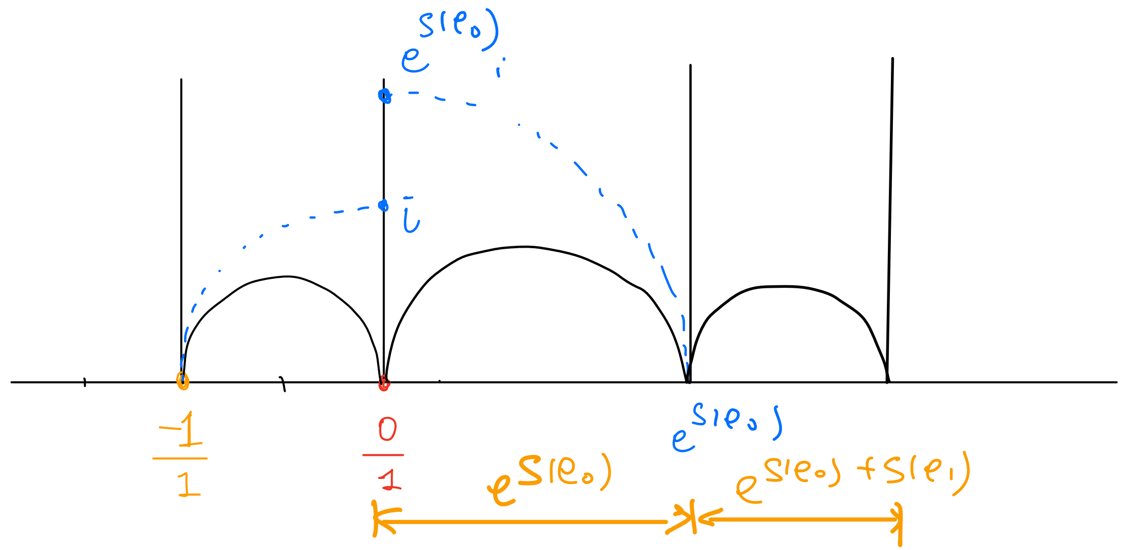
$S_\varphi : E \to \mathbb{R}$

Shear coordinates









Fact: from $S : E \to \mathbb{R}$. we can reconstruct a map $\varphi_S : V \to S'$

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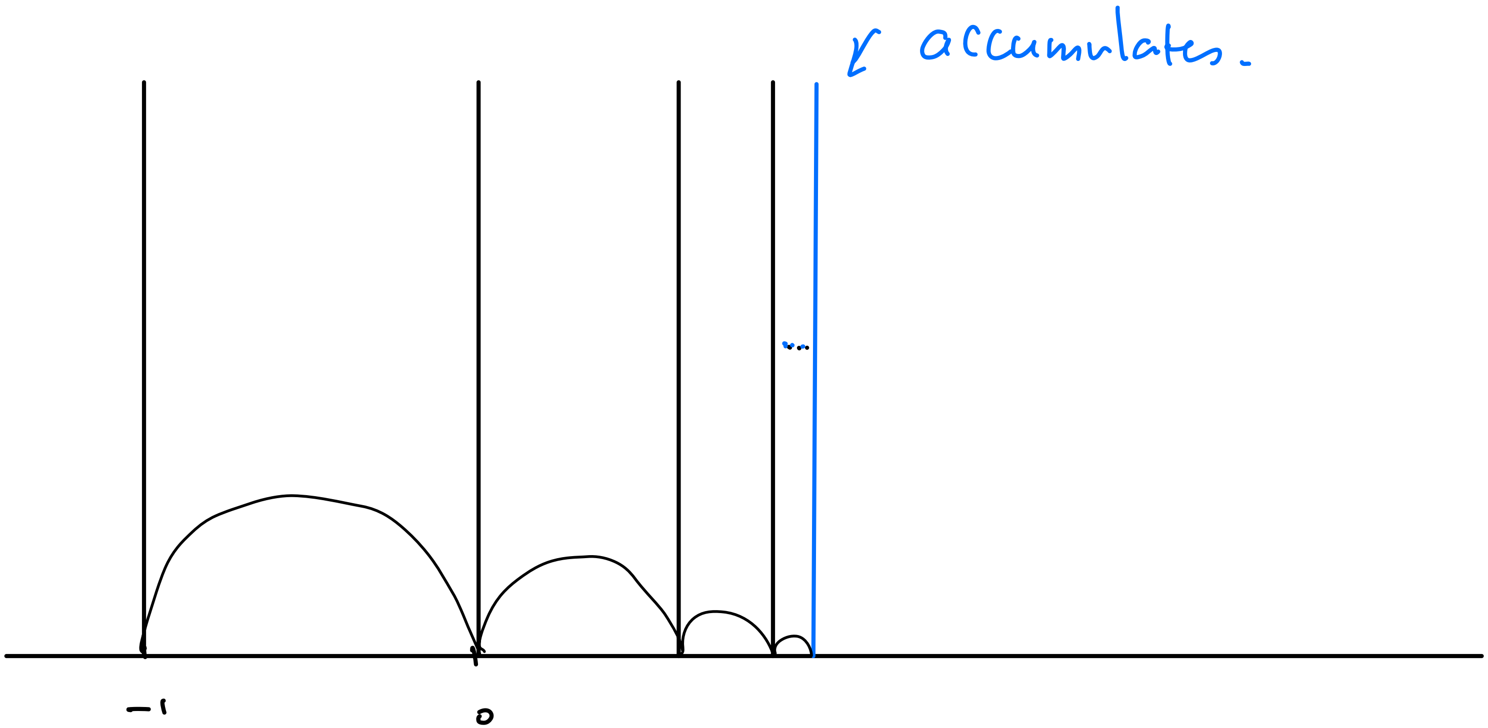
But $\varphi_S(V)$ may not be dense (does not extend to homeomorphism)



