On the regular side of Random conformal geometry

through the lens of large deviations

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smooth, analytic — more regular — fractal — discrete.

Energy minimizers

Finite energy deterministic

Large deviations

Continuum random

Scaling limit

discrete random

SLE, CLE

LQG GFF

Brownian Maps.

random planar map

statistical mechanics lattice models

Constant function

C^∞

Finite Dirichlet energy function

$\frac{1}{2}$-Hölder

Brownian motion

$(\frac{1}{2}-\varepsilon)$ Hölder-continuous

Simple random walk

W is a.c. $\quad I(W) := \frac{1}{2}\int_0^\infty \dot{W}^2(t)\,dt$, $\quad I(W) = \infty$ otherwise.

Cauchy-Schwarz

$$W(t) - W(s) = \int_s^t \dot{W}(r)\,dr \le \sqrt{\int_s^t \dot{W}^2(r)\,dr} \cdot \sqrt{t-s} \le \sqrt{I(W)}\,\sqrt{t-s}$$

Lecture 1. (Chordal SLE_{0+})

- Loewner energy and SLE_{0+} large deviations

  Energy reversibility from SLE reversibility

- Multichordal Loewner energy

  Multiple SLE_0 $\longleftrightarrow$ rational functions

  Multiple SLE_{0+} large deviations

  BPZ equation

  $\log \det_\zeta \Delta$

Lecture 2 (SLE$_{0+}$ loops, GFF, LQG, MoT)

- Loewner energy and equivalent descriptions

  Dirichlet energy of $\log|f'|$

  Weil-Petersson quasicircles

- Welding, flowline, complex identity

  A dictionary

  Quantum zipper - imaginary geometry

  Mating of trees

Lecture 3 (Radial SLE_∞)

- Radial SLE_∞ Large deviations
  Loewner-Kufarev energy.

- Foliation of Weil-Petersson quasicircles
  and energy duality

References

1 [1] **Y. Wang** The energy of a deterministic Loewner chain: Reversibility and interpretation via SLE_{0+} JEMS 21(7) (2019)

2 [2] **S. Rohde, Y. Wang** The Loewner energy of loops and regularity of driving functions IMRN (2019)

2 [3] **Y. Wang** Equivalent Descriptions of the Loewner Energy Invent. Math. 218(2) (2019)

2 [4] **F. Viklund, Y. Wang** Interplay between Loewner and Dirichlet energies via conformal welding and flow-lines GAFA 30 (2020)

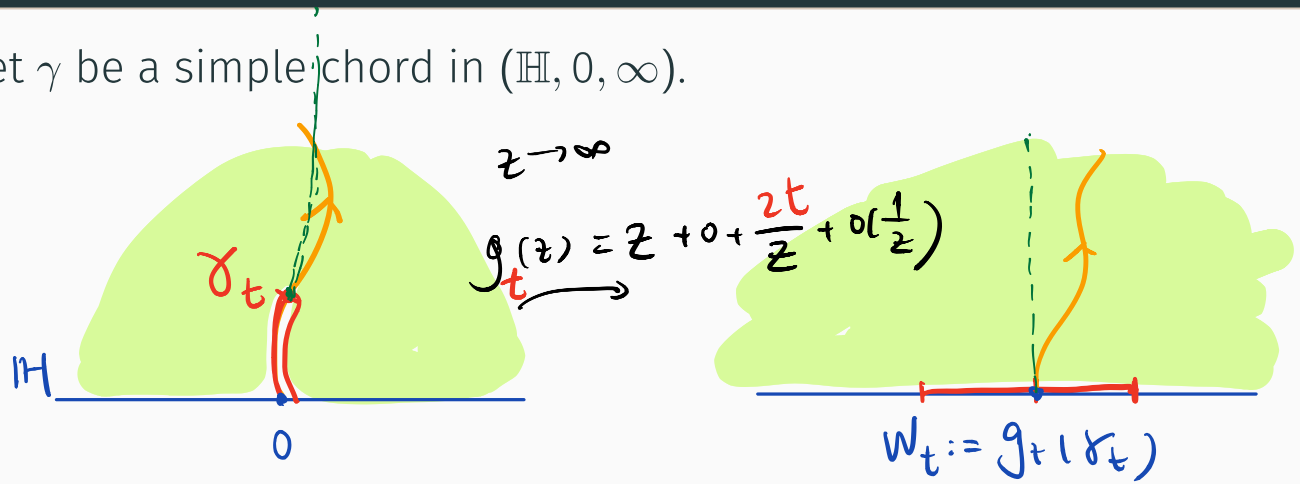
3 [5] **M. Ang, M. Park, Y. Wang** Large deviations of radial SLE_∞ EJP 25(102) (2020)

1 [6] **E. Peltola, Y. Wang** Large deviations of multichordal SLE0+, real rational functions, and zeta-regularized determinants of Laplacians Preprint (2020)

3 [7] **F. Viklund, Y. Wang** The Loewner-Kufarev Energy and Foliations by Weil-Petersson Quasicircles Available soon (2020)

Chordal Loewner chains

Let γ be a simple chord in $(\mathbb{H}, 0, \infty)$.

