On the regular side of Random conformal geometry

through the lens of large deviations (2/3)

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smooth analytic Energy minimizers	$\xleftarrow{\text{minimize}}$ more regular ↓ Finite energy deterministic	$\xleftarrow{\text{large deviations}}$ fractal ↓ Continuum random
hyperbolic geodesic Rational function	Loewner energy fixed boundary data Multichordal energy or potential varying boundary data	$\xleftarrow{\kappa\to 0_+}$ SLE Multichordal SLE

Lecture 2 (SLE$_{0+}$ loops, GFF, LQG, MoT)

- Loewner energy and equivalent descriptions
  - Dirichlet energy of $\log |f'|$
  - Weil-Petersson quasicircles
- Welding, flowline, complex identity

A dictionary

Quantum zipper - imaginary geometry

Mating of trees

References

[1] **Y. Wang** The energy of a deterministic Loewner chain: Reversibility and interpretation via SLE_{0+} JEMS 21(7) (2019)

[2] **S. Rohde, Y. Wang** The Loewner energy of loops and regularity of driving functions IMRN (2019)

[3] **Y. Wang** Equivalent Descriptions of the Loewner Energy INVENT. MATH. 218(2) (2019)

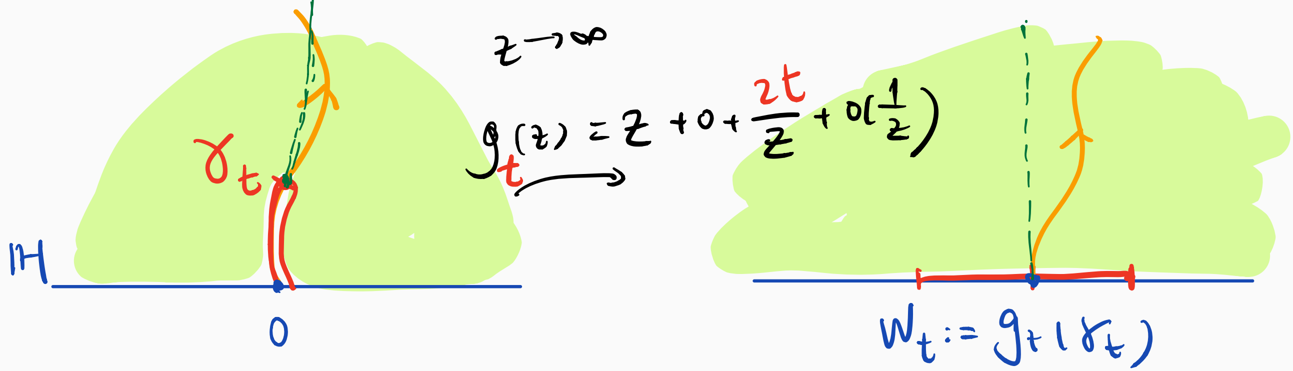
[4] **F. Viklund, Y. Wang** Interplay between Loewner and Dirichlet energies via conformal welding and flow-lines GAFA 30 (2020)

[5] **M. Ang, M. Park, Y. Wang** Large deviations of radial SLE_{∞} EJP 25(102) (2020)

[6] **E. Peltola, Y. Wang** Large deviations of multichordal SLE0+, real rational functions, and zeta-regularized determinants of Laplacians PREPRINT (2020)

[7] **F. Viklund, Y. Wang** The Loewner-Kufarev Energy and Foliations by Weil-Petersson Quasicircles Available soon (2020)

Let γ be a simple chord in $(\mathbb{H}, 0, \infty)$.

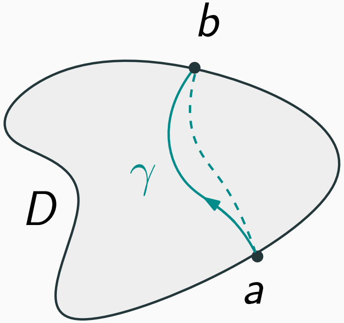


- γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of γ.
- $W_0 = 0$, W is continuous.
- The curve γ can be recovered from W using Loewner's differential equation: $\partial_t g_t(z) = 2/(g_t(z) - W_t)$, $g_0(z) = z$.
- We say that γ is the **chordal Loewner curve** driven by W.

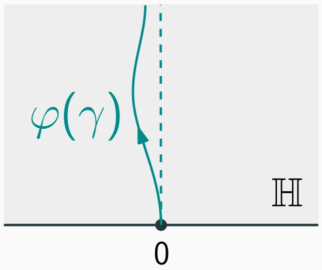
Introduced by *[Loewner '23 Math. Ann.]*.

