Quasiconformal deformation of Loewner driving function and variation of the Loewner energy

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Overview

- Quasiconformal deformation
- Chordal Loewner driving function
- Variation of the Loewner energy
- Consequences
  - → Optimal curve and piecewise geodesic property.
  - → Variation of Brownian loop measure

Quasiconformal deformation

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$$\delta I = \sum_{i,j} \int_{\mathbb{C}} T_{ij} (g^{-1} \delta g)_{ij}$$

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then (T_{ij}) is symmetric and trace free.

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in $(z, \bar{z})$ coordinate $(T_{ij}) = \begin{pmatrix} T_{zz} & 0 \\ 0 & \overline{T_{zz}} \end{pmatrix}$ $\quad g^{-1}\delta g = \begin{pmatrix} \bar{\mu} & s \\ s & \mu \end{pmatrix}$

$\Rightarrow$ $$\boxed{\delta I = 2\,\mathrm{Re}\left(\int_{\mathbb{C}} T_{zz} \cdot \mu(z)\, dg(z) \right)}$$

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T holomorphic SET

$$\Rightarrow \quad \boxed{\delta I = 2\,\mathrm{Re}\left[\int_{\mathbb{C}} T_{zz} \cdot \mu(z)\, dg(z)\right)}$$

Quasiconformal mapping

- Let $\mu : \mathbb{C} \to \mathbb{R}$ measurable, compactly supported and $\|\mu\|_\infty < \infty$
- Let $\varepsilon_0 > 0$, such that $\|\varepsilon_0 \mu\|_\infty < 1$
- for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$

Measurable Riemann mapping theorem:

$\exists!\ w_\varepsilon : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

such that
- $$\frac{\bar{\partial} w_\varepsilon}{\partial w_\varepsilon} = \varepsilon\mu$$
- $w_\varepsilon(z) - z = O(1)$ as $z \to \infty$
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(other solution is $PSL(2,\mathbb{C}) \circ w_\varepsilon$)

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- $\dfrac{\bar{\partial} w_\varepsilon}{\partial w_\varepsilon} = \varepsilon \mu$ $\qquad \dfrac{\bar{\partial} w_\varepsilon}{\partial w_\varepsilon} = \begin{cases} \varepsilon \mu(z) & \text{in } \mathbb{H} \\ \varepsilon \bar{\mu}(\bar{z}) & \text{in } \mathbb{H}^* \end{cases}$
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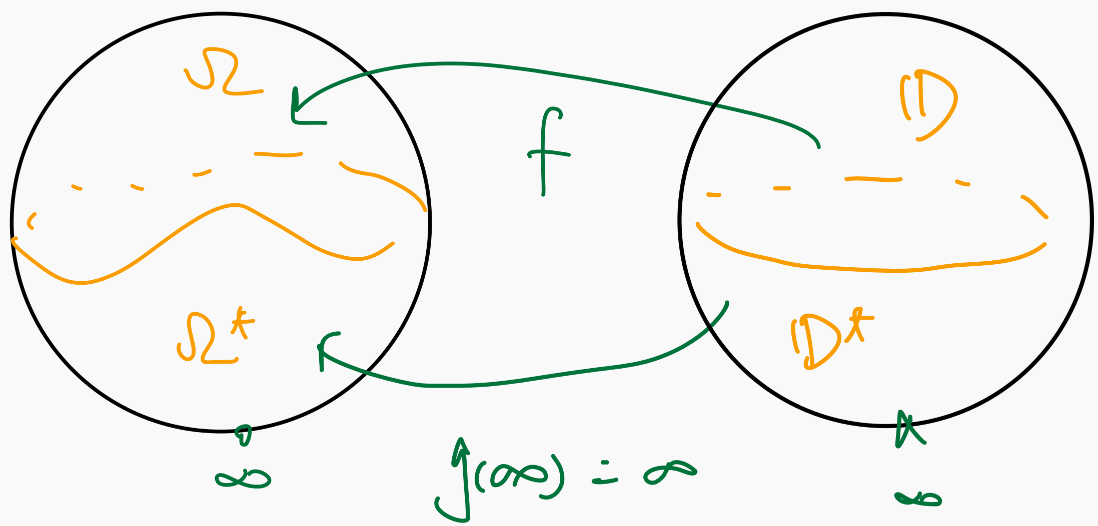
Loewner energy

$$I^L(\gamma, dz^2) = I^L(\gamma) := \frac{1}{\pi}\int_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 + \frac{1}{\pi}\int_{\mathbb{D}^*} \left|\frac{g''}{g'}\right|^2 + 4\log\left|\frac{f'(0)}{g'(\infty)}\right|$$



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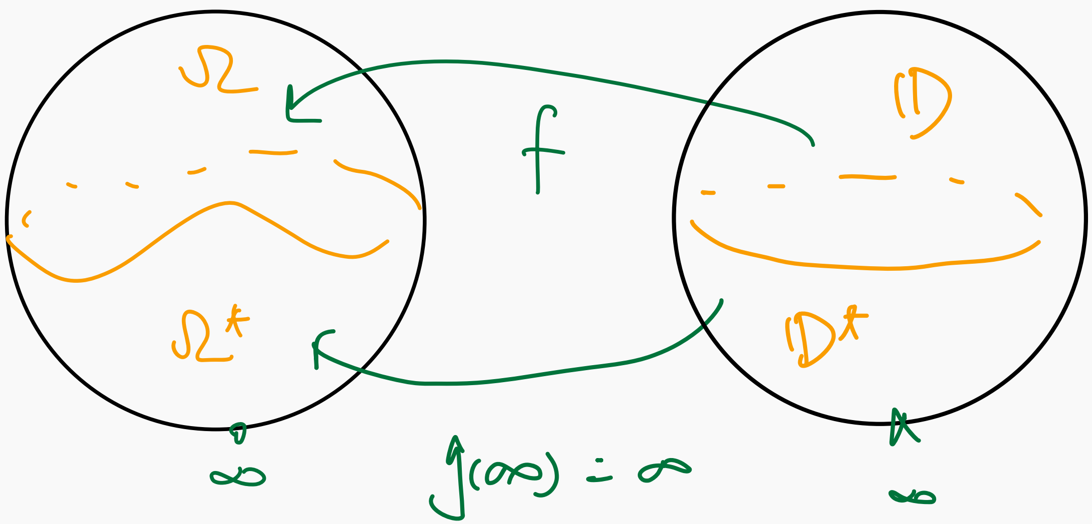

Thm (W. 19)

$$= \frac{1}{2}\int_{-\infty}^{\infty} \dot{\lambda}_t^2 \, dt$$

where $\lambda : \mathbb{R} \to \mathbb{R}$ is the Loewner driving of γ

Question

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} I^L(\gamma, w_\varepsilon^* dz^2) \overset{?}{=} 2\,\mathrm{Re}\int_{\mathbb{C}} \overline{T(z)}\, \mu(z)\, d^2 z$$

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Thm (Takhtajan-Teo)

When $\mu \in L^2(\Omega, \rho_\Omega)$ ← hyperbolic area measure

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} I^L(w_\varepsilon(\gamma)) = -\frac{4}{\pi}\,\mathrm{Re}\int_\Omega S(f^{-1})\,\mu(z)\,d^2z$$