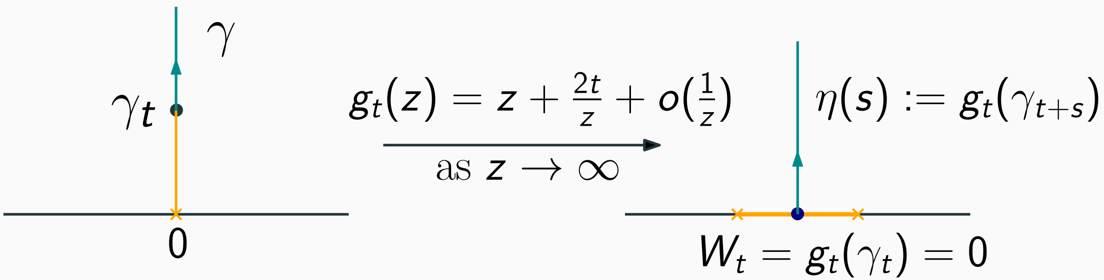


- γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of γ.
- $W_0 = 0$, W is continuous.
- The curve γ can be recovered from W using Loewner's differential equation: $\partial_t g_t(z) = 2/(g_t(z) - W_t)$, $g_0(z) = z$.
- We say that γ is the **chordal Loewner curve** driven by W.

Introduced by *[Loewner '23 Math. Ann.]*.

A trivial example

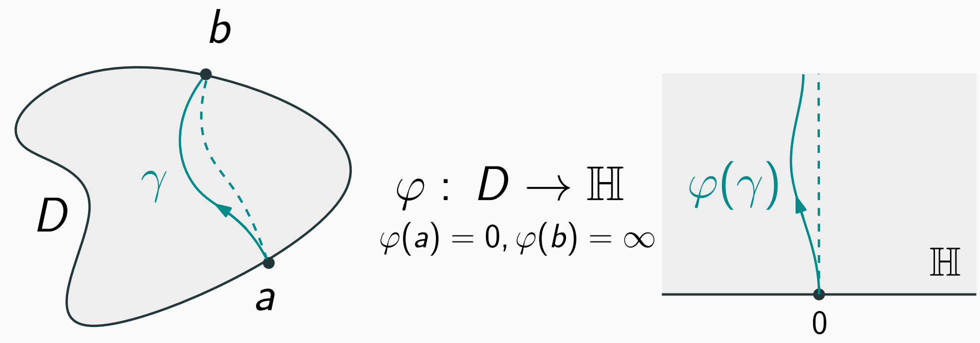
- If $W \equiv 0$, then $\gamma = i\mathbb{R}_+$.



- When the curve is driven by $W = \sqrt{\kappa}B$ where B is 1-d Brownian motion, the curve generated is the **Schramm-Loewner Evolution of parameter** κ (SLE_κ). It is introduced by *[Schramm '00 Isr.JM]*.

The chordal Loewner energy (W. [1])

$D \subset \mathbb{C}$ a simply connected domain, a, b are two boundary points of D.



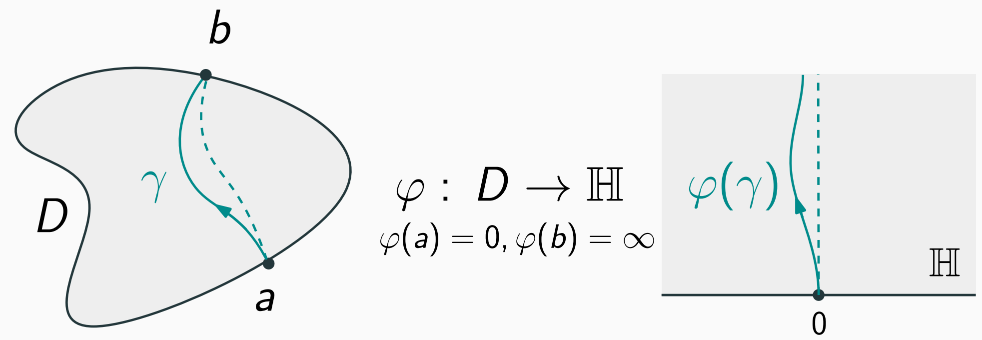
We define the **Loewner energy of a simple chord γ in** (D, a, b) to be