The chordal Loewner energy (W. [1])

$D \subset \mathbb{C}$ a simply connected domain, a, b are two boundary points of D.



$$\varphi : D \to \mathbb{H}$$
$$\varphi(a) = 0, \varphi(b) = \infty$$

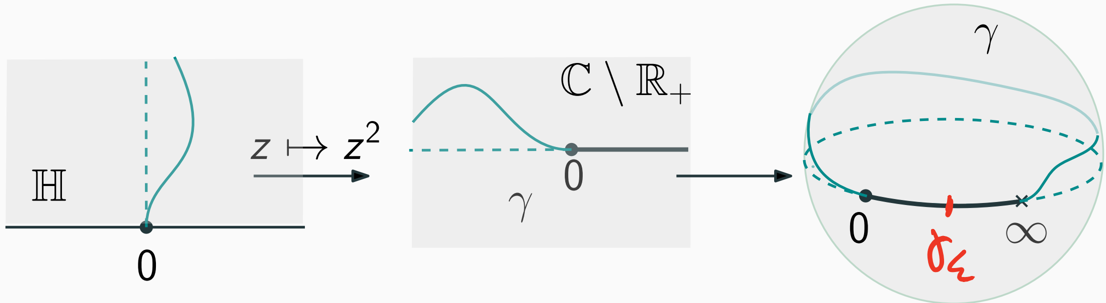


We define the **Loewner energy of a simple chord γ in** (D, a, b) to be

$$I_{D,a,b}(\gamma) := I_{\mathbb{H},0,\infty}(\varphi(\gamma)) := I(W) := \frac{1}{2}\int_0^\infty W'(t)^2 dt$$
$$= \sup_{0=t_0<t_1<\cdots<t_n} \frac{1}{2}\sum_{i=1}^{n} \frac{(W(t_i) - W(t_{i-1}))^2}{t_i - t_{i-1}}$$

where W is the driving function of $\varphi(\gamma)$.

Loewner loop energy (Rohde, W. [2])



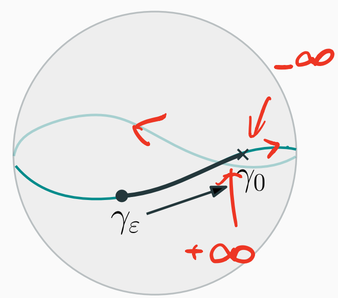
$$I^L(\gamma \cup \mathbb{R}_+, \infty) := I_{\mathbb{C}\setminus\mathbb{R}_+,0,\infty}(\gamma).$$

More generally, we define the **Loewner energy of a simple loop** $\gamma : [0,1] \mapsto \hat{\mathbb{C}}$ rooted at $\gamma_0 = \gamma_1$ to be

$W : \mathbb{R} \longrightarrow \mathbb{R}$

$$I^L(\gamma, \gamma_0) := \lim_{\varepsilon \to 0} I_{\hat{\mathbb{C}}\setminus\gamma[0,\varepsilon],\gamma_\varepsilon,\gamma_0}(\gamma[\varepsilon, 1]).$$

$= \frac{1}{2}\int_{-\infty}^{\infty} \dot{W}^2(t)\, dt$



Remarks: $I^L(\gamma, \gamma_0) = 0$ if and only if γ is a circle.

If $\varphi \in \mathrm{PSL}(2, \mathbb{C})$, then $I^L(\varphi(\gamma), \varphi(\gamma_0)) = I^L(\gamma, \gamma_0)$.

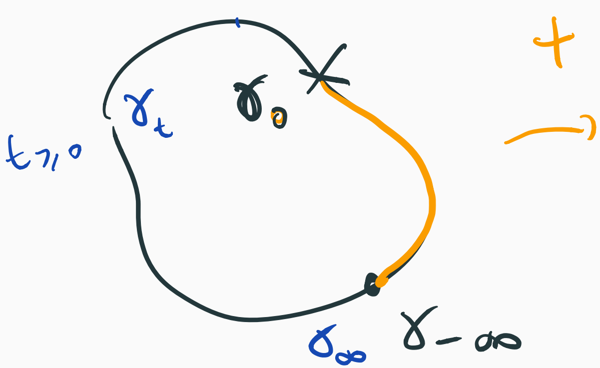
Root-invariance.

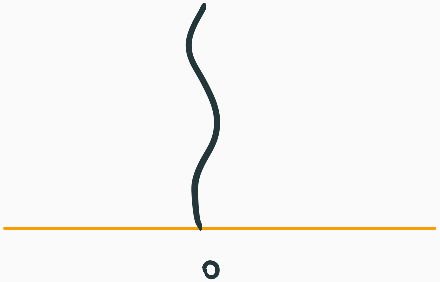
Theorem (Rohde, W. [2])

The Loewner loop energy is **independent** *of the choice of the root.*

$\Rightarrow$ $I^L(\gamma)$ measures how round is γ.

