On the regular side of Random conformal geometry

through the lens of large deviations (3/3)

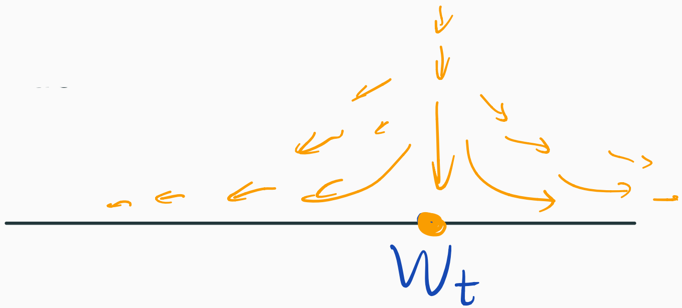
Yilin Wang (MIT)

October 22, 2020

smooth, analytic Energy minimizers $\xleftarrow{\text{minimize}}$	more regular ↓ Finite energy deterministic $\xleftarrow{\text{large deviations}}$	fractal ↓ Continuum random	
hyperbolic geodesic Rational function	Loewner energy $\xleftarrow{\kappa \to 0}$ Multichordal energy or potential	SLE Multichordal SLE	Lecture 1
circle / line 0 function $dA(z)$	Loop Loewner energy $\xleftarrow{\kappa \to 0}$ $\mathcal{D}_D(\varphi)$ $\quad D \subset \mathbb{C}$ $\underbrace{e^{2\varphi}}_{L^1_{loc}} \underbrace{dA(z)}_{\text{Euclidean}} \,/\, e^{2i\varphi}$ $\xleftarrow{\gamma \to 0}$	SLE loop GFF LQG / IG	Lecture 2
	Loewner Kufarev energy $\xleftarrow{\kappa \to \infty}$ Foliation of WP-q.c	SLE	Today

What happens when $\kappa \longrightarrow \infty$?

Chordal SLE



$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}$$

$$g_0(z) = z$$

where $W_t = \sqrt{\kappa}\, B_t$

$\tau(z)$ = max solution time

$K_t = \{z \in \mathbb{H} \mid \tau(z) \le t\}$

$H_t = \{z \in \mathbb{H} \mid \tau(z) > t\}$

[Rohde-Schramm]

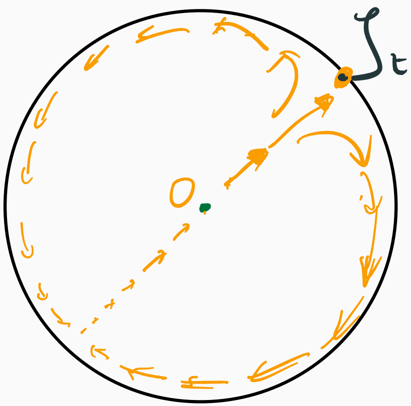
when $\kappa \geq 8$. $K_t = \gamma[0,t]$

where γ is a space-filling curve.

Easy: $\kappa \to \infty$. $g_t(z) \longrightarrow z$.

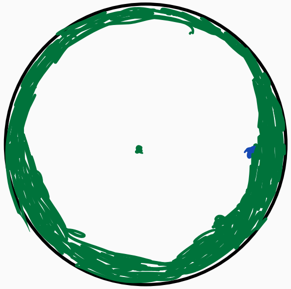
Issue: normalized at a boundary point

Radial Loewner chain



Driving function $t \mapsto \xi_t \in S^1$

$$\begin{cases} \partial_t g_t(z) = g_t(z) \dfrac{\xi_t + g_t(z)}{\xi_t - g_t(z)} \\ g_0(z) = z \end{cases}$$



$\tau(z)$ = max solution time.