$$I_{D,a,b}(\gamma) := I_{\mathbb{H},0,\infty}(\varphi(\gamma)) := I(W) := \frac{1}{2}\int_0^\infty W'(t)^2 dt$$

$$= \sup_{0=t_0<t_1<\cdots<t_n} \frac{1}{2}\sum_{i=1}^{n} \frac{(W(t_i) - W(t_{i-1}))^2}{t_i - t_{i-1}}$$

where W is the driving function of $\varphi(\gamma)$.

Properties



- For $c > 0$, we have $I_{\mathbb{H},0,\infty}(\gamma) = I_{\mathbb{H},0,\infty}(c\gamma)$.
  $\Rightarrow$ The Loewner energy is well-defined in (D, a, b).
- $I_{D,a,b}(\gamma) = 0 \Leftrightarrow W \equiv 0 \Leftrightarrow \gamma$ is the *hyperbolic geodesic* connecting a and b.

  (SLE_κ has H. dim $1+\frac{\kappa}{8}$)
- $I_{D,a,b}(\gamma) < \infty$, then γ is a rectifiable *[Friz & Shekhar PTRF '15]*.
- If and only if characterization by Weil-Petersson quasicircles *[W. Invent. '19]*.

SLE$_\kappa$ vs. Loewner energy

"$\sqrt{\kappa}B$ has the distribution on $C^0(\mathbb{R}_+, \mathbb{R})$ with density

$$p(W) \propto \exp(-I(W)/\kappa)\mathcal{D}W."$$

However, $I(B) = \infty$ *with probability* 1.

The Schilder's theorem states that $I(W)$ is also the **large deviation rate function** for Brownian motion $\sqrt{\kappa}B$ as $\kappa \to 0$. Loosely speaking,

$$\text{"}\mathrm{P}(\sqrt{\kappa}B \text{ stays close to } W) \approx \exp\left(-\frac{I(W)}{\kappa}\right).\text{"}$$

Large deviations of chordal SLE

[W., JEMS] [Peltola-W. '20]

Assume $D = \mathbb{D}$. We endow $\mathcal{X}(D; a, b)$ with the ~~product~~ topology induced from the Hausdorff metric.

Theorem

The family of laws $(\mathbb{P}^\kappa)_{\kappa>0}$ of the chordal SLE_κ curves γ^κ satisfies the following LDP with good rate function I_D:

For any subset A of $\mathcal{X}(D; a, b)$, we have

$$-\inf_{\gamma\in A^o} I_D(\gamma) \le \varliminf_{\kappa\to 0+} \kappa \log \mathbb{P}^\kappa[\gamma^\kappa \in A^o]$$
$$\le \varlimsup_{\kappa\to 0+} \kappa \log \mathbb{P}^\kappa[\gamma^\kappa \in \overline{A}] \le -\inf_{\gamma\in\overline{A}} I_D(\gamma)$$

and the sub-level set $(I_D)^{-1}[0, c]$ is compact for any $c \ge 0$.

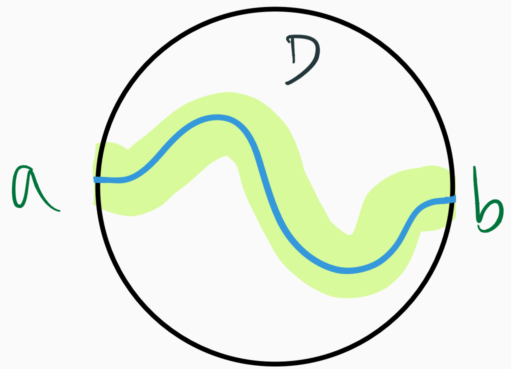
Fix $\gamma \in \mathcal{X}(D, a, b)$, let $A^\varepsilon = \{\gamma' \mid \mathrm{dist}(\gamma, \gamma') < \varepsilon\}$