[2] **S. Rohde, Y. Wang** The Loewner energy of loops and regularity of driving functions IMRN (2019)





Equivalent descriptions

[3] **Y. Wang** Equivalent Descriptions of the Loewner Energy INVENT. MATH. 218(2) (2019)

I. Dirichlet energy of log-derivatives of conformal maps

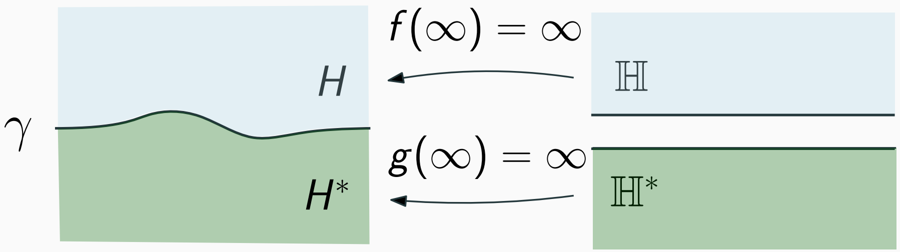
For $D \subset \mathbb{C}$, we write

$$\mathcal{D}_D(\varphi) := \frac{1}{\pi} \int_D |\nabla \varphi(z)|^2 \, dz^2.$$

Theorem (W. [3])

If γ passes through ∞, we have the identity

$$I^L(\gamma, \infty) = \mathcal{D}_{\mathbb{H}}(\log |f'|) + \mathcal{D}_{\mathbb{H}^*}(\log |g'|).$$

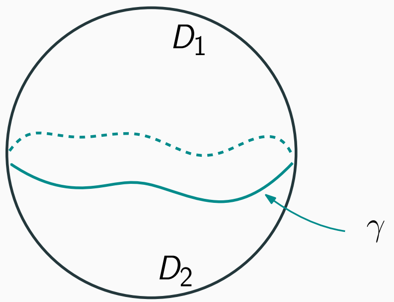


The identity is related to SLE/GFF couplings but the proof is purely analytic.

Further connection to SLE/GFF couplings is studied in *[Viklund, W. 4]*.

II. Loewner Energy vs. Determinants: the set-up

- $g_0(z) = \frac{4}{(1+|z|^2)^2}\,\mathrm{d}z^2$ denotes the spherical metric on $\hat{\mathbb{C}} \simeq S^2$;
- $g = e^{2\varphi} g_0$ be a metric conformally equivalent to g_0, $\varphi \in C^\infty(S^2, \mathbb{R})$;
- γ a C^∞ **smooth** simple loop in S^2;
- D_1 and D_2 two connected components of $S^2 \backslash \gamma$;
- $\Delta_g(D_i)$ the Laplace-Beltrami operator with Dirichlet boundary condition on D_i.



$$\mathcal{H}(\gamma, g) := \log \det{}_\zeta \Delta_g(S^2) - \log \mathrm{Area}_g(S^2) - \log \det{}_\zeta \Delta_g(D_1) - \log \det{}_\zeta \Delta_g(D_2)$$

Theorem (W. [3])

If $g = e^{2\varphi} g_0$, we have:

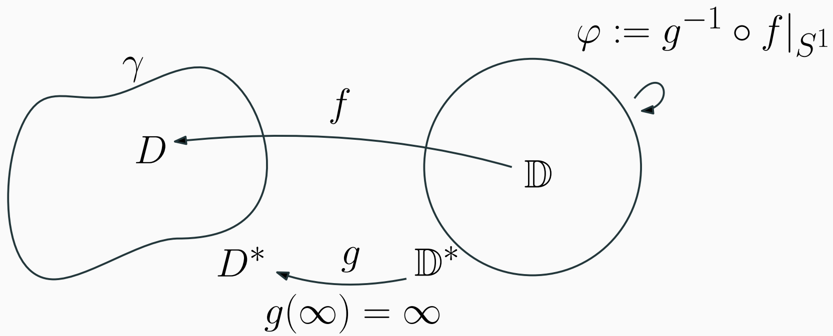
1. $\mathcal{H}(\cdot, g) = \mathcal{H}(\cdot, g_0)$, *i.e. $\mathcal{H}$ only depends on the conformal class of g;*
2. *Let γ be a smooth Jordan curve on S^2. We have the identity*

$$I^L(\gamma, \gamma(0)) = 12\mathcal{H}(\gamma, g_0) - 12\mathcal{H}(S^1, g_0).$$

Proof Sketch.

Based on the **Polyakov-Alvarez formula** which computes explicitly $\log \Delta_{g_0}(D_1) - \log \Delta_{g_0}(\mathbb{D}_1)$ in terms of scalar curvatures, geodesic curvatures, and $\log |f'|$ of a conformal map $f : \mathbb{D}_1 \to D_1$.