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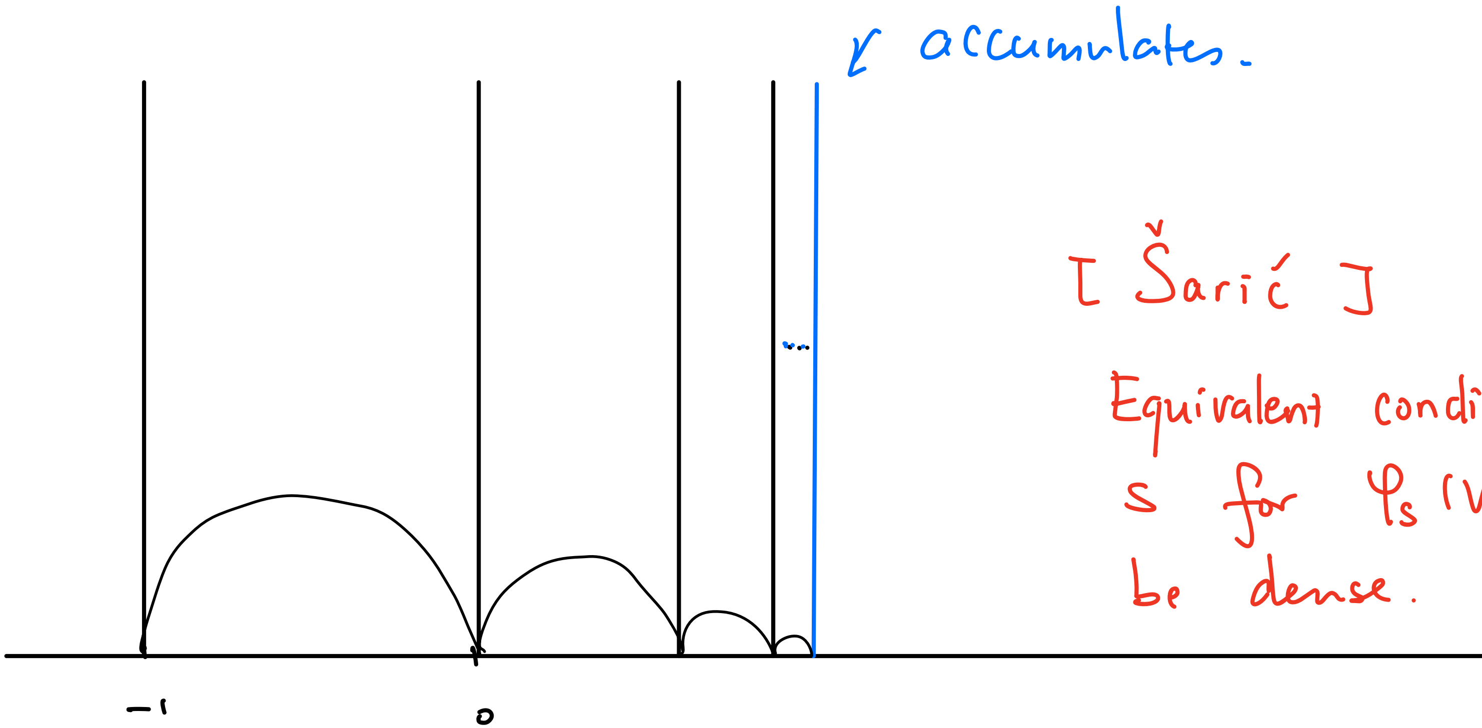


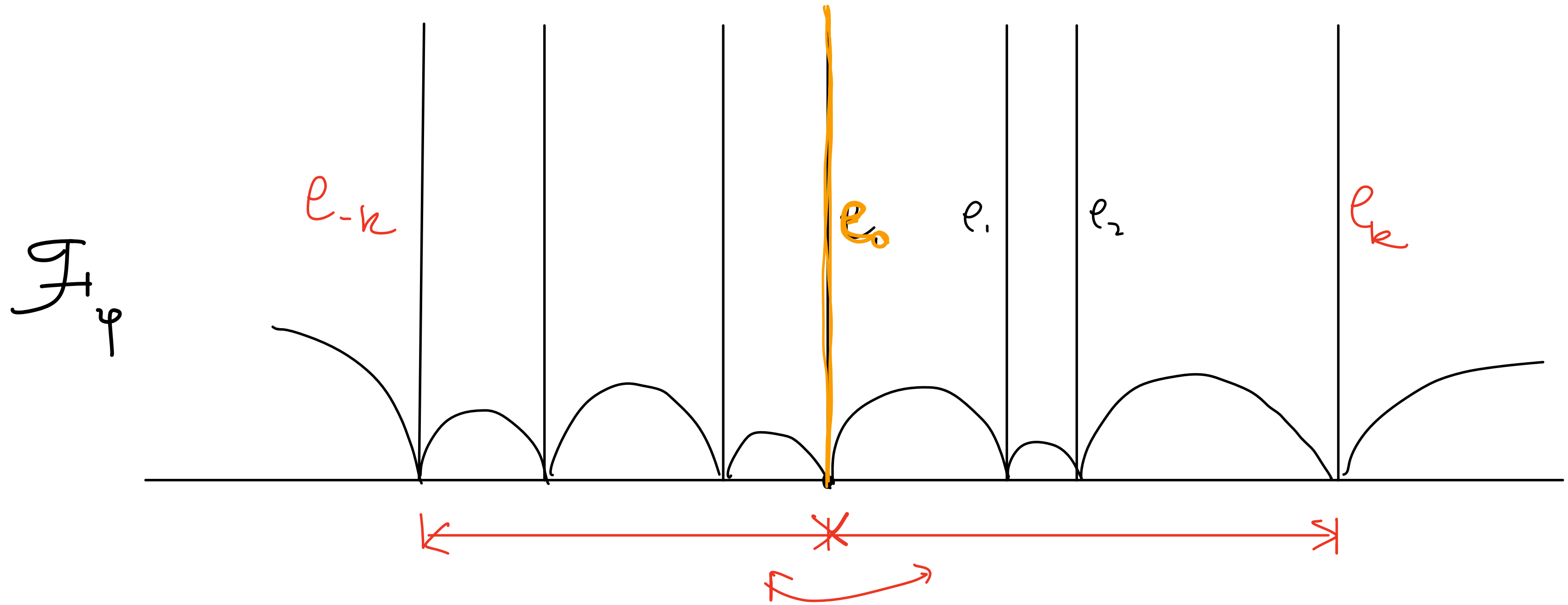
[Šarić]

Equivalent condition on s for $\varphi_s(V)$ to be dense.

Thm (Šarić)

$S : E \to \mathbb{R}$ is induced from a quasisymmetric homeomorphism iff $\exists\, M > 1$. s.t. $\forall e \in E$, $\forall k \geq 1$