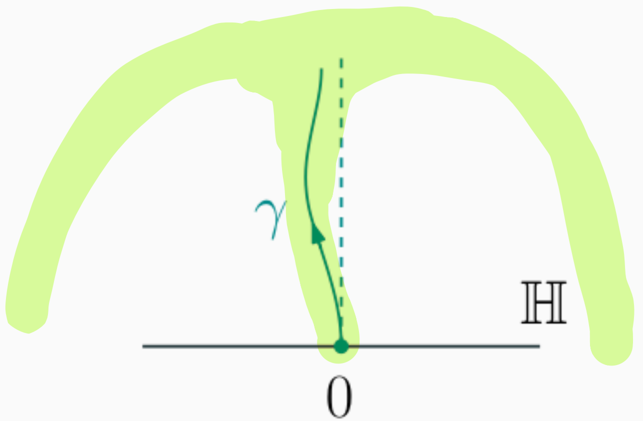
with countable exception of ε

$$-\inf_{\gamma \in A^o} I_D(\gamma) = -\inf_{\gamma \in \bar{A}} I_D(\gamma)$$

LDP $\Rightarrow$ $\mathbb{P}(SLE_\kappa \in A^\varepsilon) \underset{\kappa \to 0}{\sim} \exp\left(-\frac{1}{\kappa} \inf_{\gamma' \in A} I_D(\gamma')\right)$

$$\approx \exp\left(-\frac{I_D(\gamma)}{\kappa}\right)$$





Consequence: Energy reversibility

Theorem (Energy reversibility [W. JEMS] **)**

We have $I_{D,a,b}(\gamma) = I_{D,b,a}(\gamma)$. *It is equivalent to $I_{\mathbb{H},0,\infty}(\gamma) = I_{\mathbb{H},0,\infty}(-1/\gamma)$.*

This deterministic result is based on (combined with the large deviation result):

Theorem (SLE reversibility [Zhan '08 AOP])

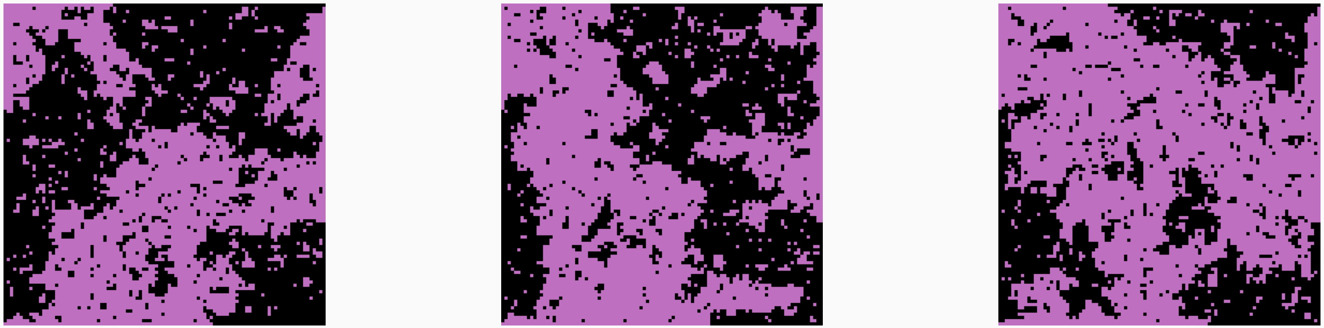
For $\kappa \le 4$, the law of the trace of SLE_κ in (D, a, b), is the same as the law of SLE_κ in (D, b, a).

In fact, the decay rate as $\kappa \to 0$ of the probability of SLE_κ stays close to γ is the same as the decay rate of being close to $-1/\gamma$.

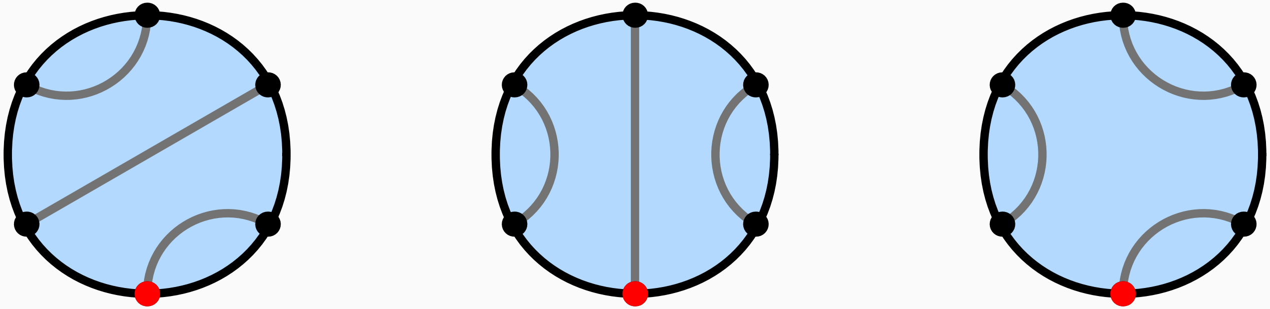
A proof without SLE will appear in Lecture 2.