Use the identity between the Dirichlet energy of $\log |f'|$ and I^L. $\square$



Theorem [Takhtajan & Teo '06 Memoir AMS]

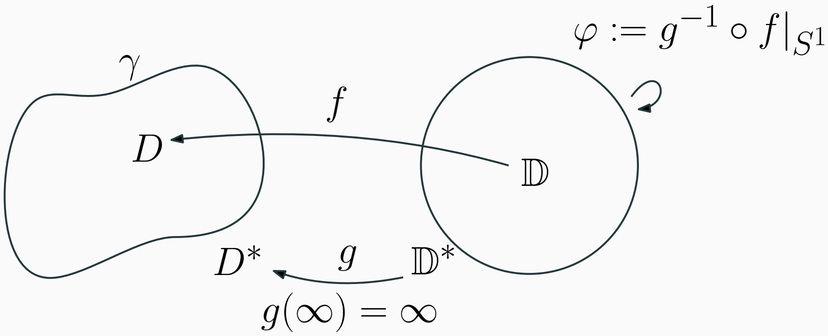
The **universal Liouville action** S_1 :

$\int_{\mathbb{D}} \log |f'|^2$

$$S_1(\gamma) := \int_{\mathbb{D}} \left|\frac{f''}{f'}(z)\right|^2 dz^2 + \int_{\mathbb{D}^*} \left|\frac{g''}{g'}(z)\right|^2 dz^2 + 4\pi \log \left|\frac{f'(0)}{g'(\infty)}\right|$$

is a **Kähler potential** for the Weil-Petersson metric.

$S_1(\gamma) < \infty \overset{\text{def}}{\Longleftrightarrow} \gamma$ is a Weil-Petersson quasicircle.



Theorem (W. [3])

A bounded simple loop γ has finite Loewner energy if and only if $[\varphi] \in T_0(1)$. Moreover,

$$I^L(\gamma) = \mathrm{S}_1([\varphi])/\pi.$$

Remark: This is proved using the identity with $\det_\zeta \Delta$, but there is no more regularity assumption.

WEIL-PETERSSON CURVES, CONFORMAL ENERGIES, β-NUMBERS, AND MINIMAL SURFACES

CHRISTOPHER J. BISHOP

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding midpoints
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
12	Menger curvature
13	biLipschitz involutions
14	between disjoint disks
15	thickness of convex hull
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder
21	closure of smooth curves in $T_0(1)$
22	P_φ^- is Hilbert-Schmidt
23	double hits by random lines
24	finite Loewner energy
25	large deviations of SLE(0^+)
26	Brownian loop measure

Here

The names of 26 characterizations of Weil-Peterson curves

Interplay between Loewner and Dirichlet energies:

conformal welding & flow-lines (joint with F. Viklund, KTH)

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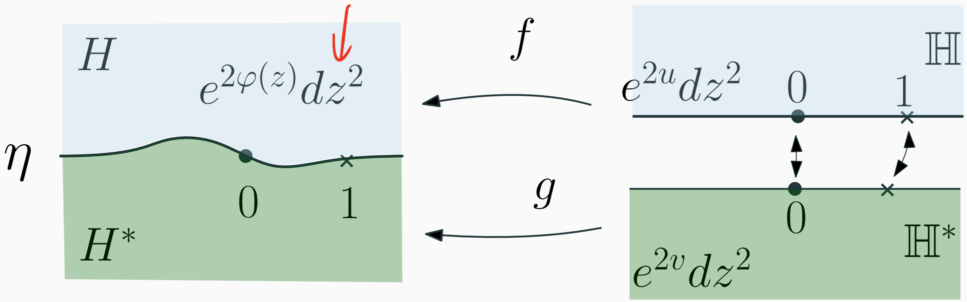
[4] **F. Viklund, Y. Wang** Interplay between Loewner and Dirichlet energies via conformal welding and flow-lines GAFA 30 (2020)

Cutting and welding identity

Real-valued

Let $\varphi \in \mathcal{E}(\mathbb{C}) \subset W^{1,2}_{loc}(\mathbb{C}) \subset VMO(\mathbb{C})$, f, g conformal maps from $\mathbb{H}, \mathbb{H}^*$ onto H, H^* fixing ∞.