e_0



$\frac{1}{M} \leq$ ratio of length $\leq M$

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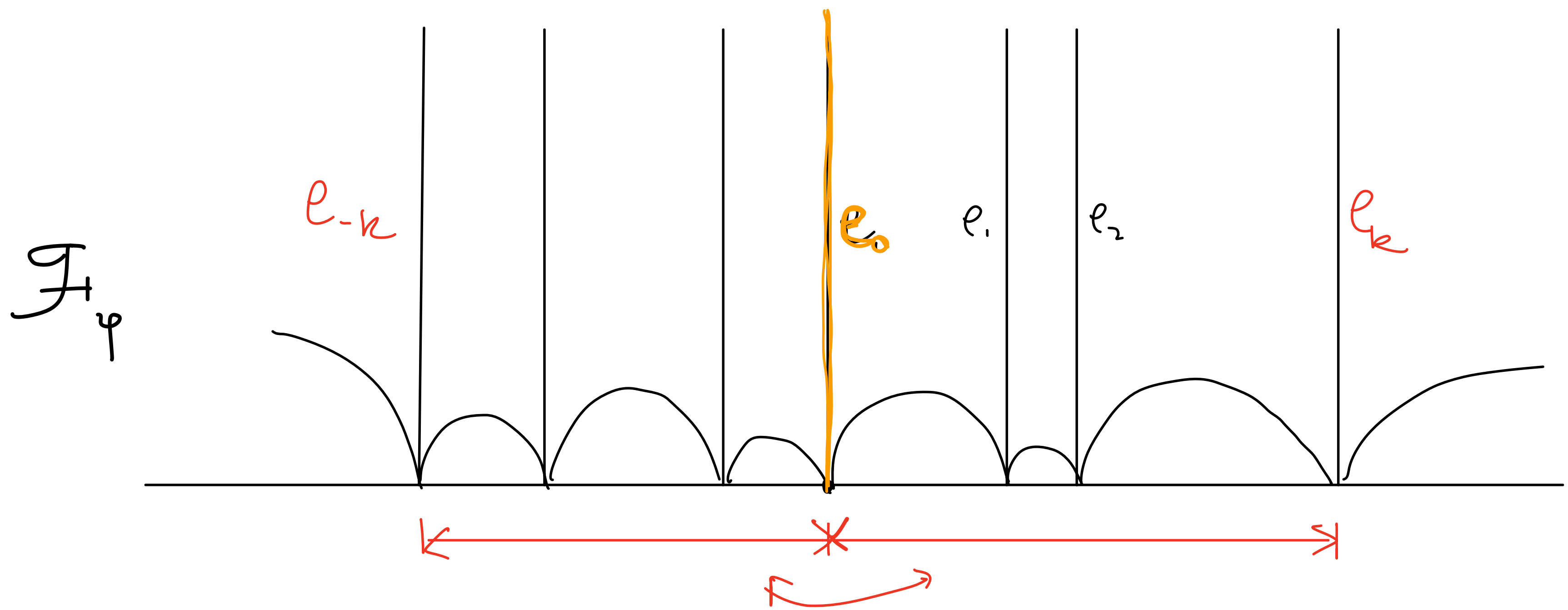

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can be expressed using shears

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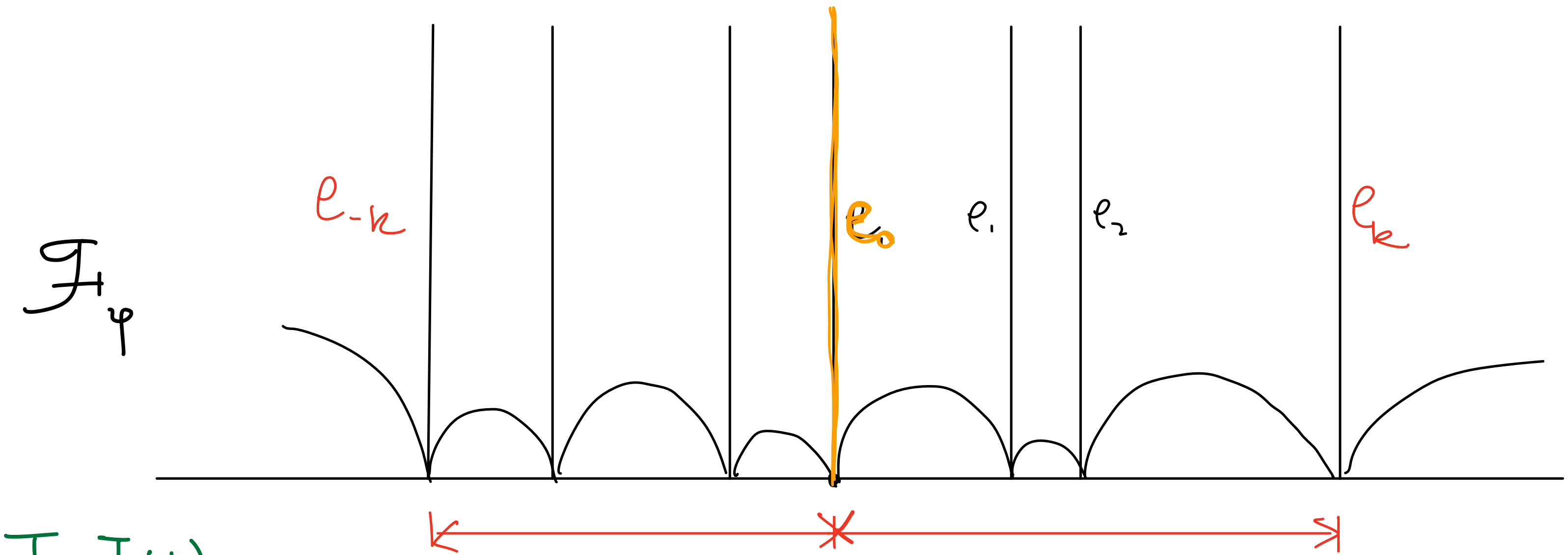

$T_{Id} T(1)$

$= \{\text{Zygmund vector fields}\}$ can be expressed using shears

(Not known in terms of Fourier series)

Weil-Petersson universal Teichmüller space

$WP(S^1) = \{ \varphi : S^1 \to S^1 \mid$ fixing $1, -1, i$

admitting q.c extension

with $\mu \in L^2(\mathbb{D}, \rho_{hyp}) \}$

$\subset T(1)$

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[Takhtajan-Teo]: (Bowick-Rajeev, Nag-Sullivan, Kirillov-Yuriev ...)

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- Kähler-Einstein, complete
- $T_0(1)$ is a topological group

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[Shen]
$$= \{ \varphi : S^1 \to S^1 \mid \log|\varphi'| \in H^{1/2}(S^1) \}$$
fixing ± 1, i

... 30 equivalent definitions ...

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$$\| u \|^2_{H^{1/2}(S^1)} = \int_{S^1} \int_{S^1} \frac{|u(s) - u(t)|^2}{|s-t|^2} \, ds \, dt$$

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$$C^{1,\alpha}(S^1) \subset WP(S^1) \iff \alpha > \frac{1}{2}$$

Q: Can we characterize $WP(S^1)$ using shears?

Q: What is a natural L^2 subspace in shear coordinates?

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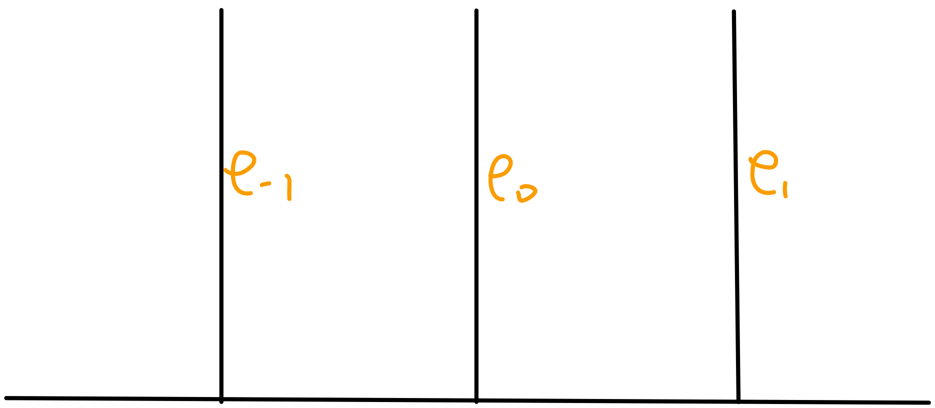
Q: What is a natural L^2 subspace in shear coordinates?