Multichordal SLE: Examples

Simulations of multichordal SLE_3:

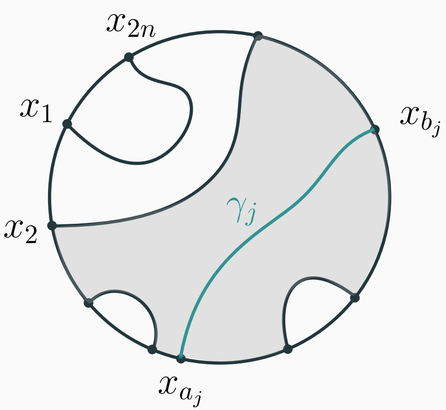


The corresponding link pattern α:



Multichordal SLE$_\kappa$: Characterization by conditioning

Fix $\alpha = \{\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_n, b_n\}\}$ a planar link pattern. There are $C_n = \frac{1}{n+1}\binom{2n}{n}$ of those link patterns.



Multichordal SLE is characterized as γ_j being the chordal SLE$_\kappa$ in $\hat{D}_j$ (gray) for all $j = 1, \dots, n$.

Cardy, Werner, Dubédat, Lawler, Kozdron, Bauer, Bernard, Kytölä, Sheffield, Miller, Wu, Peltola, Beffara, etc. The existence and uniqueness are obtained when $\kappa \in (0, 4)$ (see next slide).

When $\kappa = 0$, multichordal SLE$_0$ is naturally defined as a *geodesic multichord*. We will give a proof for the existence later.

Multichordal SLE$_\kappa$: Radon-Nikodym derivatives

ind. SLE $\exp\left(-\sum \frac{c(\kappa)}{2} \mu^{\text{loops}}\left(\underbrace{\ell \mid \ell \cap \gamma_i \neq \emptyset}\right)\right)$ " $\mathbb{E}$(# chords touched)

Multichordal SLE can be obtained by weighting n independent SLE$_\kappa$

$$\exp\left(\frac{c(\kappa)}{2} m_D(\gamma_1, \ldots, \gamma_n)\right),$$

for $\kappa < \frac{8}{3}$

$(c < 0)$

where

$$c(\kappa) = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa} \sim_{\kappa \to 0+} -\frac{24}{\kappa},$$

and $m_D(\gamma_1, \ldots, \gamma_n) \geq 0$ is expressed in terms of the *Brownian loop measure* introduced by Lawler, Schramm, and Werner.

We have, $m_D(\gamma) = 0$ if $n = 1$ and m_D is conformally invariant.

$m_D(\gamma_1, \ldots, \gamma_n) = \sum_{p=2}^{n} \mu_D^{\text{loop}}(\ell \mid \ell \cap \gamma_i \neq \emptyset$ for at least p chords $\gamma_i)$

$\in [0, \infty)$ " $\mathbb{E}$(# chords $- 1$)