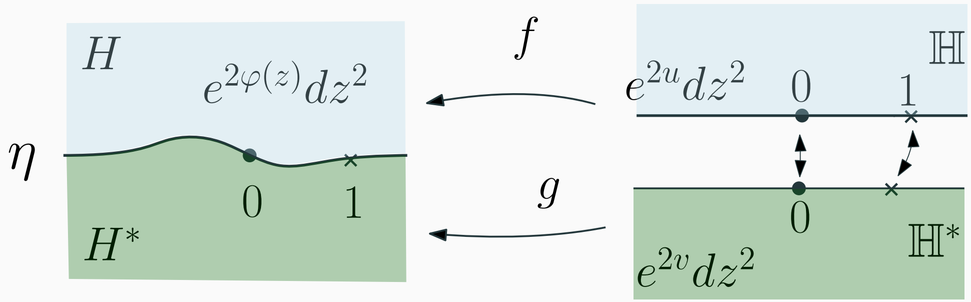
Euclidean area measure



We have $e^{2\varphi} \in L^1_{loc}(\mathbb{C})$ and the transformation law:

$$u(z) = \varphi \circ f(z) + \log|f'(z)|, \quad v(z) = \varphi \circ g(z) + \log|g'(z)|,$$

such that $e^{2u}dz^2 = f^*(e^{2\varphi}dz^2)$, $e^{2v}dz^2 = g^*(e^{2\varphi}dz^2)$.



Theorem (cutting)

We have the identity

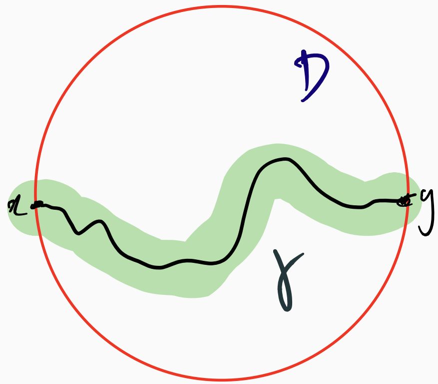
$$\mathcal{D}_{\mathbb{C}}(\varphi) + I^L(\eta) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v).$$

Proof: Check directly (very short proof).

SLE$_{0+}$ & Loewner energy

From Lecture 1

For chordal SLE



$$\mathbb{P}(\mathrm{SLE}_\kappa \text{ stays close to } \gamma) \sim \exp\left(-\frac{I_D(\gamma)}{\kappa}\right)$$

as $\kappa \to 0+$.

$\sqrt{\kappa}$ GFF & Dirichlet energy in $\mathbb{D}$

$$\mathcal{D}_D(\varphi) := \frac{1}{\pi} \int_D |\nabla \varphi(z)|^2 \, dz^2.$$

the action functional/large deviation rate function of (a small parameter γ times) the **Gaussian free field** (GFF)$\in H^{-\varepsilon}(D)$

$$\text{“}\mathrm{P}(\sqrt{\kappa} GFF \text{ stays close to } 2\varphi) \approx e^{-\mathcal{D}(\varphi)/\kappa}, \quad \text{as } \kappa \to 0.\text{”}$$

Large deviation heuristics

SLE/GFF $\gamma := \sqrt{\kappa}$	**Finite energy**
SLE_κ loop.	Finite energy Jordan curve, η.
Free boundary GFF $\gamma\Phi$ on $\mathbb{H}$ (on $\mathbb{C}$).	$2u$, $u \in \mathcal{E}(\mathbb{H})$ (2φ, $\varphi \in \mathcal{E}(\mathbb{C})$).
γ-LQG on quantum plane $\approx e^{\gamma\Phi} dz^2$.	$e^{2\varphi} dz^2$, $\varphi \in \mathcal{E}(\mathbb{C})$.
γ-LQG on quantum half-plane on $\mathbb{H}$	$e^{2u} dz^2$, $u \in \mathcal{E}(\mathbb{H})$.
SLE_κ cuts an independent quantum plane $e^{\gamma\Phi} dz^2$ into ind. quantum half-planes $e^{\gamma\Phi_1}, e^{\gamma\Phi_2}$.	Finite energy η cuts $\varphi \in \mathcal{E}(\mathbb{C})$ into $u \in \mathcal{E}(\mathbb{H})$, $v \in \mathcal{E}(\mathbb{H}^*)$ and $I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v)$.

SLE/GFF $\Rightarrow$ one may expect that under appropriate topology and for small κ,

$$\text{“}\mathrm{P}(\mathrm{SLE}_\kappa \text{ loop stays close to } \eta, \sqrt{\kappa}\Phi \text{ stays close to } 2\varphi)$$

$$= \mathrm{P}(\sqrt{\kappa}\Phi_1 \text{ stays close to } 2u, \sqrt{\kappa}\Phi_2 \text{ stays close to } 2v)\text{”}$$