Guess 1: $\mathcal{S} := \{ s \in L^2(E, \mathbb{R}) \}$

Fact: $s \in L^2(E, \mathbb{R}) \not\Rightarrow$ homeomorphism

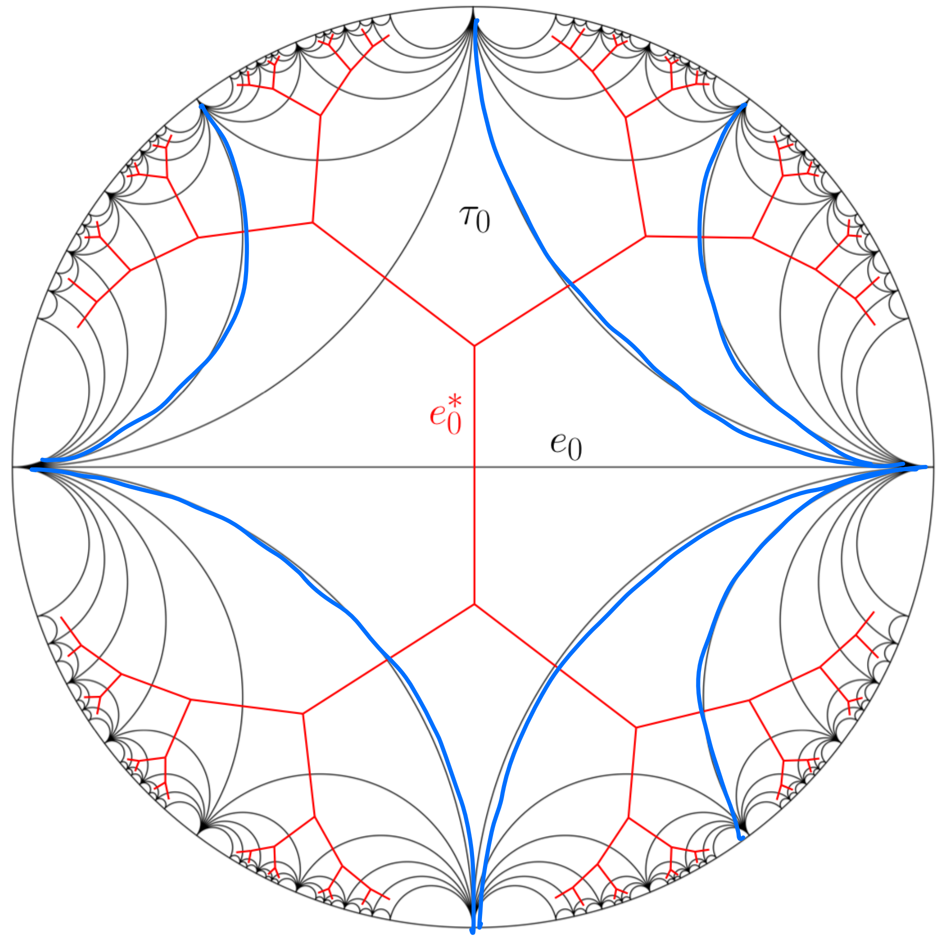


Example: For $n \geq 1$

$s(e_n) = -\dfrac{1}{n^\alpha}$, 0 otherwise

$\alpha \in (\frac{1}{2}, 1)$

Take one step back. Consider finite support shear.



Take one step back. Consider finite support shear.

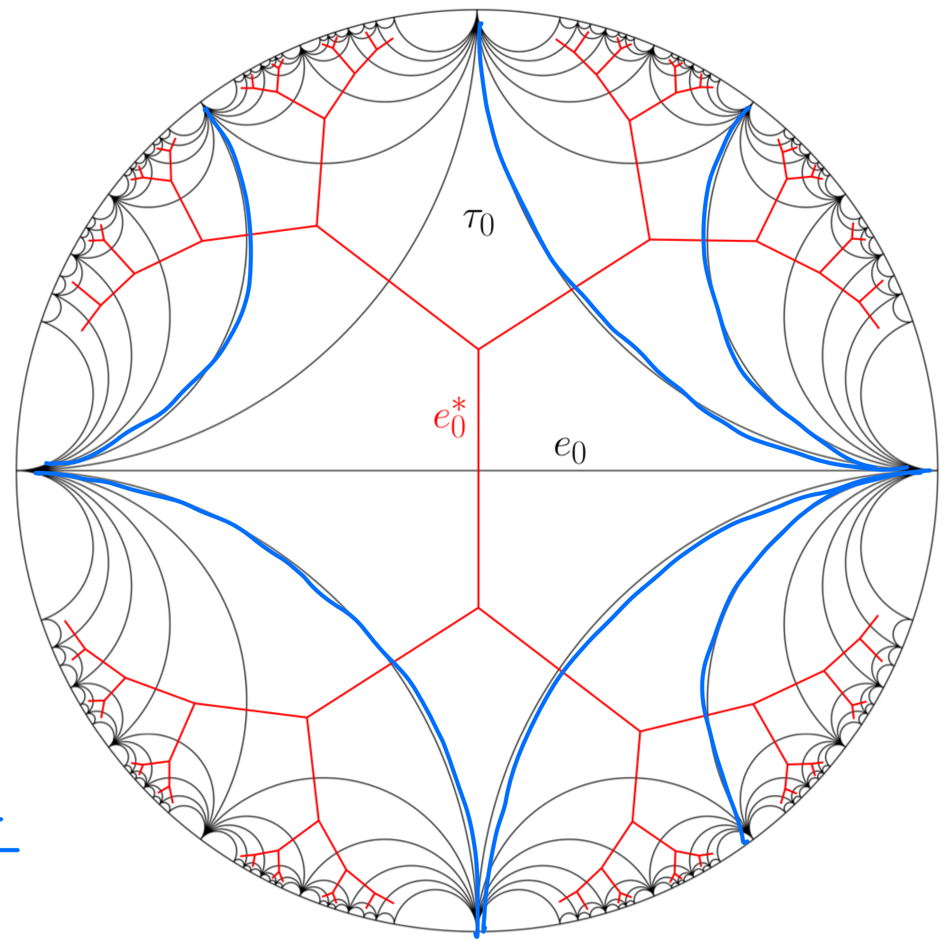
Lemma: If s has finite support, then s induces a piecewise Möbius homeomorphism φ. In this case,