Schwarzian

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Understand this identity using Loewner driving function

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Our goal:

Understand this identity using Loewner driving function

$\Rightarrow$ Justifies the identity between the 2 expressions of Loewner energy

$\Rightarrow$ Important to show I^L is a Kähler potential on $T_0(1)$ and almost all identities of I^L

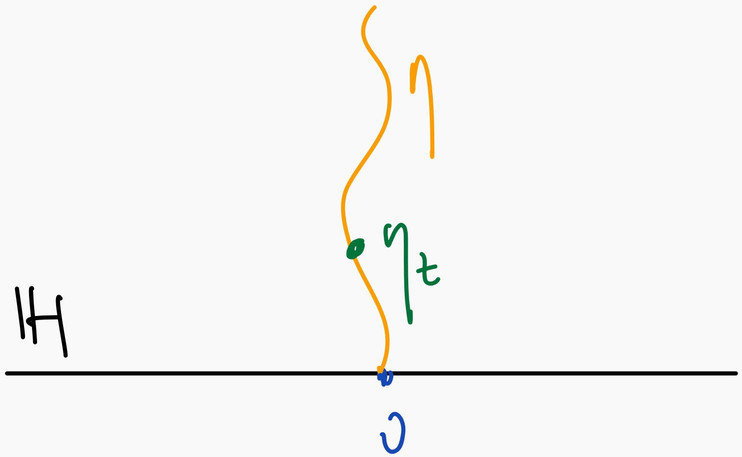
(Fredholm determinant, Renormalized volume)

$\Rightarrow$ Understand the role of Loewner driving function in Teichmüller theory

$\Rightarrow$ More applications

Deforming Loewner driving function

Chordal Loewner driving function

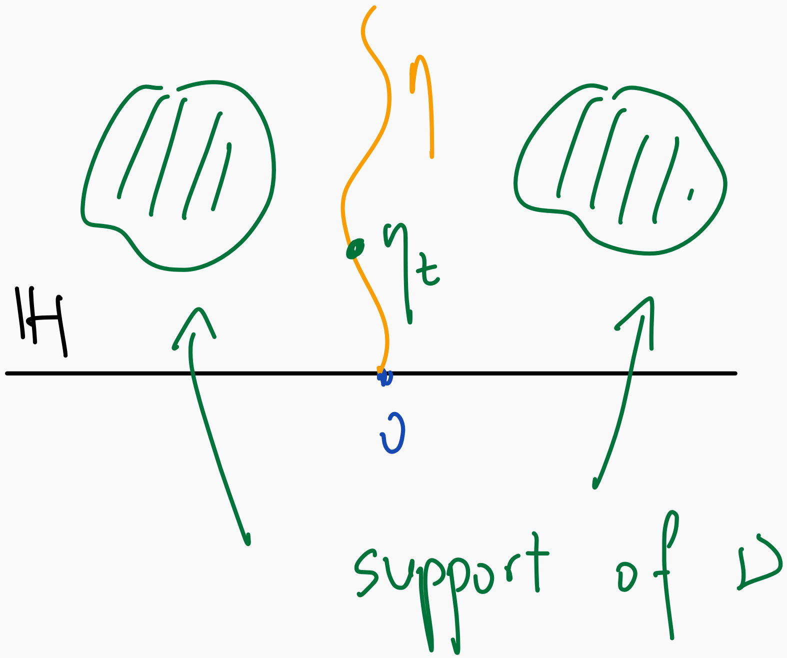


$\lambda : \mathbb{R}_+ \longrightarrow \mathbb{R}$

driving function of η

$\mathrm{hcap}(\eta[0,t]) = 2t = 2a_t$

Chordal Loewner driving function



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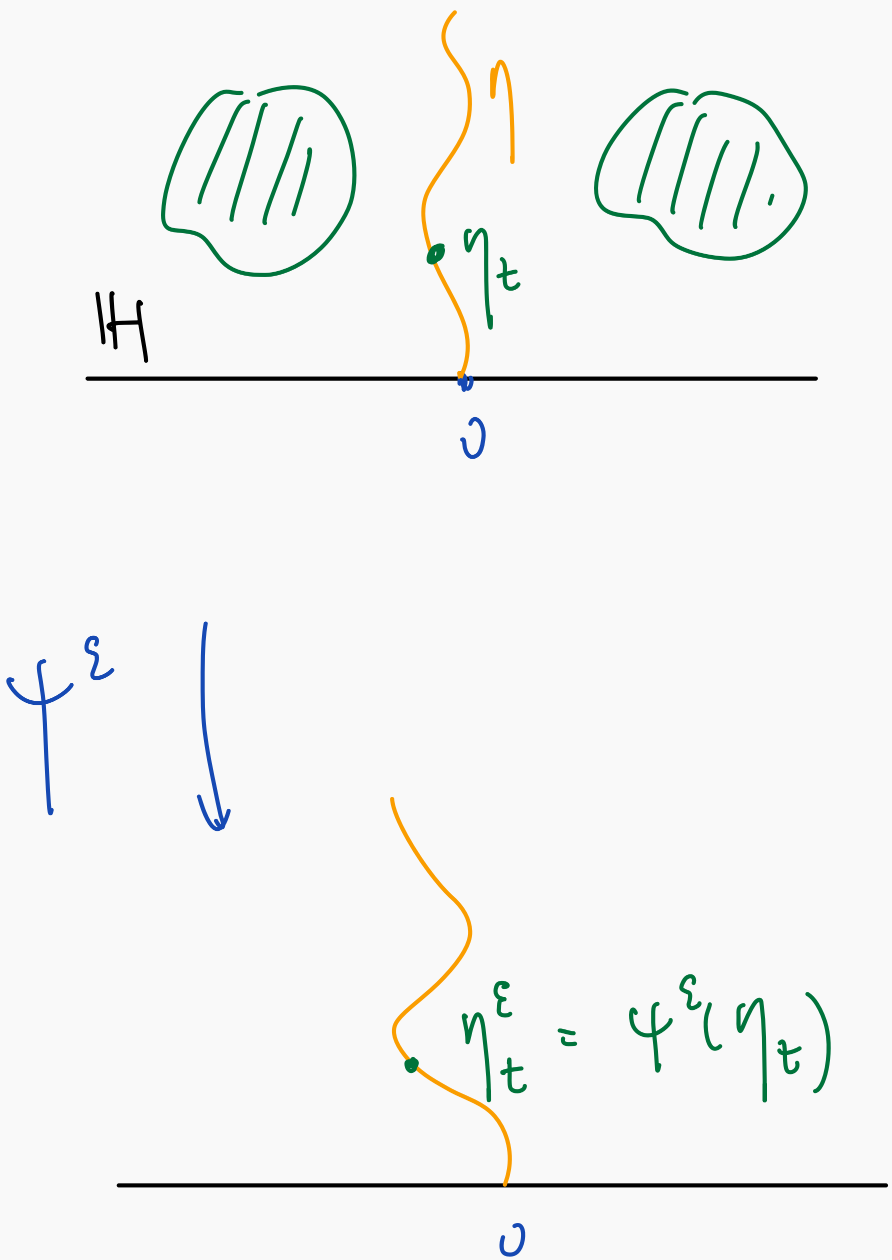