$K_t = \{z \in \mathbb{D} \mid \tau(z) \le t\}$

$D_t = \{z \in \mathbb{D} \mid \tau(z) > t\}$

g_t : conformal $D_t \longrightarrow \mathbb{D}$, with $\begin{cases} g_t(0) = 0 \\ g_t'(0) = e^t \end{cases}$

Radial SLE$_\infty$

$$\zeta_t := e^{iB_{\kappa t}} \qquad \text{Let } \kappa \to \infty.$$

$$t \rightsquigarrow t + \Delta t$$

$$\Delta g_t(z) \simeq \int_t^{t+\Delta t} g_t(z) \int_{S^1} \frac{\zeta + g_t(z)}{\zeta - g_t(z)} \overset{\text{Dirac}}{\delta_{e^{iB_{\kappa t}}}}(\zeta)\, ds$$

$$\simeq g_t(z) \int_S \frac{\zeta + g_t(z)}{\zeta - g_t(z)} \Big(\underbrace{L^{\kappa}_{t+\Delta t}(\zeta) - L^{\kappa}_t(\zeta)}_{\text{Brownian local time}} \Big)$$

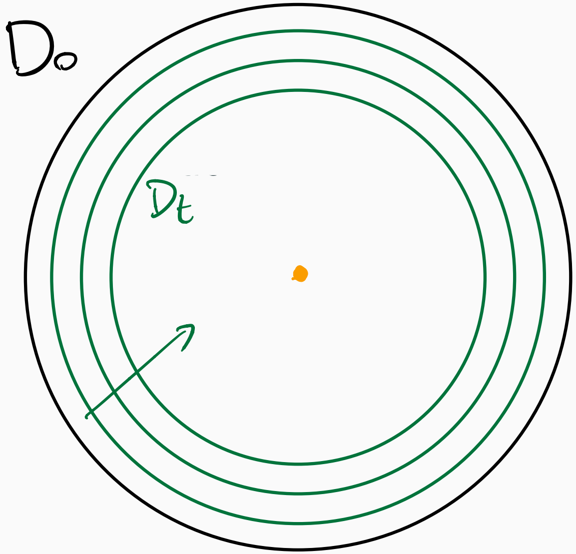
$$\xrightarrow{\kappa \to \infty} \Delta t \; g_t(z) \int_{S^1} \frac{\zeta + g_t(z)}{\zeta - g_t(z)} \frac{|d\zeta|}{2\pi}$$

$$= \Delta t \; g_t(z)$$

$$\Rightarrow \quad g_t(z) = e^t z : D_t \to \mathbb{D}$$

Loewner-Kufarev equation

(Measure-driven Loewner chain)



$$\mathcal{N}_+ := \{\rho \in \mathcal{M}(S^1 \times \mathbb{R}_+),\ \rho(S^1 \times I) = |I|\}$$

Disintegration $\rightsquigarrow$ $t \mapsto \rho_t \in \mathrm{Prob}(S^1)$ measurable

Loewner-Kufarev equation

$$\begin{cases} \partial_t g_t(z) = g_t(z) \displaystyle\int_{S^1} \frac{\zeta + g_t(z)}{\zeta - g_t(z)}\, d\rho_t(\zeta) \\ g_0(z) = z \end{cases}$$

Same $\varphi(z)$

D_t, K_t

Continuity

Lemma (e.g. J-Viklund-Sola-Turner, Miller-Sheffield)

Weak convergence on compact

$\mathcal{N}_+ \longrightarrow f$ is a homeomorphism.

where $f(t,z) := g_t^{-1}(z)$ (with unif. convergence on compact)

$\mathbb{R}_+ \times \mathbb{D} \to \mathbb{D}$ unif. on compact Caratheodory convergence of Loewner chain.

Convergence/LDP on $\mathcal{N}_+ \Longleftrightarrow$ Loewner chain

Brownian local time $\Longleftrightarrow$ SLE

Loewner-Kufarev energy S_+

For $\rho \in \mathcal{N}_+$, $\rightsquigarrow$ $(\rho_t)_{t \geq 0}$

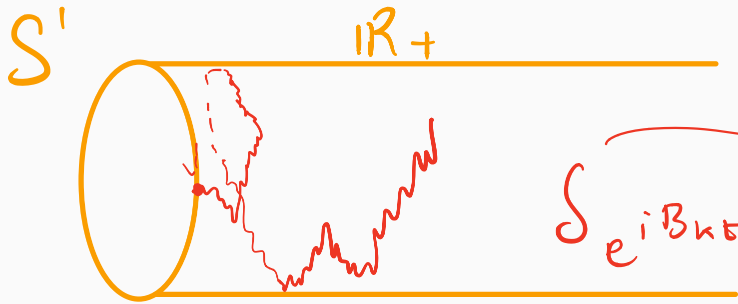
$$S_+(\rho) := \int_0^\infty L(\rho_t)\, dt$$

$$L(\rho_t) = \frac{1}{2} \int_{S^1} |\nu_t'(\theta)|^2 \, d\theta$$

if $\rho_t = \nu_t^2(\theta)\, d\theta$ and $L(\rho_t) = \infty$ otherwise.