Large deviation heuristics, cont'd

From the large deviation principle and the independence of SLE and Φ, one expects

$$\begin{aligned}
&\lim_{\kappa\to 0} -\kappa \log \mathrm{P}(\mathrm{SLE}_\kappa \text{ stays close to } \eta,\ \sqrt{\kappa}\Phi \text{ stays close to } 2\varphi) \\
&= \lim_{\kappa\to 0} -\kappa \log \mathrm{P}(\mathrm{SLE}_\kappa \text{ stays close to } \eta) + \lim_{\kappa\to 0} -\kappa \log \mathrm{P}(\sqrt{\kappa}\Phi \text{ stays close to } 2\varphi) \\
&= I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi).
\end{aligned}$$

Similarly, the independence between Φ_1 and Φ_2 gives

$$\begin{aligned}
&\lim_{\kappa\to 0} -\kappa \log \mathrm{P}(\sqrt{\kappa}\Phi_1 \text{ stays close to } 2u,\ \sqrt{\kappa}\Phi_2 \text{ stays close to } 2v) \\
&= \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v).
\end{aligned}$$

$$\Longrightarrow\ I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v).$$

Conversely

One expects the density of an independent couple $(\mathrm{SLE}_\kappa, \sqrt{\kappa}\,\mathrm{GFF})$ has density

$$\begin{aligned}\rho(\eta, 2\varphi) &\propto \exp(-I^L(\eta)/\kappa)\exp(-\mathcal{D}_{\mathbb{C}}(\varphi)/\kappa)\\ &= \exp(-\mathcal{D}_{\mathbb{H}}(2u)/\kappa)\exp(-\mathcal{D}_{\mathbb{H}^*}(2v)/\kappa)\end{aligned}$$

the identity on the action functional also suggests the SLE/GFF coupling.

Converse operation : conformal welding

Assume $\mathcal{D}_{\mathbb{H}}(u) < \infty$, $\mathcal{D}_{\mathbb{H}^*}(v) < \infty$

$\Rightarrow$ $u|_{\mathbb{R}}$ and $v|_{\mathbb{R}}$ $\in H^{1/2} \subset VMO(\mathbb{R})$

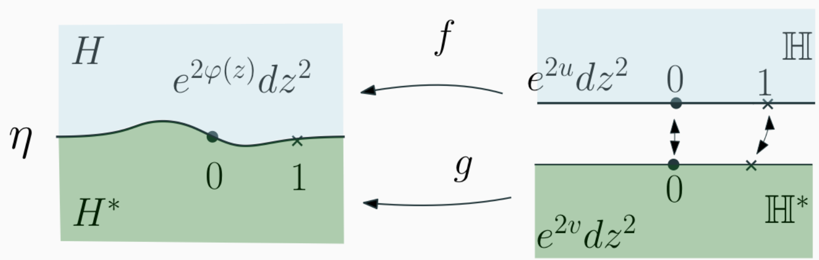
traces (Sobolev space)

There exists a unique normalized solution (η, f, g) to the welding homeomorphism induced by e^u and e^v, and the curve obtained has finite Loewner energy.

$L^1_{loc}(\mathbb{R})$

Moreover, φ defined from the **transformation law** *is in $\mathcal{E}(\mathbb{C})$, therefore the welding identity holds:*

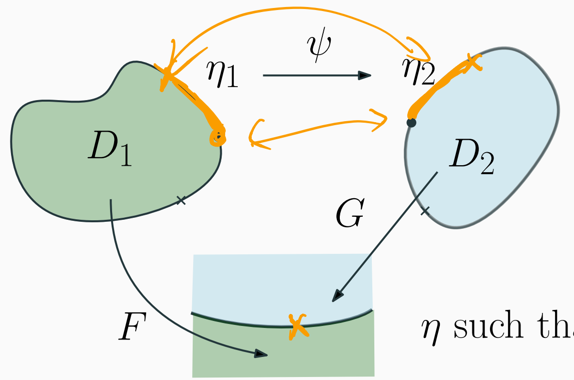
$$I^L(\eta) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v) - \mathcal{D}_{\mathbb{C}}(\varphi).$$



Application: arclength conformal welding

Assume η_1, η_2 are rectifiable Jordan curves and $|\eta_1| = |\eta_2|$.

$\psi : \eta_1 \to \eta_2$ preserves arclength.



- *[Huber 1976]* The solution does not always exist.
- *[Bishop 1990]* If the solution exists, η can be a curve of positive area and the solution is not unique.
- *[David 1982, Zinsmeister 1982, Jerison-Kenig 1982]* If η_1 and η_2 are **chord-arc**, then the solution exists and is unique, and is a quasicircle.
- *[Bishop 1990]* But the Hausdorff dimension of η can take any value in $1 < d < 2 \implies$ not rectifiable.
- We show : The class of finite energy curves is **closed** under arclength welding.