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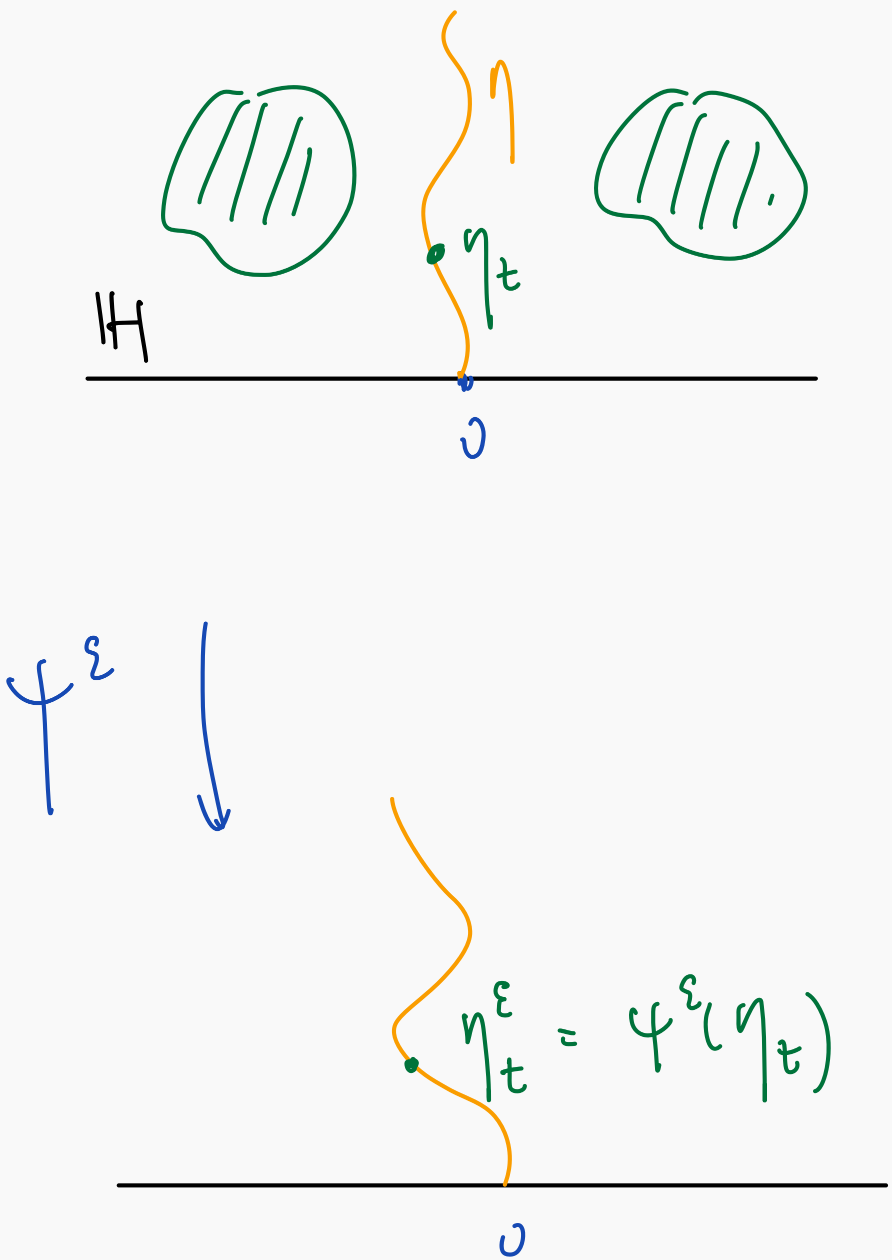

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λ^ε driving function of η^ε

$\mathrm{hcap}(\eta^\varepsilon_{[0,t]}) =: 2a^\varepsilon_t$

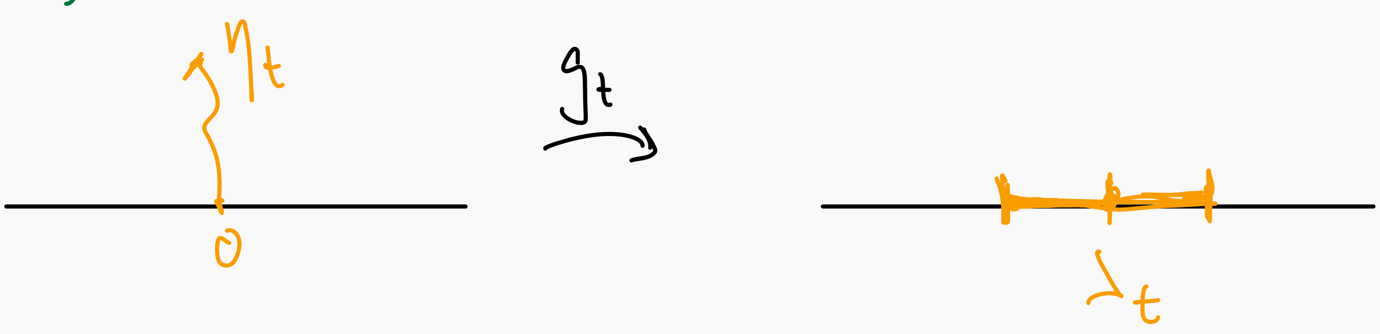
Deformation of λ

Thm (Sung - W.)

$$\left.\frac{\partial \lambda_t^{\varepsilon\nu}}{\partial \varepsilon}\right|_{\varepsilon=0} = -\frac{2}{\pi}\operatorname{Re}\int_{\mathbb{H}} \nu(z)\left(\frac{g_t'(z)^2}{g_t(z)-\lambda_t} - \frac{1}{z}\right)\mathrm{d}^2 z$$

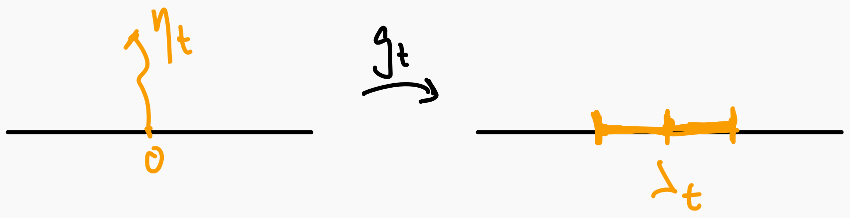
$$\left.\frac{\partial a_t^{\varepsilon\nu}}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{1}{\pi}\operatorname{Re}\int_{\mathbb{H}} \nu(z)\left(g_t'(z)^2 - 1\right)\mathrm{d}^2 z$$