- $S_+(\rho) = 0 \iff \rho_t =$ the uniform measure on S^1.

LDP of $\int e^{iB_{\kappa t}}\,dt$ or SLE_∞



ρ^κ

$$\int e^{iB_{\kappa t}}\,dt \in \mathcal{N}_+$$

Thm (Ang - Park - W. '20)

S_+ is the LDP rate function of radial SLE_κ as $\kappa \to \infty$.

$\forall A \in \mathcal{N}_+$,

$$-\inf_{\rho \in \mathring{A}} S_+(\rho) \le \varliminf_{\kappa \to \infty} \frac{1}{\kappa} \log \mathbb{P}[\rho^\kappa \in \mathring{A})$$

$$\le \varlimsup_{\kappa \to \infty} \frac{1}{\kappa} \log \mathbb{P}[\rho^\kappa \in \bar{A}] \le -\inf_{\bar{A}} S_+(\rho)$$

Whole-plane Loewner-Kufarev Chain

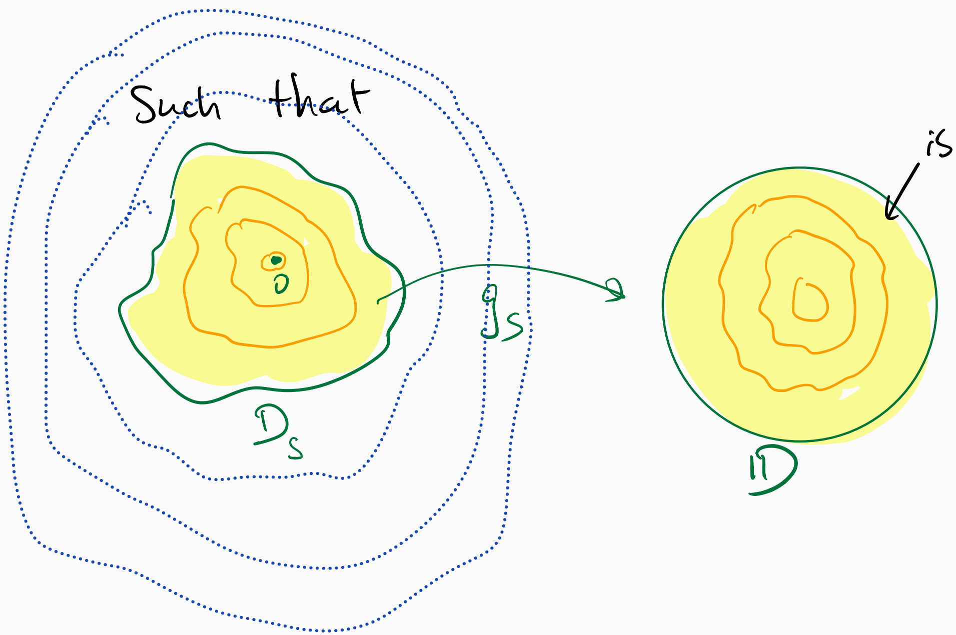
(More symmetric)

$\rho \in \mathcal{N}$ same as $\mathcal{N}_+$ but on $S^1 \times \mathbb{R}$

$\rightsquigarrow (\rho_t)_{t \in \mathbb{R}} \rightsquigarrow (D_t)_{t \in \mathbb{R}} \rightsquigarrow g_t : D_t \longrightarrow \mathbb{D}$

with $g_t'(0) = e^t$

Such that



is the radial Loewner-Kufarev chain generated by $(\rho_{t+s})_{t \geq 0}$

What happens when $S(\rho) < \infty$?