How does the energy change under the arclength welding operation?

$$I^L(\eta) \quad ?? \quad I^L(\eta_1) + I^L(\eta_2)$$

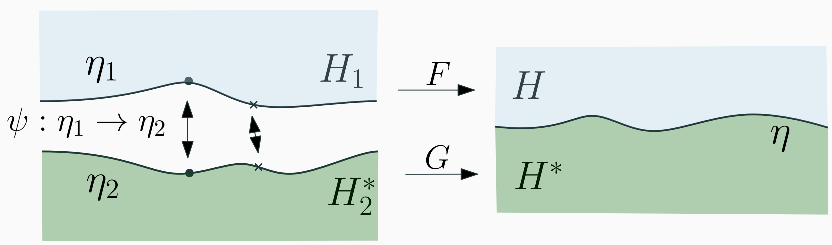
Arclength welding of finite energy domains

Assume $I^L(\eta_1) < \infty, I^L(\eta_2) < \infty$, both passing through ∞. Let H_i, H_i^* be the two connected components of $\mathbb{C} \setminus \eta_i$.

Corollary (sub-additivity)

Let η (resp. $\tilde{\eta}$) be the arclength welding curve of the domains H_1 and H_2^* (resp. H_2 and H_1^*). Then η and $\tilde{\eta}$ have finite energy. Moreover,

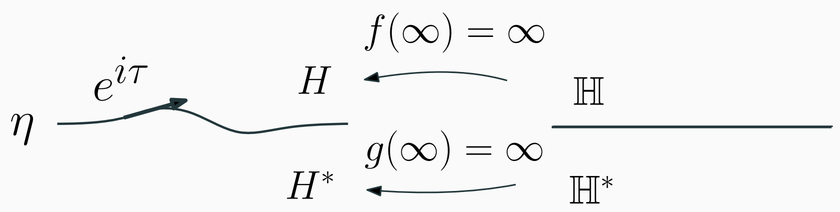
$$I^L(\eta) + I^L(\tilde{\eta}) \le I^L(\eta_1) + I^L(\eta_2).$$



Proof: Energy dissipated through welding.

Winding identity

Assume η is rectifiable.



We denote by

$$\mathcal{P}[\tau](z) = \begin{cases} \arg f'(f^{-1}(z)) & z \in H; \\ \arg g'(g^{-1}(z)) & z \in H^* \end{cases}$$

which is the Poisson integral of τ in $\mathbb{C}$.

Flow-line identity

$\text{Im} \log f'$ $\quad\quad$ $\text{Re}_\eta \log f'$

Notice that $\arg(f')$ has the same Dirichlet energy as $\log |f'|$. We have the identity

$$I^L(\eta) = \mathcal{D}_{\mathbb{H}}(\arg f') + \mathcal{D}_{\mathbb{H}^*}(\arg g') = \mathcal{D}_{\mathbb{C}}(\mathcal{P}[\tau]).$$

Consequence: $I^L(\eta) < \infty \Leftrightarrow \eta$ is chord-arc and $\tau \in H^{1/2}(\eta)$.

Flow-line identity, cont'd

Corollary (Flow-line identity)

Conversely, if $\varphi \in \mathcal{E}(\mathbb{C}) \cap C^0(\hat{\mathbb{C}})$, then for all $z_0 \in \mathbb{C}$, there is a unique solution to the differential equation

$$\eta'(t) = e^{i\varphi(\eta(t))}, \forall t \in \mathbb{R} \quad \text{and} \quad \eta(0) = z_0$$

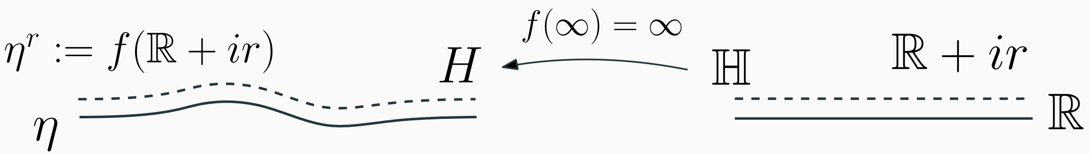
is an infinite arclength parametrized simple curve and

$$\mathcal{D}_{\mathbb{C}}(\varphi) = I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi_0),$$

where $\varphi_0 = \varphi - \mathcal{P}[\varphi|_\eta]$. $\mathcal{D}_{\mathbb{C}}(\mathcal{P}(\varphi|_\eta))$

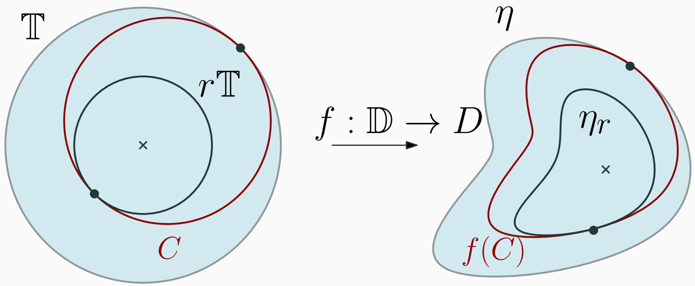
SLE/GFF counterpart (imaginary geometry): The flow-lines of $e^{i\sqrt{\kappa}GFF/2}$ is an SLE_κ curve. Conditioning on the flow-line, φ_0 is an 0-boundary GFF.