(g_t) is the forward Loewner chain



Proof :

$$g_t(z) = z + \frac{\mathrm{hcap}(\eta[0,t])}{z} + o\left(\frac{1}{z}\right)$$



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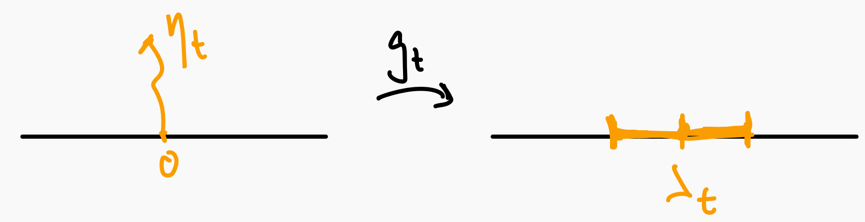
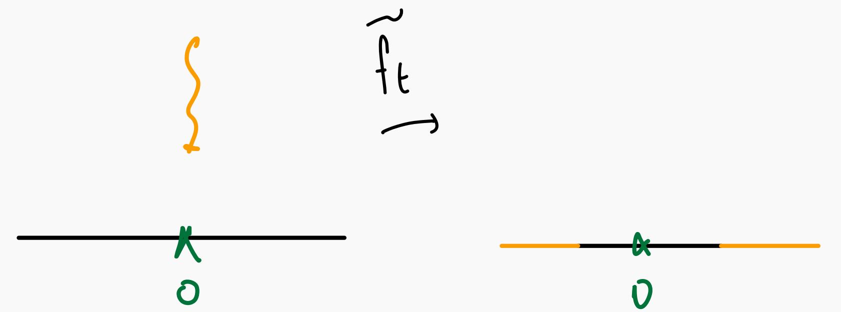


$$\tilde{f}_t(w) := \frac{-1}{g_t(-\frac{1}{z}) - \lambda_t}$$



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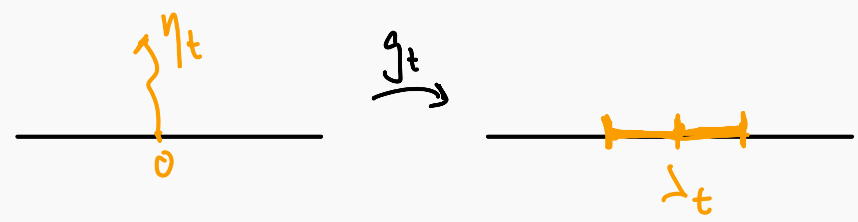

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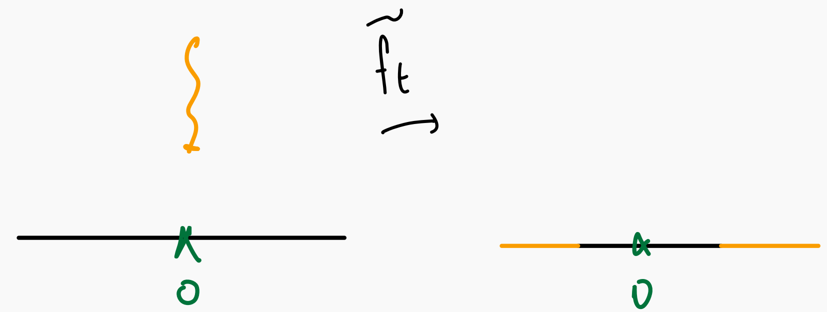

Lemma

$$\lambda_t = -\frac{1}{2} \boxed{\frac{\tilde{f}_t''(0)}{\tilde{f}_t'(0)}} \leftarrow \text{pre-Schwarzian} \quad N(f) = \frac{f''}{f'}$$

$$\mathrm{hcap}(\eta[0,t]) = -\frac{1}{6} \boxed{S(\tilde{f}_t)(0)} \leftarrow \text{Schwarzian} \quad S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

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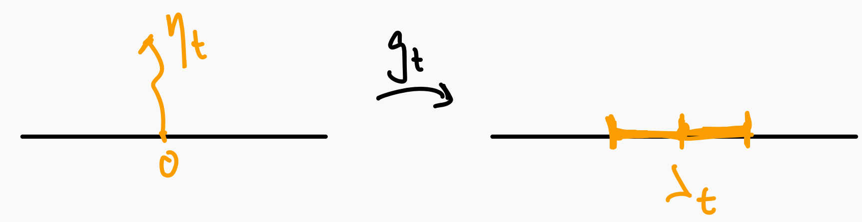

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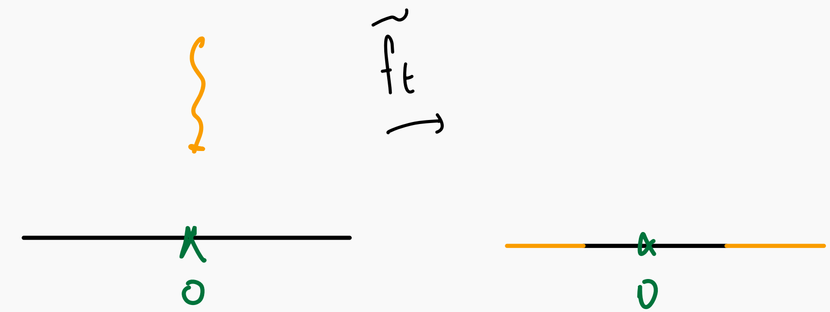

Lemma

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$$\operatorname{hcap}(\eta[0,t]) = -\frac{1}{6}\boxed{S(\tilde{f}_t)(0)}$$ ← Schwarzian $S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$

Proof : Expand $\tilde{f}_t$ near 0. ∎

Chain rules

$$\mathcal{N}(f\circ g) = \mathcal{N}(f)\circ g \cdot g' + \mathcal{N}(g) \qquad \forall f, g \text{ conformal}$$

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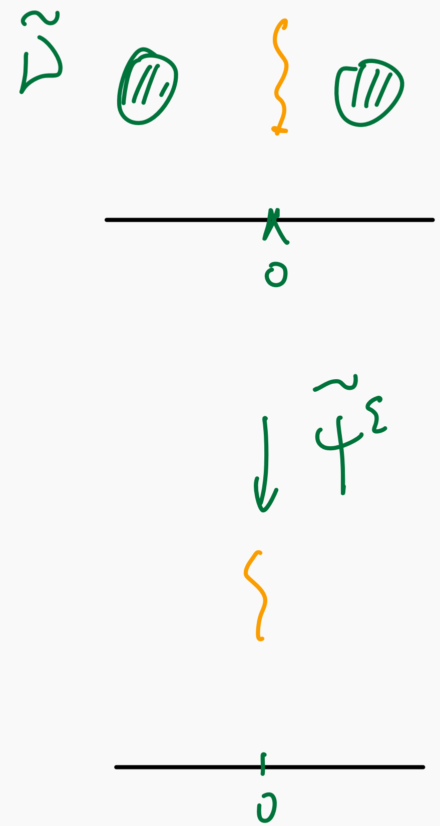

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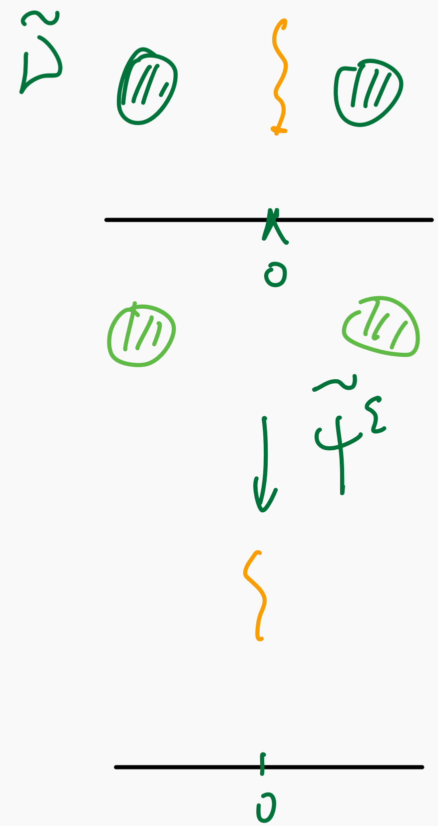