$\varphi \in WP(S^1)$

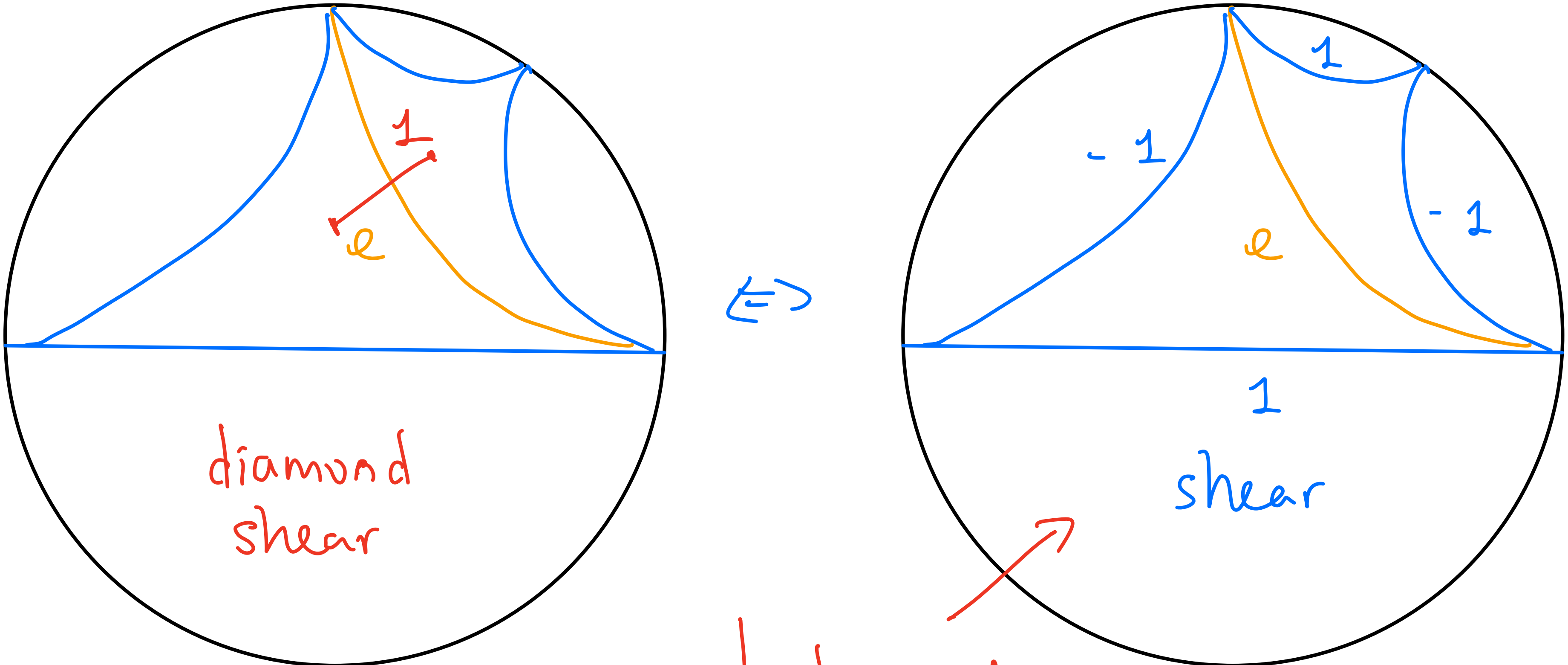
$\iff$ S is balanced.

$\iff$ φ is C^1.