Geodesic multichord

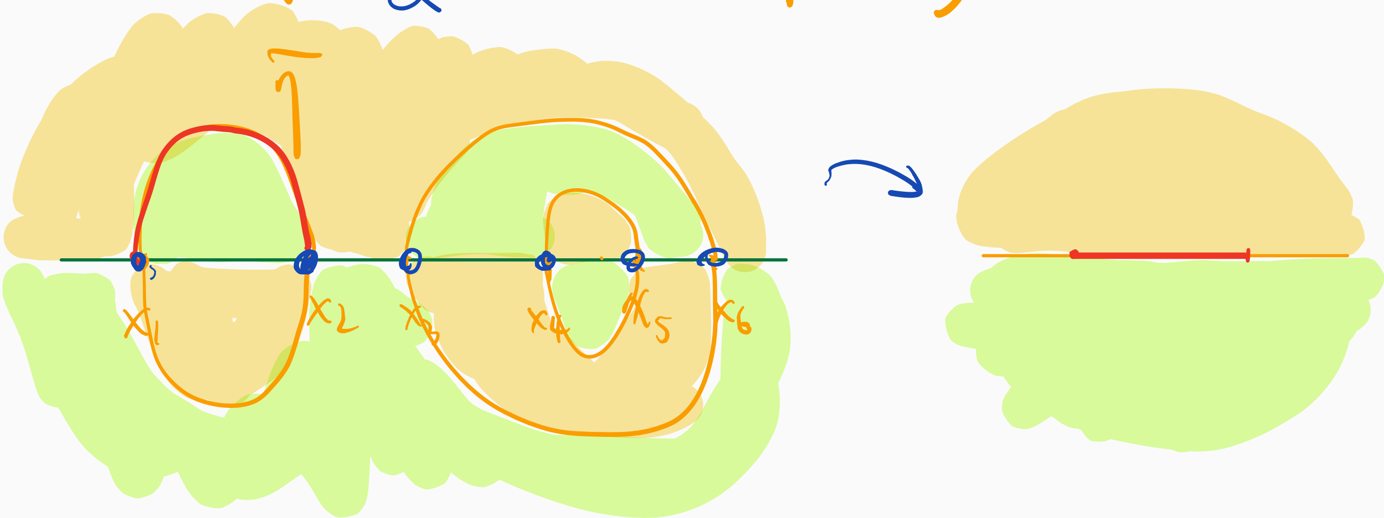
Multiple SLE,

We consider $D = \mathbb{H}$.

Theorem

Let $(\eta_1, \cdots, \eta_n)$ be a geodesic multichord in $\mathcal{X}_\alpha(\mathbb{H}; x_1, \ldots, x_{2n})$. The union of η_js, its complex conjugate, and the real line is the real locus of a rational function of degree $n+1$ with critical points $\{x_1, \ldots, x_{2n}\}$.

$f := \frac{P}{Q}$ $\qquad$ $f^{-1}(\mathbb{R})$



Catalan number

$\frac{P}{Q} \Rightarrow \mathrm{Span}_{\mathbb{C}}\{P, Q\} \subset \mathbb{C}^{n+2} = \text{Polynomial } (n+1)$. $\frac{aP+bQ}{cP+dQ}$

Span $\{aP+bQ,\ cP+dQ\}$

Theorem (Goldberg, Adv. Math. '91)

Let $z_1, \ldots, z_{2n}$ be $2n$ distinct complex numbers. There are at most C_n rational functions of degree $n+1$ with critical points $z_1, \ldots, z_{2n}$ up to $PSL(2, \mathbb{C})$ post-composition (conformal automorphisms of $\hat{\mathbb{C}}$).

Assuming the existence of geodesic multichord in $\mathcal{X}_\alpha(\mathbb{H}; x_1, \ldots, x_{2n})$:

Corollary $\Delta(P, Q) := P'Q - Q'P$

If all critical points of a rational function are real, then it is a real rational function up to $PSL(2, \mathbb{C})$ post-composition.

This result is first proved by [Eremenko-Gabrielov, Annals '02].

Existence is given by a minimizer Multichordal Loewner potential

Multichordal Loewner potential

Let $\overline{\gamma} := (\gamma_1, \ldots, \gamma_n)$. The *Loewner potential* of $\overline{\gamma}$ is given by

Allows relating different bdry data

$$\mathcal{H}_D(\overline{\gamma}) := \frac{1}{12}\sum_{j=1}^{n} I_D(\gamma_j) + m_D(\overline{\gamma}) - \frac{1}{4}\sum_{j=1}^{n} \log P_{D;x_{a_j},x_{b_j}},$$

Nice formula using $\det\Delta$

where $P_{D;x,y}$ is the *Poisson excursion kernel*, defined via

$$P_{D;x,y} := |\varphi'(x)||\varphi'(y)|P_{\mathbb{H};\varphi(x),\varphi(y)}, \qquad \text{and} \qquad P_{\mathbb{H};x,y} := |y-x|^{-2},$$

and where $\varphi\colon D \to \mathbb{H}$ is a conformal map.

When $n = 1$,

$$\mathcal{H}_D(\gamma) = \frac{1}{12} I_D(\gamma) - \frac{1}{4}\log P_{D;a,b}.$$

Minimized for fixed (D, a, b) by hyperbolic geodesic.

Properties of potential

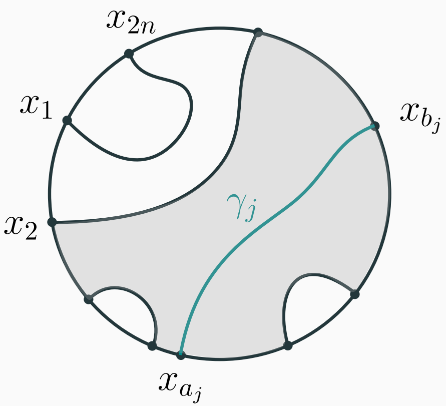
Lemma

The multichord $\overline{\gamma}$ has finite potential in D if and only if $I_D(\gamma_j) < \infty$ for all $j \in \{1, \ldots, n\}$ and all $\gamma_1, \ldots, \gamma_n$ are pairwise disjoint.