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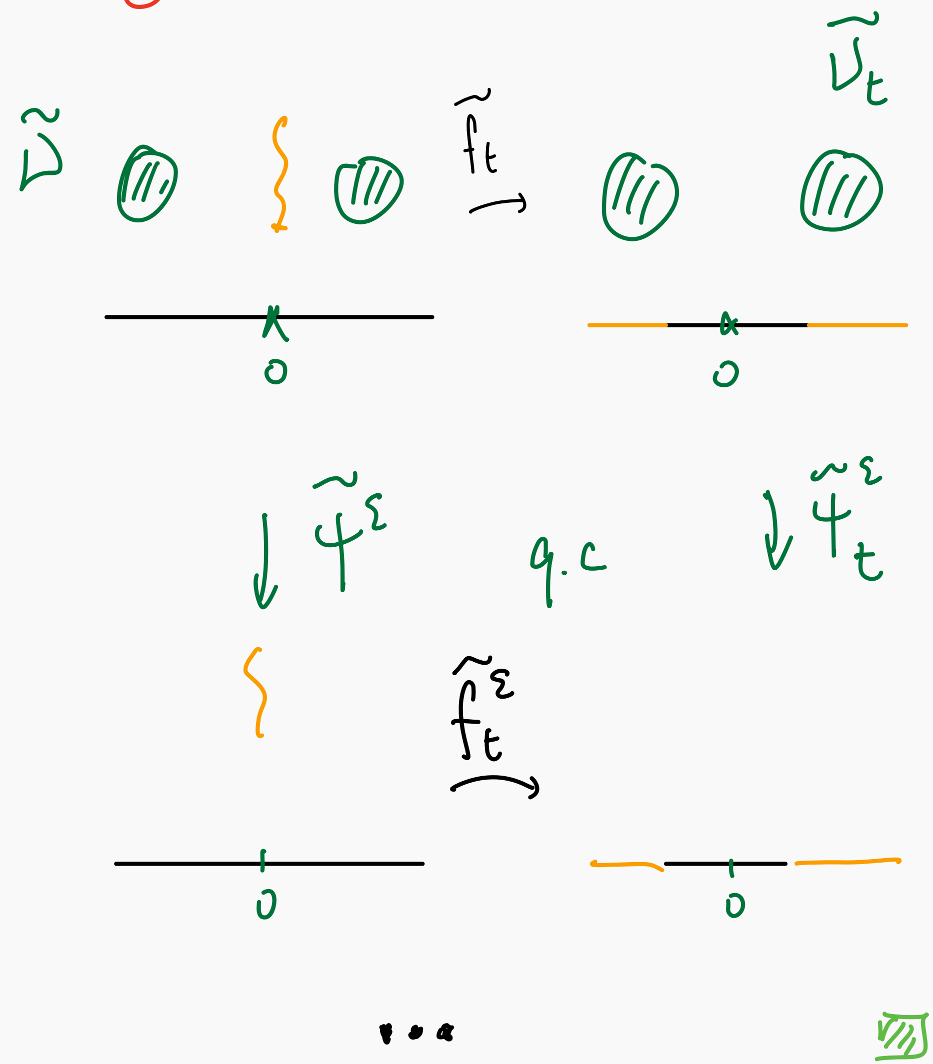

Variation of the Loewner energy

Distortion of the Loewner energy

Def : $I^L_{\mathbb{H}}(\gamma) := \frac{1}{2}\int_0^\infty \frac{\dot{\lambda}_t^2}{\dot{a}_t}\,dt$ | If capacity-parametrized $\dot{a}_t \equiv 1$

Dirichlet energy w.r.t capacity param.

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Corollary 2.4. *Let $T \in (0,\infty)$. Suppose λ_t is absolutely continuous on $[0,T]$ and $\dot{\lambda}_t \in L^2([0,T])$. Then,*

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} I^L(\eta^{\varepsilon\nu}[0,T]) = \frac{12}{\pi}\,\mathrm{Re}\int_{\mathbb{H}} \nu(z)\int_0^T \dot{\lambda}_t \frac{f_t'(z)^2}{f_t(z)^3}\,\mathrm{d}t\,\mathrm{d}^2 z. \tag{2.24}$$

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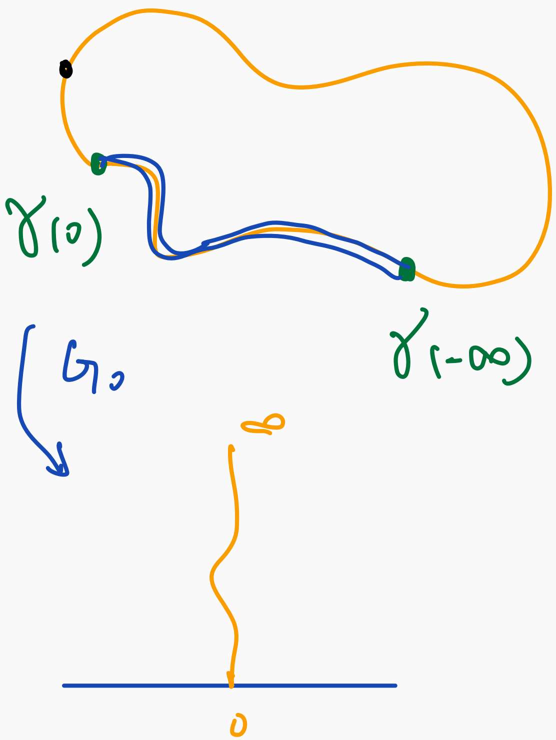
Proof: Combine the variation of λ_t and a_t. ∎