Sum over any fan $= 0$

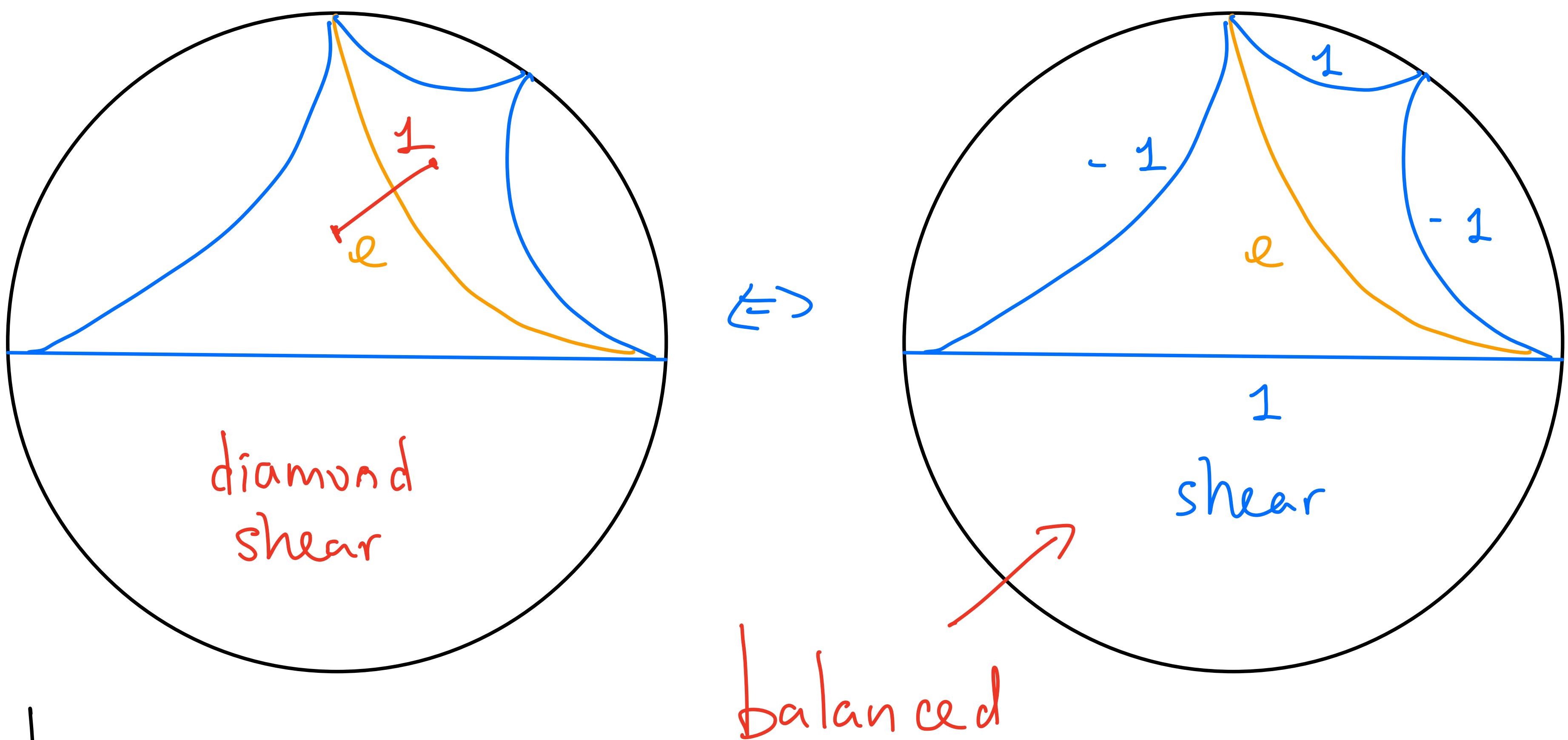


A unit <u>diamond shear</u> around $e = e^*$



balanced

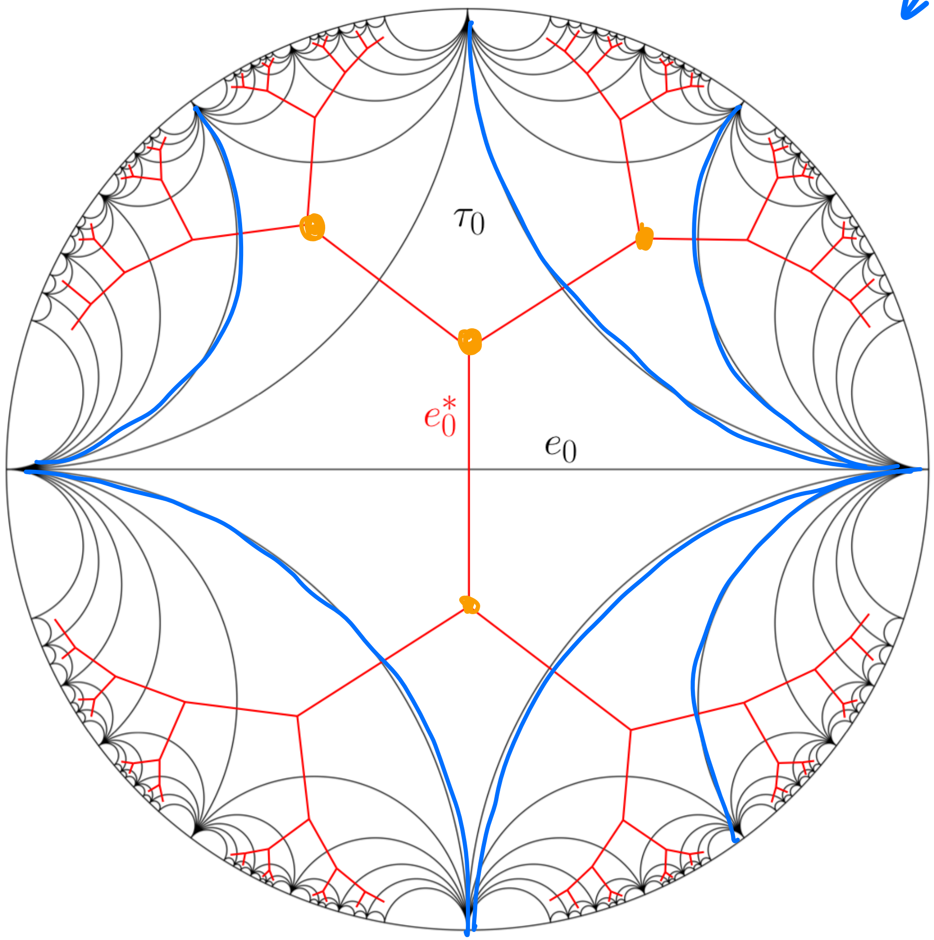
A unit <u>diamond shear</u> around $e \simeq e^*$

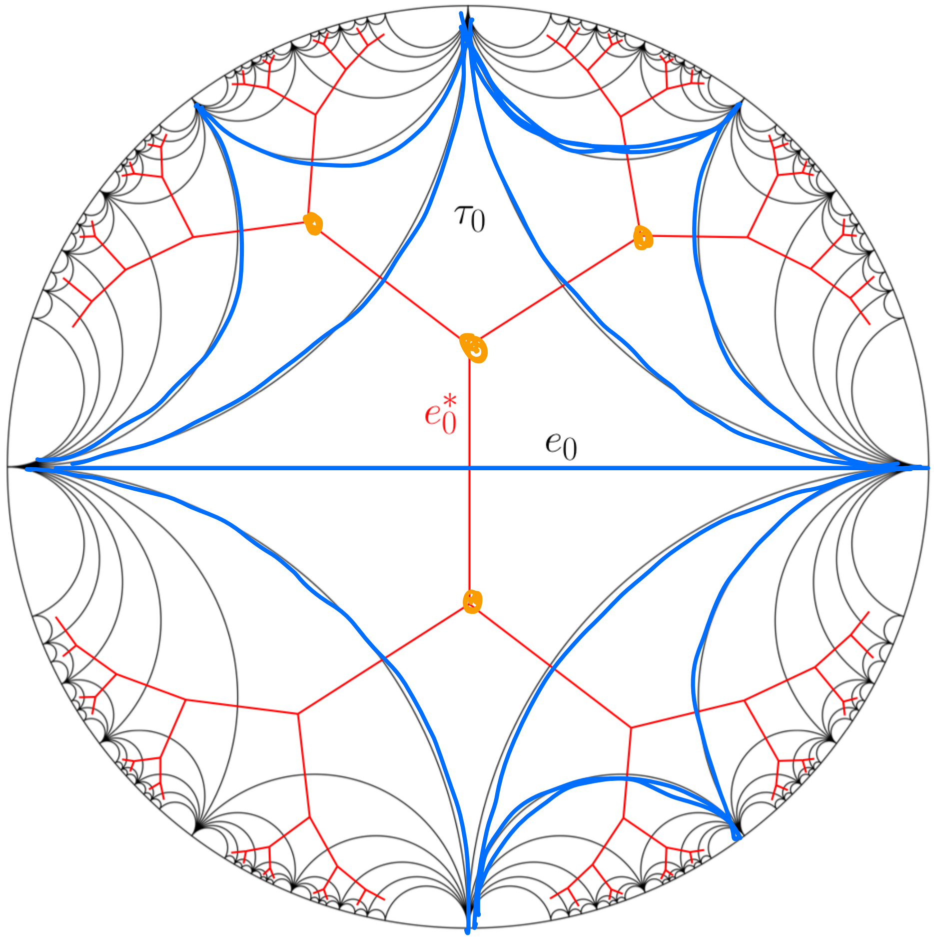


<u>Lemma</u>

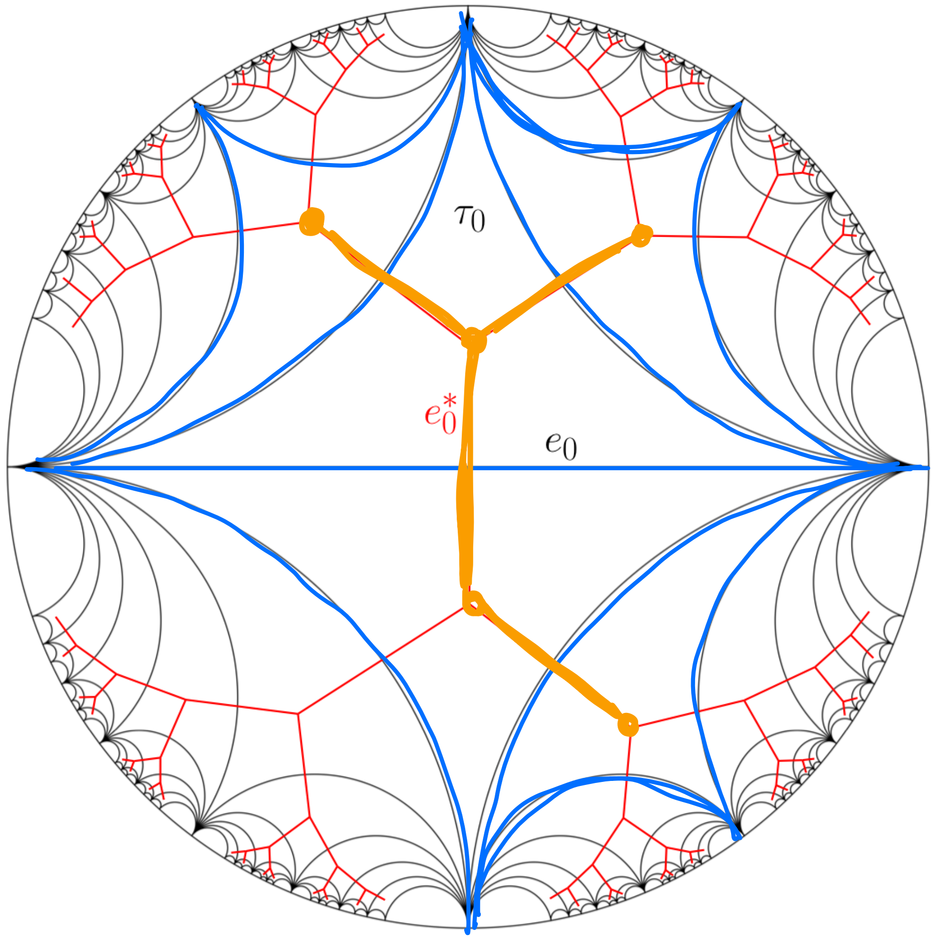
Any finite balanced shear is finite linear combination of diamond shears.