$$S(\rho) = \int_{-\infty}^{\infty} L(\rho_t)\, dt$$

$$L(\rho_t) = \frac{1}{2}\int_{S^1} (\nu_t')^2 \, d\theta$$

$$\rho_t = \nu_t^2 \, d\theta$$

Loewner energy

$\Updownarrow$ $I^L < \infty$

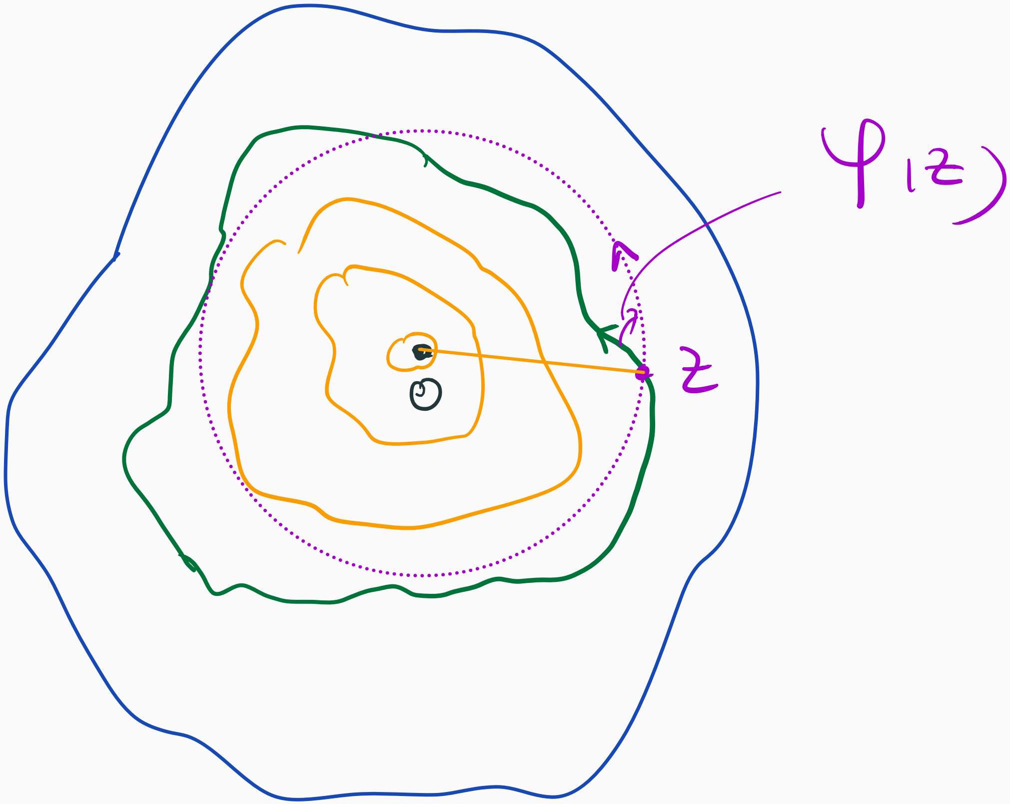
Thm (Viklund – W.)

If $S(\rho) < \infty$, then ∂D_t is a Weil-Petersson quasicircle, $\forall t \in \mathbb{R}$. And $\bigcup_{t \in \mathbb{R}} \partial D_t = \mathbb{C} \setminus \{0\}$

$t \mapsto \partial D_t$ is continuous for sup-norm.

Foliation of $\mathbb{C} \setminus \{0\}$ by Weil-Petersson quasi-circles

Winding function φ



Energy duality

Thm (Viklund – W.)

$$I^L(\gamma) = \mathcal{D}_{\mathbb{C}}(\varphi) \quad \left(= \frac{1}{\pi}\int_{\mathbb{C}} |\nabla \varphi|^2 \, dA(z) \right)$$