Expression nicer for Jordan curve →

Loewner energy for Jordan curve



Choose $\gamma(-\infty) \in \gamma$

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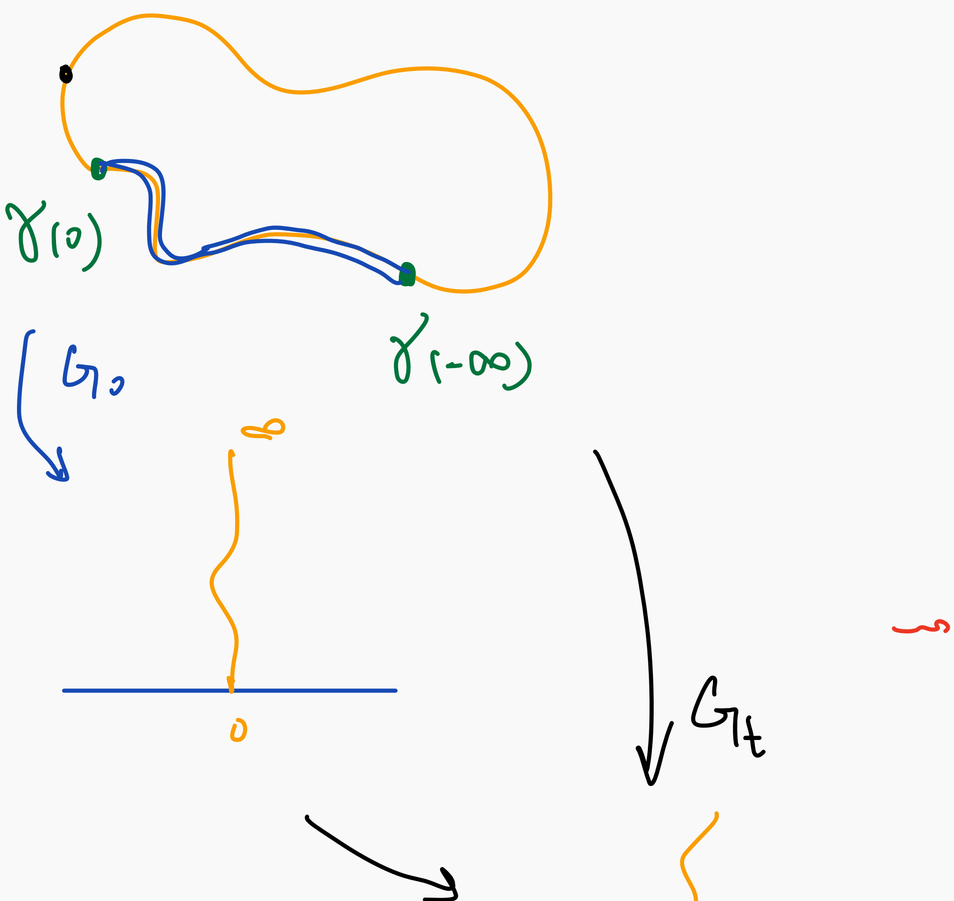
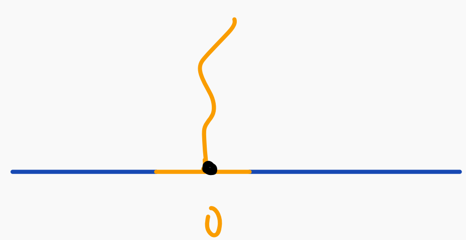
a conformal map

$G_0 : \hat{\mathbb{C}} \setminus \gamma_{[-\infty, 0]} \to \mathbb{H}$

Loewner energy for Jordan curve





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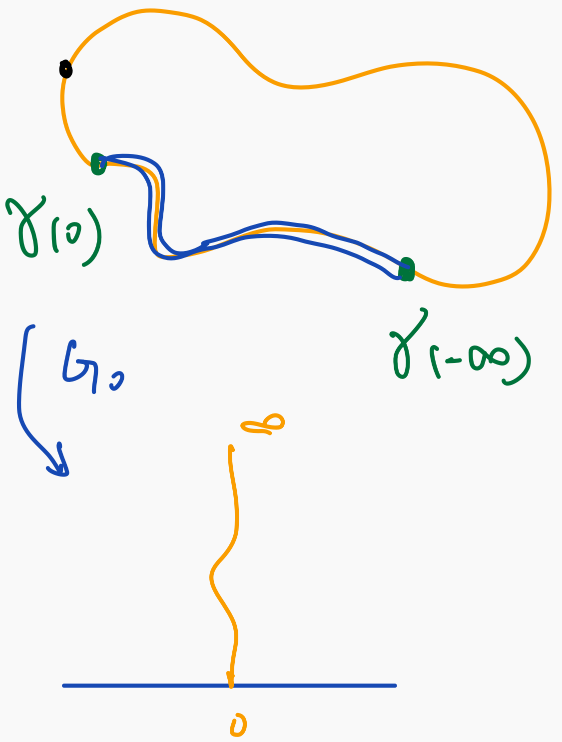
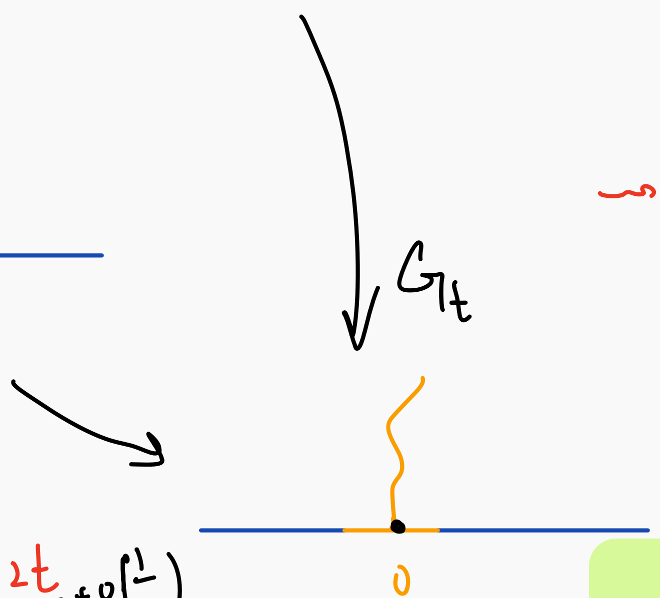
for all $t \in \mathbb{R}$

Loewner energy for Jordan curve





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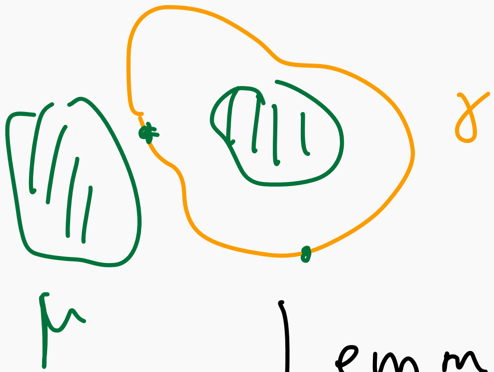
$$\boxed{I^L(\gamma) := \frac{1}{2}\int_{-\infty}^{\infty} \frac{(\dot{\lambda}_t)^2}{\dot{a}_t}\, dt}$$