support of s





convex hull of supp(s)

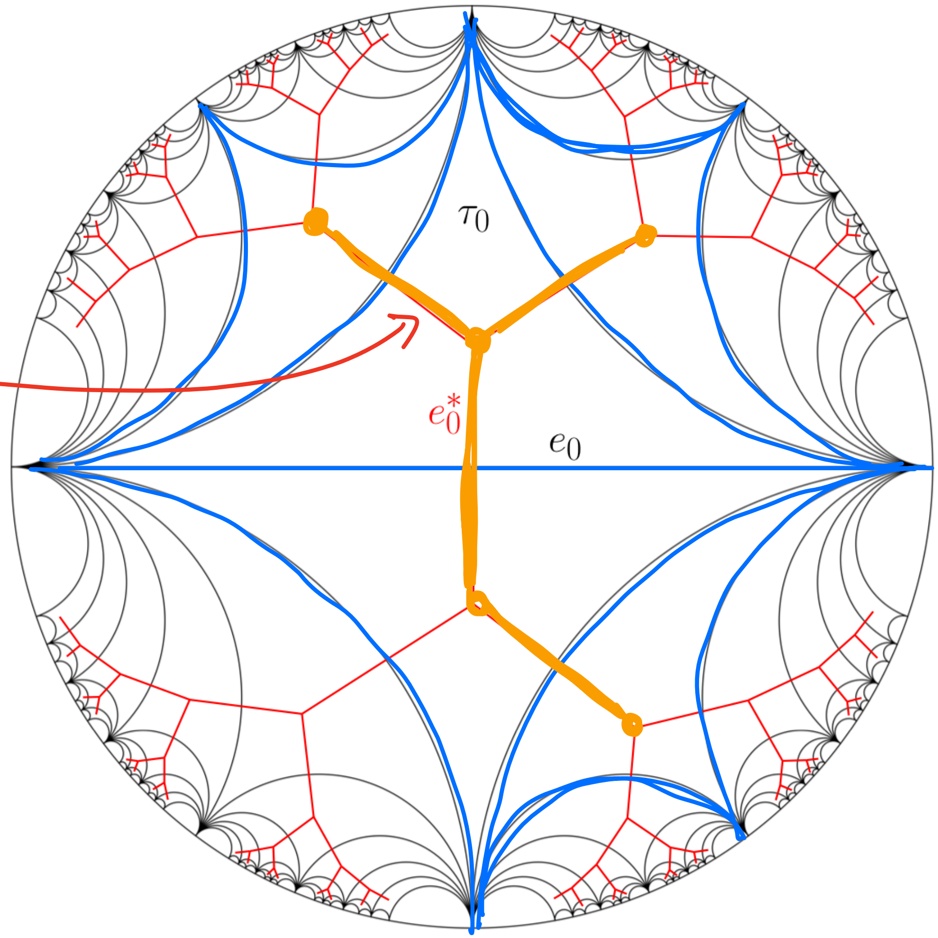


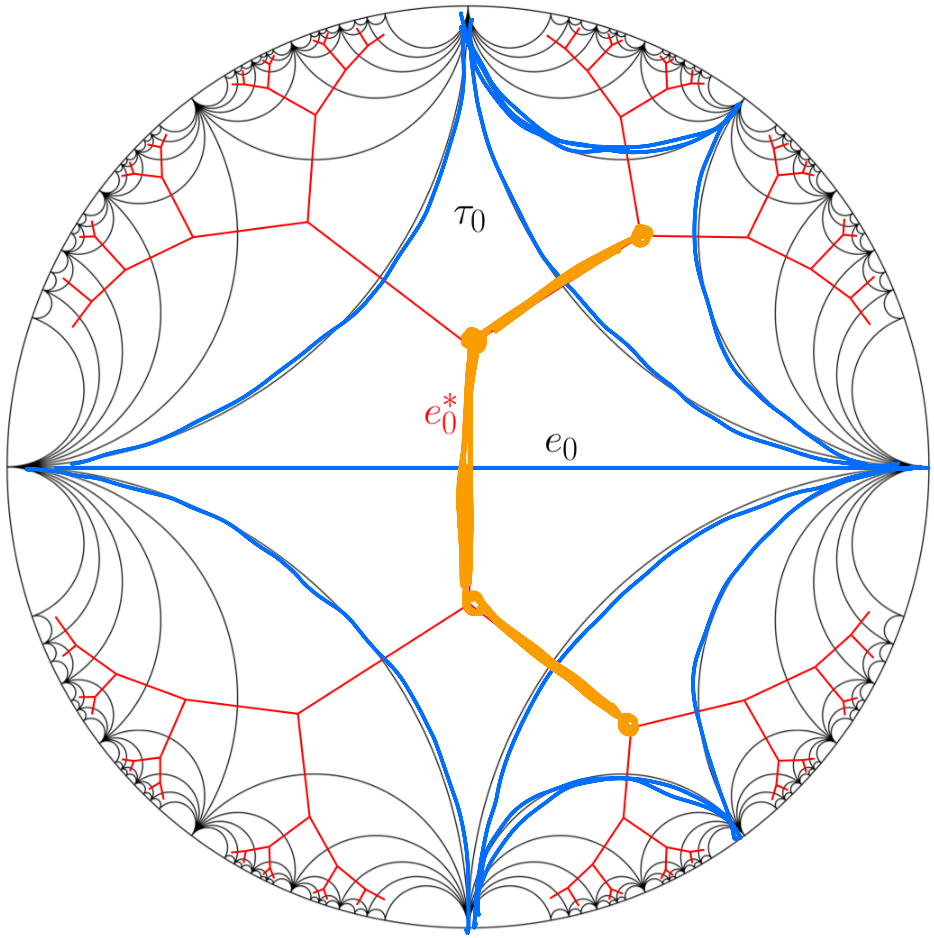
Support of
diamond
shear
$\Theta : E^* \to \mathbb{R}$