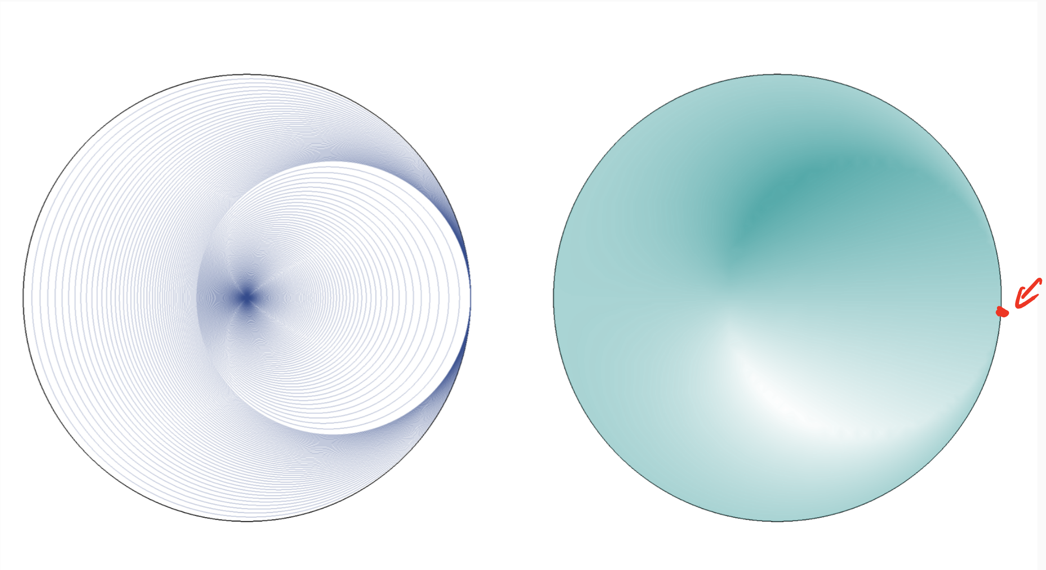
One is finite iff the other is finite

- Inspired by SLE duality

$$\kappa \Leftrightarrow \frac{16}{\kappa}$$

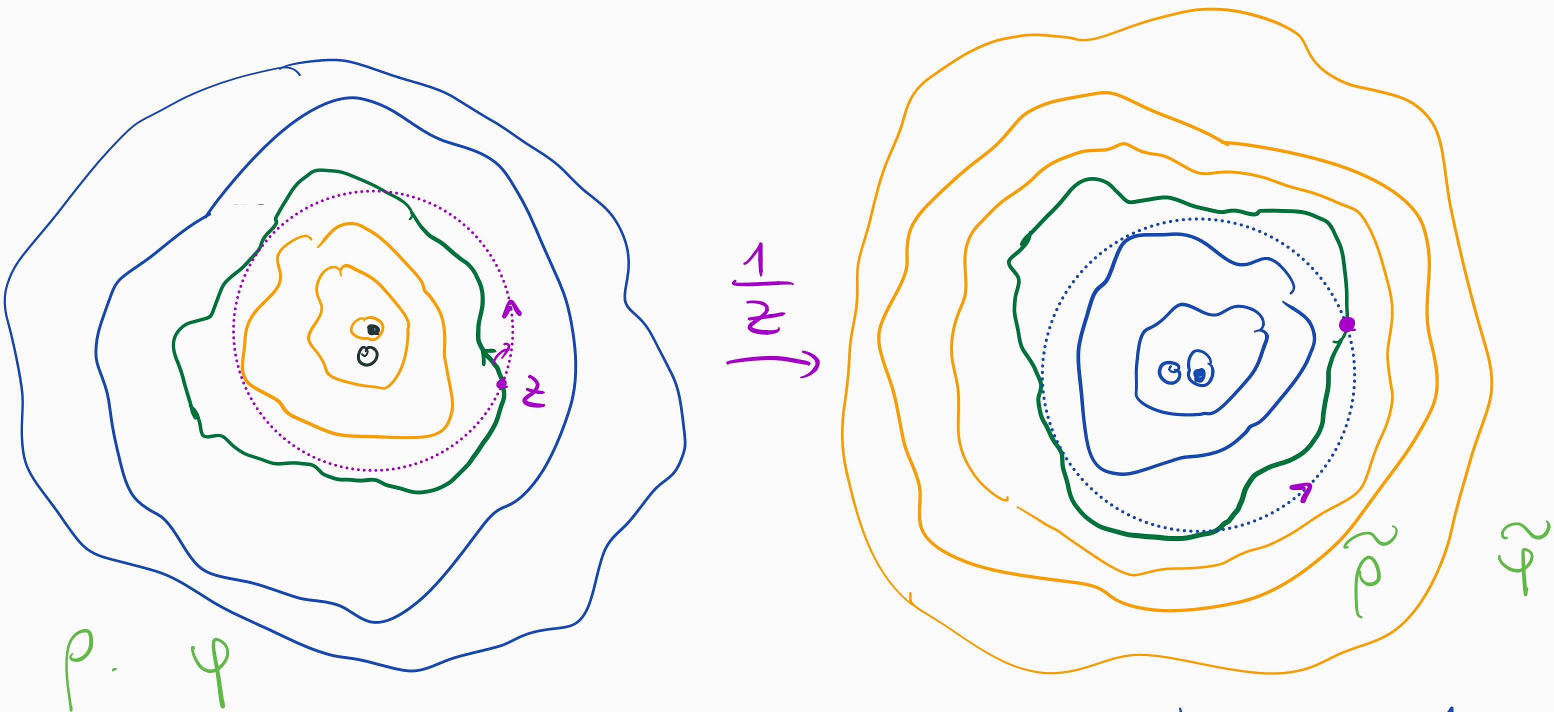
Example

$$\rho_t(d\theta) = \begin{cases} \frac{1}{\pi}\sin^2\left(\frac{\theta}{2}\right)d\theta & \text{for } t \in [0,1] \\ \frac{1}{2\pi}d\theta & \text{otherwise} \end{cases}$$



$\nabla\varphi$ singularity

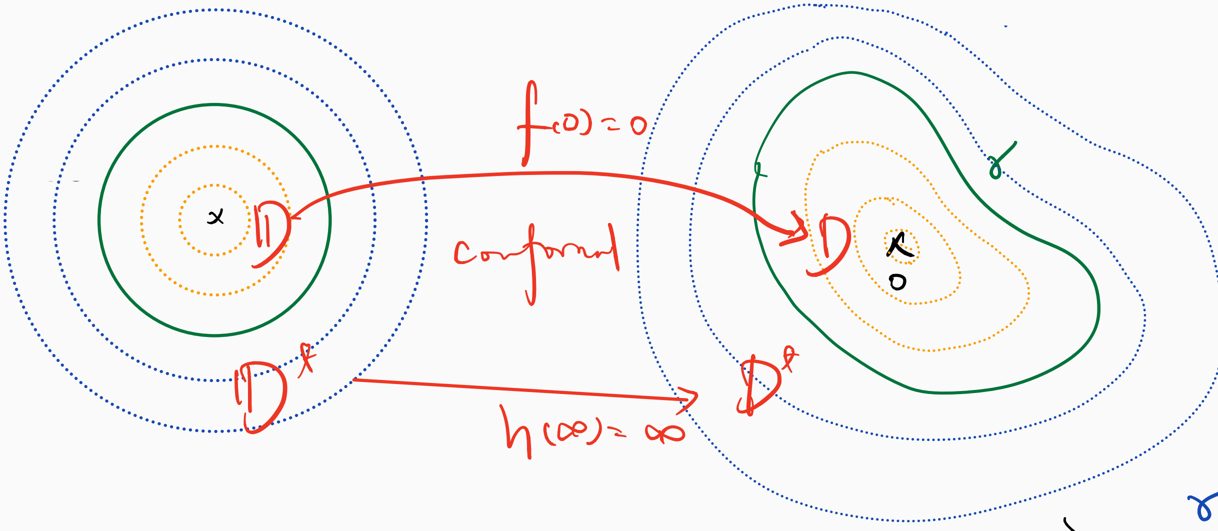
Energy reversibility



$$\tilde{\varphi}(z) = \varphi(1/z)$$

Cor (VW) $\quad S(\rho) = S(\tilde{\rho})$

$S \leftrightarrow I^L$



$\hookrightarrow \rho^\gamma \in \mathcal{N}$

Thm (V.W.)

$$16\, S(\rho^\gamma) = I^L(\gamma) + \underbrace{2\log\left|\frac{h'(\infty)}{f'(0)}\right|}_{\geq 0}.$$

φ is harmonic in $\mathbb{C} \setminus \gamma$.

Weil-Petersson quasicircle Characterization

Cor. (V. W.)

A Jordan curve γ separating 0 and ∞ is WP

$\Longleftrightarrow$ γ can be realized as leaf in the foliation generated by a measure $\rho \in \mathcal{N}$ with $S(\rho) < \infty$.

And $I^L(\gamma) \leq 16\, S(\rho)$.

WEIL-PETERSSON CURVES, CONFORMAL ENERGIES, β-NUMBERS, AND MINIMAL SURFACES

CHRISTOPHER J. BISHOP

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding midpoints
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
12	Menger curvature
13	biLipschitz involutions

14	between disjoint disks
15	thickness of convex hull
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder
21	closure of smooth curves in $T_0(1)$
22	P_φ^- is Hilbert-Schmidt
23	double hits by random lines
24	finite Loewner energy
25	large deviations of SLE(0^+)
26	Brownian loop measure

27 Leaf in $<\infty$ Loewner chain driven by $S(\rho)<\infty$

Here

Key step of the proof of $16\, S(\rho) = \mathcal{D}(\varphi)$

- Dirichlet energy is conformally invariant
- Loewner chain explores the strength of conformal invariance.