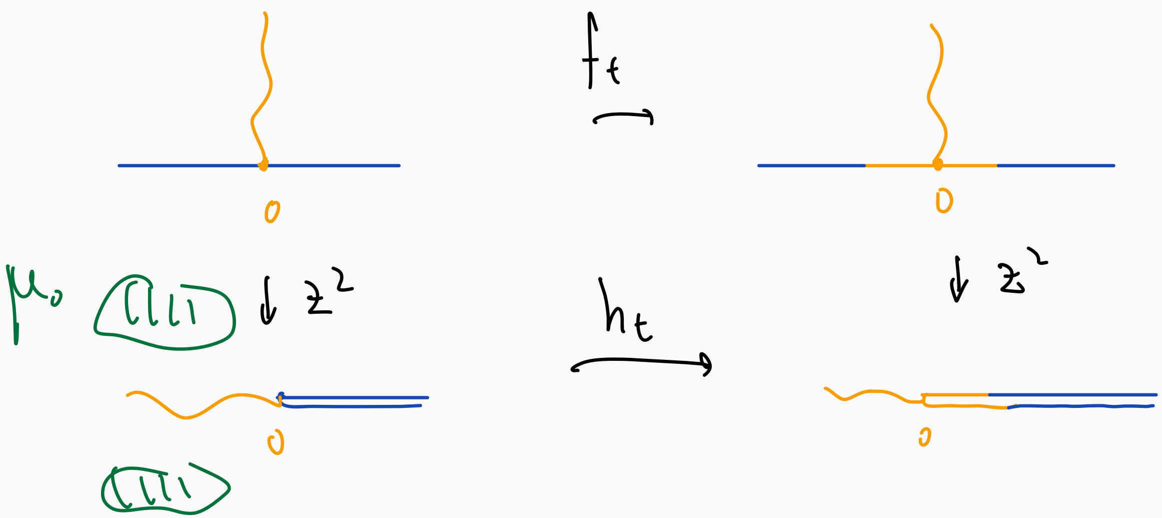
Distorsion of the Loewner energy of Jordan curve



$\mu \in L^\infty_c(\hat{\mathbb{C}} \setminus \gamma)$

Lemma

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{(\dot{\lambda}_t^\varepsilon)^2}{2\dot{a}_t^\varepsilon} = \frac{3}{\pi} \mathrm{Re} \int_{\mathbb{C}} \mu_0(z)\, \dot{\lambda}_t \frac{h_t'(z)^2}{h_t(z)^{5/2}}\, dz$$



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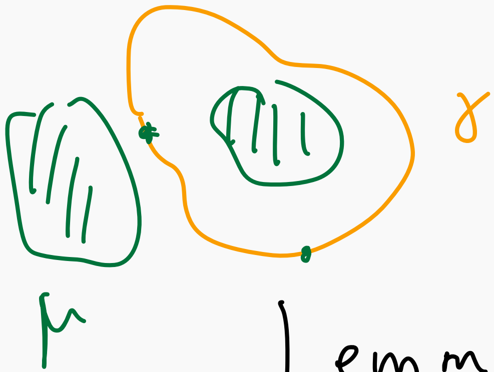

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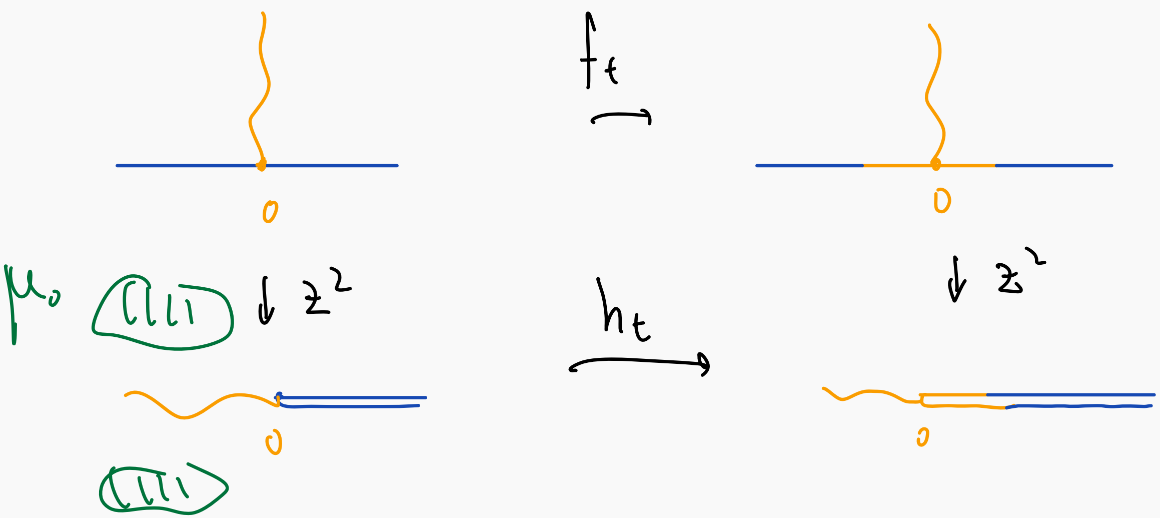
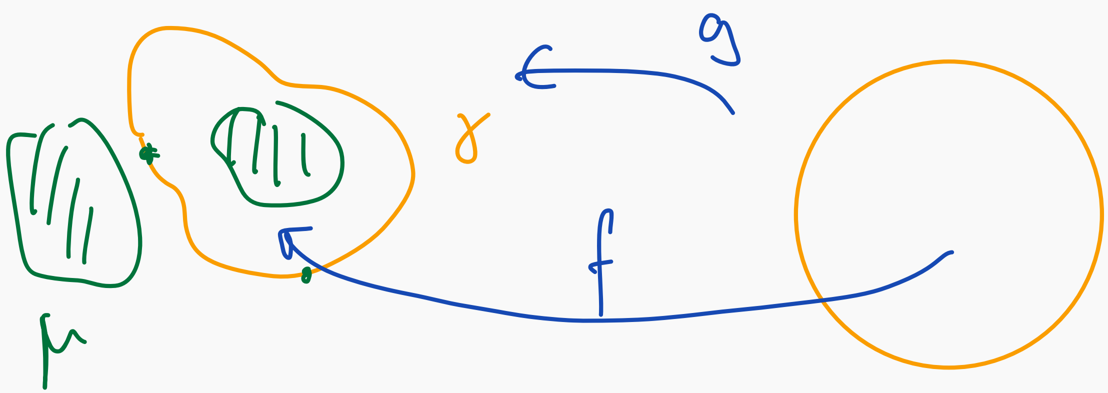
$$= \frac{-4}{\pi}\,\mathrm{Re}\int_{\mathbb{C}} \mu_0(z) \left(- \frac{d\, S h_t(z)}{dt}\right) dz$$



Distorsion of the Loewner energy of Jordan curve



$\bar\partial w^\varepsilon = \varepsilon\mu\,\partial w^\varepsilon$

Thm (Sung, W)

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} I^L(\gamma^{\varepsilon\mu}) = -\frac{4}{\pi}\,\mathrm{Re}\left[\int_\Omega \mu(z)\mathscr{S}[f^{-1}](z)\,\mathrm{d}^2 z + \int_{\Omega^*} \mu(z)\mathscr{S}[g^{-1}](z)\,\mathrm{d}^2 z\right]$$