Lemma

If $\mathcal{H}_D(\overline{\gamma}) < \infty$, then there exists $K \in [1, \infty)$, depending only on $\mathcal{H}_D(\overline{\gamma})$, and a K-quasiconformal map φ such that $\gamma_j = \varphi(\rho_j)$ for all $j \in \{1, \ldots, n\}$, $\varphi(D) = D$, and φ extends continuously to $\overline{D}$ and equals the identity function on ∂D.

Properties of potential, cont'd



Lemma (Cascade relation of $\mathcal{H}$)

For each $j \in \{1, \ldots, n\}$, we have

$$\mathcal{H}_D(\overline{\gamma}) = \mathcal{H}_{\hat{D}_j}(\gamma_j) + \mathcal{H}_D(\gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1} \ldots, \gamma_n).$$

Corollary

Any minimizer of $\mathcal{H}_D$ in $\mathcal{X}_\alpha(D; x_1, \ldots, x_{2n})$ is a geodesic multichord.

Using the quasiconformal map:

Corollary Existence

There exists a geodesic multichord $\overline{\eta}$ in $\mathcal{X}_\alpha(D; x_1, \ldots, x_{2n})$.

Multichordal Loewner energy

We also define the minimal potential:

$$\mathcal{M}_D^\alpha(x_1, \ldots, x_{2n}) := \min_{\overline{\gamma} \in \mathcal{X}_\alpha(D; x_1, \ldots, x_{2n})} \mathcal{H}_D(\overline{\gamma}) = \mathcal{H}_D(\overline{\eta}) > -\infty.$$

Note that the minimal potential depends on the marked points $x_1, \ldots, x_{2n} \in \partial D$ as well as on the link pattern α.

The *multichordal Loewner energy* of $\overline{\gamma} \in \mathcal{X}_\alpha(D; x_1, \ldots, x_{2n})$ is

$$I_D^\alpha(\overline{\gamma}) := 12(\mathcal{H}_D(\overline{\gamma}) - \mathcal{M}_D^\alpha(x_1, \ldots, x_{2n})) \geq 0.$$

When $n = 1$, this energy coincides with the Loewner energy I_D.

(Peltola - W)

Large deviations of Multichordal SLE_{0+}

Assume $D = \mathbb{D}$. We endow $\mathcal{X}_\alpha(D; x_1, \ldots, x_{2n})$ with the product topology induced from the Hausdorff metric.

Theorem

The family of laws $(\mathbb{P}^\kappa_\alpha)_{\kappa>0}$ of the multichordal SLE$_\kappa$ *curves $\overline{\gamma}^\kappa$ satisfies the following LDP in $\mathcal{X}_\alpha(D; x_1, \ldots, x_{2n})$ with good rate function I^α_D:*

Proof:
- LDP of single SLE_{0+}
- $\exp\left(\frac{c(\kappa)}{2} m_D(\overline{\gamma})\right) \sim \exp\left(-\frac{12 m_D(\overline{\gamma})}{\kappa}\right)$.

Driving function of a geodesic multichord

Let $\overline{\eta}$ be the minimizer of $\mathcal{H}_{\mathbb{H}}$ in $\mathcal{X}_\alpha(\mathbb{H}; x_1, \ldots, x_{2n})$ and

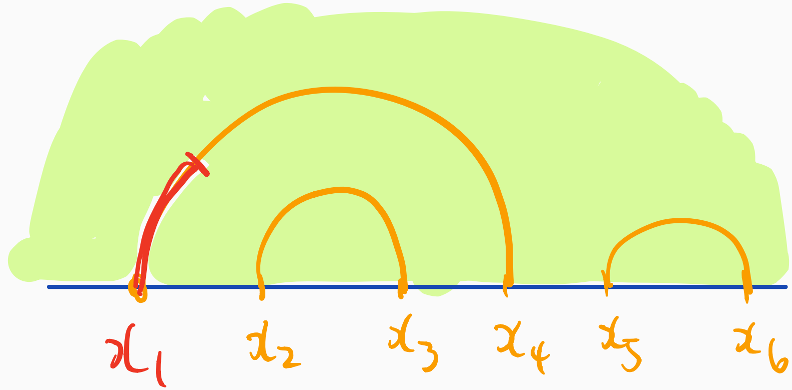
$$\mathcal{U}(x_1, \ldots, x_{2n}) := 12\mathcal{M}^\alpha_{\mathbb{H}}(x_1, \ldots, x_{2n}) = 12\mathcal{H}_{\mathbb{H}}(\overline{\eta}).$$