Assume ρ generates a foliation. $\mathcal{D}(\phi) < \infty$

$$\phi = \underset{\substack{\uparrow \\ \text{harmonic} \\ \text{in } D_t}}{\phi^{h,t}} + \phi^{0,t} \in W_0^{1,2}(D_t)$$

Disintegration isometry

Under very weak assumption on ρ.

Thm (V W).

$\iota : (W_0^{1,2}(\mathbb{D}), \mathcal{D}^{1/2}) \longrightarrow L^2(S^1 \times \mathbb{R}_+, 2\rho)$

$\partial_n \phi^{0,t} \circ g_t^{-1} \cdot J_t$

$$\phi \mapsto \frac{1}{2\pi}\int_{\mathbb{D}} \Delta(\phi \circ f_t)(z) P_{\mathbb{D}}(z, e^{i\theta})\, \mathrm{d}A(z).$$

is an bijective isometry with inverse operator

$$\varkappa[u](w) = 2\pi \int_0^{\tau(w)} \underbrace{P_{\mathbb{D}}[u_t \rho_t](g_t(w))}_{\text{harmonic function in } D_t}\, \mathrm{d}t, \qquad u_t(\cdot) := u(\cdot, t).$$

GFF $\rightarrow$ white noise decomposition. generalize [Hedenmalm - Nieminen]

If φ = winding function.

$$\rho = \mathcal{U}_t^2(\theta)\, d\theta\, dt$$

Show $i(\varphi)(\theta, t) = \dfrac{-2\mathcal{U}_t'}{\mathcal{U}_t}$.

$\mathcal{D}(\varphi)$

$\|$

$$\|i(\varphi)\|^2_{L^2(2\rho)} = \int_0^\infty \int_{S^1} \frac{4(\mathcal{U}_t')^2}{\mathcal{U}_t^2} 2\mathcal{U}_t^2(\theta)\, d\theta\, dt$$

$$= 16 \int_0^\infty L(\rho_t)\, dt = 16 S_+(\rho). \quad \boxed{M}$$

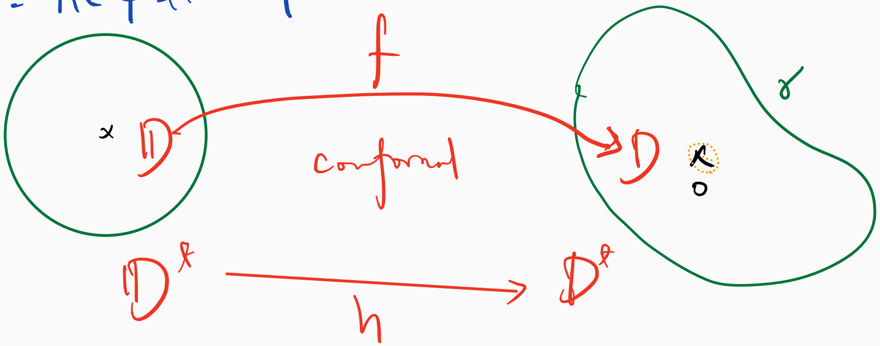
Complex identity

Proposition 1.5 (Complex identity)**.** *Let ψ be a complex valued function on $\mathbb{C}$ with $\mathcal{D}_{\mathbb{C}}(\psi) = \mathcal{D}_{\mathbb{C}}(\operatorname{Re}\psi) + \mathcal{D}_{\mathbb{C}}(\operatorname{Im}\psi) < \infty$ and let γ be a Weil-Petersson quasicircle compatible with $\operatorname{Im}\psi$. If*

$$\zeta(z) := \psi \circ f(z) + \log \frac{f'(z)z}{f(z)} \quad \textit{and} \quad \xi(z) := \psi \circ h(z) + \log \frac{h'(z)z}{h(z)},$$

then $\mathcal{D}_{\mathbb{C}}(\psi) = \mathcal{D}_{\mathbb{D}}(\zeta) + \mathcal{D}_{\mathbb{D}^*}(\xi)$.

$\psi = \operatorname{Re}\psi + i\operatorname{Im}\psi$



Complex identity