Application: Equipotential energy monotonicity



Corollary [infinite curve]

Let $r > 0$, we have $I^L(\eta^r) \le I^L(\eta)$.



Corollary [bounded curve]

For $0 < r < 1$, we have $I^L(\eta_r) \le I^L(f(C)) \le I^L(\eta)$.

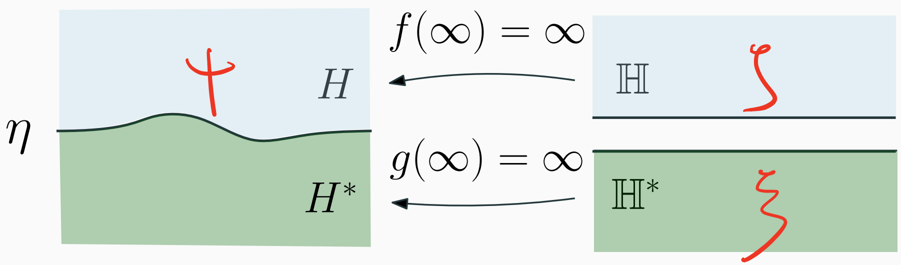
Complex identity

Corollary (Complex identity)

Let ψ be a complex-valued function on $\mathbb{C}$ with finite Dirichlet energy and $\operatorname{Im}\psi \in C^0(\hat{\mathbb{C}})$. *Let η be a flow-line of the vector field e^{ψ} and f, g the conformal maps associated to η. Then we have*

$$\mathcal{D}_{\mathbb{C}}(\psi) = \mathcal{D}_{\mathbb{H}}(\zeta) + \mathcal{D}_{\mathbb{H}^*}(\xi),$$

where $\zeta = \psi \circ f + \overline{\log f'}$, $\xi = \psi \circ g + \overline{\log g'}$.



A (very loose) dictionary

SLE/GFF with $\gamma = \sqrt{\kappa} \to 0$	Finite energy
SLE_κ loop.	Finite energy Jordan curve, η.
Free boundary GFF $\gamma\Phi$ on $\mathbb{H}$ (on $\mathbb{C}$).	$2u$, $u \in \mathcal{E}(\mathbb{H})$ (2φ, $\varphi \in \mathcal{E}(\mathbb{C})$).
γ-LQG on quantum plane $\approx e^{\gamma\Phi}dz^2$.	$e^{2\varphi}dz^2$, $\varphi \in \mathcal{E}(\mathbb{C})$.
γ-LQG on quantum half-plane on $\mathbb{H}$	$e^{2u}dz^2$, $u \in \mathcal{E}(\mathbb{H})$.
γ-LQG boundary measure on $\mathbb{R} \approx e^{\gamma\Phi/2}dx$	$e^{u(x)}dx$, $u \in H^{1/2}(\mathbb{R})$.
SLE_κ cuts an independent quantum plane into independent quantum half-planes.	Finite energy η cuts $\varphi \in \mathcal{E}(\mathbb{C})$ into $u \in \mathcal{E}(\mathbb{H}), v \in \mathcal{E}(\mathbb{H}^*)$ and $I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi) = \mathcal{D}_{\mathbb{H}}(u) + \mathcal{D}_{\mathbb{H}^*}(v)$.
Quantum zipper: isometric welding of independent γ-LQG measures on $\mathbb{R}$ produces SLE_κ.	Isometric welding of $e^u dx$ and $e^v dx$, $u, v \in H^{1/2}(\mathbb{R})$ produces a finite energy curve.
γ-LQG chaos w.r.t. Minkowski content equals the pushforward of γ-LQG measures on $\mathbb{R}$.	$e^{\varphi\|_\eta}\|dz\|$, $\varphi\|_\eta \in H^{1/2}(\eta)$, equals the pushforward of $e^u dx$ and $e^v dx$, $u, v \in H^{1/2}(\mathbb{R})$.
Bi-infinite flow-line of $e^{i\Phi/\chi} \approx e^{i\gamma\Phi/2}$ is an SLE_κ loop measurable wrt. Φ.	Bi-infinite flow-line of $e^{i\varphi}$ is a finite energy curve $\mathcal{D}_{\mathbb{C}}(\varphi) = I^L(\eta) + \mathcal{D}_{\mathbb{C}}(\varphi_0)$.
Mating of trees	Complex identity $\Leftrightarrow$ welding+flow-line.