Theorem

For each $j \in \{1, \ldots, n\}$, the Loewner driving function W of the chord η_j from x_{a_j} to x_{b_j} and the time evolutions $V^i_t = g_t(x_i)$ of the other marked points satisfy the differential equations

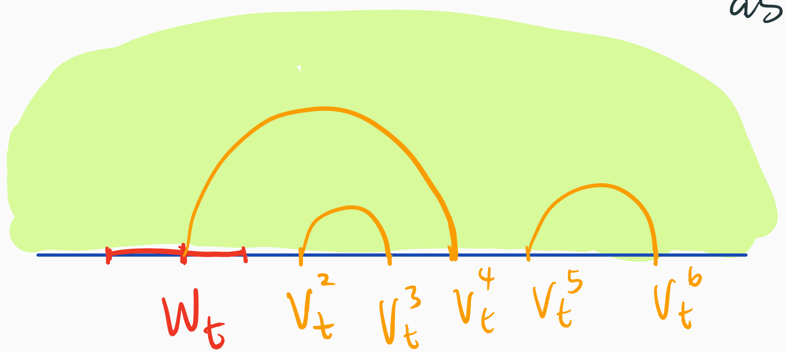
$$\begin{cases} \partial_t W_t = -\partial_{a_j}\mathcal{U}(V^1_t, \ldots, V^{a_j-1}_t, W_t, V^{a_j+1}_t, \ldots, V^{2n}_t), \qquad W_0 = x_{a_j}, \\ \partial_t V^i_t = \dfrac{2}{V^i_t - W_t}, \qquad V^i_0 = x_i, \quad \text{for } i \neq a_j, \end{cases}$$

for $0 \le t < T$, where T is the lifetime of the solution and $(g_t)_{t\in[0,T]}$ is the Loewner flow generated by η_j.



$$\downarrow \quad g_t(z) = z + \frac{2t}{z} + o\left(\frac{1}{z}\right)$$

as $z \to \infty$



$$V_t^i = g_t(x_i)$$

Semi-classical limit of BPZ equations

Recall that $\mathcal{U} := 12\mathcal{M}_{\mathbb{H}}^{\alpha}$. Let $\mathcal{Z}_\alpha(\mathbb{H}; x_1, \ldots, x_{2n}; \kappa)$ be the partition function of multiple SLE_κ in $\mathcal{X}_\alpha(\mathbb{H}; x_1, \ldots, x_{2n})$.

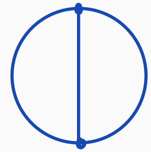
Theorem

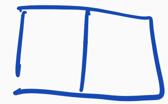
We have $\mathcal{U} = \lim_{\kappa \to 0} -\kappa \log \mathcal{Z}_\alpha(\kappa)$. For each $j \in \{1, \ldots, 2n\}$, we have

$$\frac{1}{2}(\partial_j \mathcal{U}(x_1, \ldots, x_{2n}))^2 - \sum_{i \neq j} \frac{2}{x_i - x_j} \partial_i \mathcal{U}(x_1, \ldots, x_{2n}) = \sum_{i \neq j} \frac{6}{(x_i - x_j)^2}.$$

- *It has been pointed out by Bauer-Bernard-Kytölä.*
- *This equation does not depend on α. We may wonder how many solutions are there, and what do they represent.*
- *Our proof is deterministic, by analysing directly the minimal potential.*
- *Can one relate $\mathcal{U}$ to the associated rational functions more quantitatively?*

Identity with Loewner potential





Theorem

For any smooth multichord $\overline{\gamma}$ in a bounded smooth domain D, we have

$$\mathcal{H}_D(\overline{\gamma}) = \log \det_\zeta \Delta_D - \sum_C \log \det_\zeta \Delta_C + n\lambda,$$

where the sum is taken over all connected components C of $D \setminus \bigcup_i \gamma_i$, and $\lambda \in \mathbb{R}$ is a universal constant.

We use results on $\det_\zeta \Delta$ for curvilinear domains (piecewise smooth boundary allowing corners) *[Nursultanov-Rowlett-Sher, '19]*.

The relation between $\log \det_\zeta \Delta$ and SLE was observed by Dubédat, Friedrich, Konstevich, Suhov, etc.

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- Multichordal Loewner energy

  Multiple SLE_0 and rational functions

  Multiple SLE_{0+} large deviations

  BPZ equation

  $\log \det_\zeta \Delta$ expression