Support of diamond shear $\theta : E^* \to \mathbb{R}$

decide θ on a leaf

and remove the leaf





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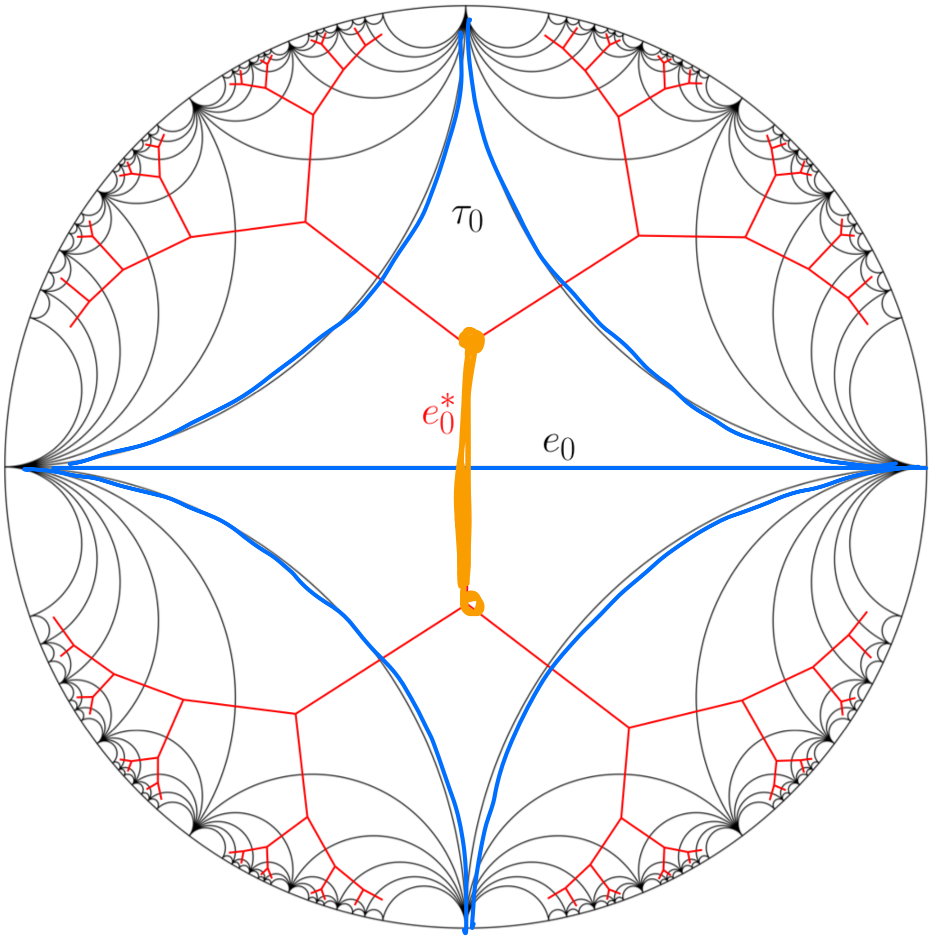
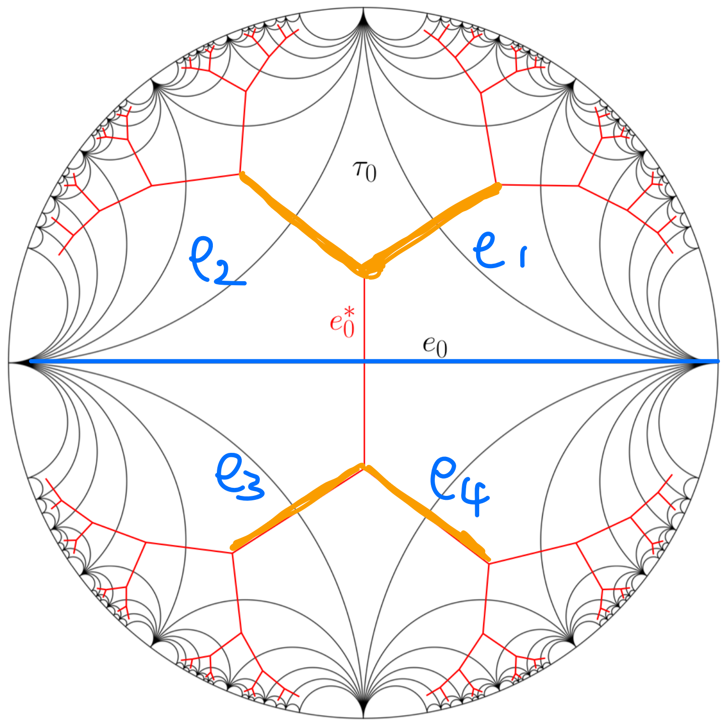


Support of diamond shear

$\Theta : E \to \mathbb{R}$

...

From $\theta : E^* \longrightarrow \mathbb{R}$. Easy to recover $s : E \to \mathbb{R}$



$$s_\theta(e_0) = -\theta(e_1) + \theta(e_2) - \theta(e_3) + \theta(e_4)$$

Guess 2:

$$\mathcal{H} := \{ s = s_\theta \mid \theta \in L^2(E^*, \mathbb{R}) \}$$

Thm (Šarić - W.-Wolfram)

$$\mathcal{C}^{1,\alpha}(S^1) \subsetneq \mathcal{H} \subsetneq WP(S^1) \subsetneq \mathcal{S}$$

$\forall\ \alpha > \frac{1}{2}$

sharp

$$\mathcal{C}^{1,\alpha}(S^1) = \{\varphi : S^1 \to S^1 \mid \log \varphi' \in C^\alpha\}$$

What is diamond shear coordinate θ ?

Lemma

If $s = s_\theta \in \mathcal{H}$, then s satisfies

- ∞-balanced condition (bi-infinite sum on each fan $= 0$)

What is diamond shear coordinate θ ?

Lemma

If $S = S_\theta \in \mathcal{H}$, then S satisfies

- ∞-balanced condition (bi-infinite sum on each fan $= 0$)
- S indues q.s. homeo φ, which is differentiable at all $v \in V = \mathbb{Q} \cup \{\infty\}$

$$\theta(\underbrace{(a,b)}_{\substack{(1)\\ E \simeq E^*}}) = \frac{1}{2}\log \varphi'(a)\varphi'(b) - \log \frac{\varphi(a) - \varphi(b)}{a-b}$$

$\rightsquigarrow$ log Λ-length (Penner)

"$\theta(e) = -\frac{1}{2}\,\mathrm{length}(\varphi(e))$"

Thm (Follows easily from Penner)

The Weil-Petersson symplectic form on $\mathcal{H}$ is given by

$$\omega_{wp} = \sum_{e \in E} d\theta(e) \wedge ds(e)$$

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The Weil-Petersson symplectic form on $\mathcal{H}$ is given by

$\cong \{\theta \in L^2\}$

$$\omega_{WP} = \sum_{e \in E} d\theta(e) \wedge ds(e)$$

(Note that $\theta \in L^2 \Rightarrow s \in L^2$)