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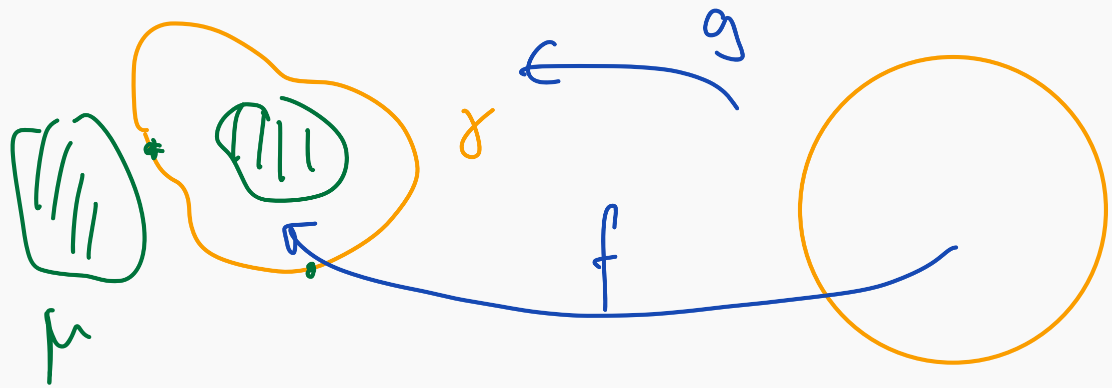

$$\bar\partial w^\varepsilon = \varepsilon\mu\, \partial w^\varepsilon$$

$$\nu := \frac{d}{d\varepsilon} w^\varepsilon$$

$$\Rightarrow \bar\partial \nu = \mu$$

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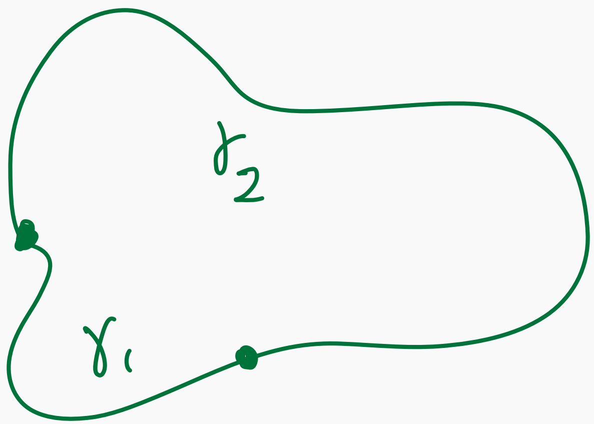
$$= -\frac{4}{\pi}\,\mathrm{Re}\left[\int_{\partial\Omega} \nu\, S(f^{-1})\, dz + \int_{\partial\Omega^*} \nu\, S(g^{-1})\, dz\right]$$

Stokes' formula

Consequences

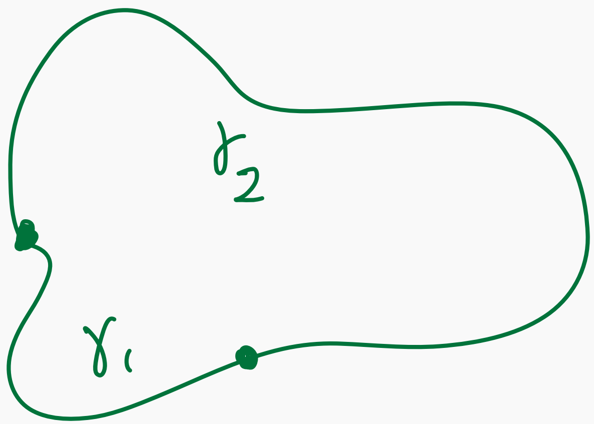
Consequences : Optimal curve



Given γ_2.

γ_1 minimizes $I^L(\gamma_1 \cup \gamma_2)$

Consequences : Optimal curve



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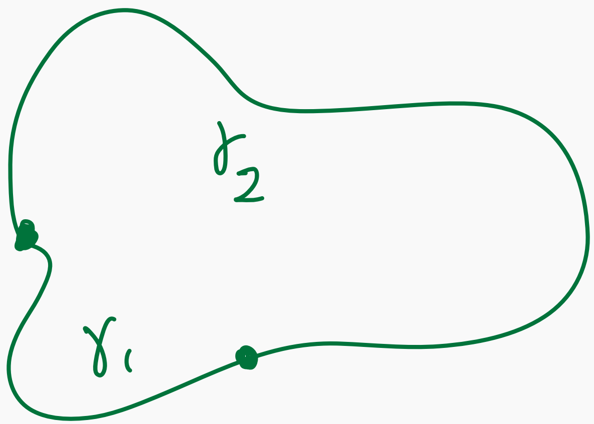

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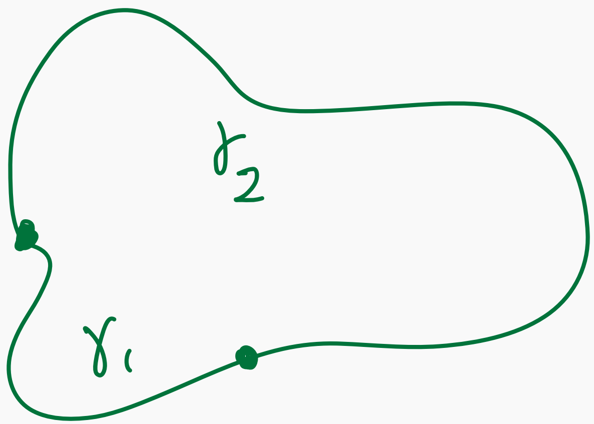

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$\Leftrightarrow \quad g^{-1} \circ f$ is Möbius on $f(\gamma_1)$

$\Leftrightarrow \quad \gamma_1$ is h. geodesic in $\hat{\mathbb{C}} \setminus \gamma_2$.

Variation of $SLE_{8/3}$ loop measure

$\mathcal{W}$: $SLE_{8/3}$ loop measure = outer boundary of Brownian loop measure

K_1 $\quad$ K_2

compact sets, disjoint

$\mathcal{W}(K_1, K_2, \hat{\mathbb{C}}) < \infty$

Möbius-invariant

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Corollary 1.6. *For every annulus A containing γ such that $A \cap \operatorname{supp}(\mu) = \varnothing$, we have*

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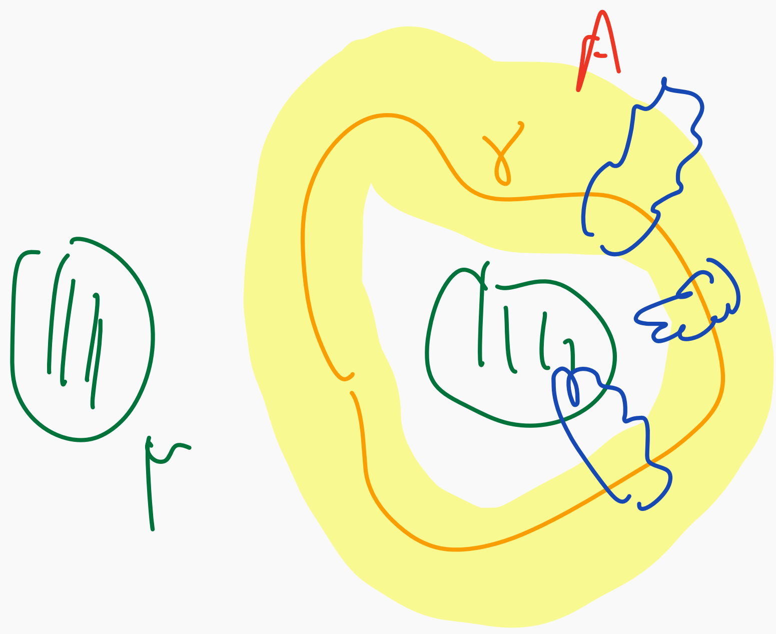

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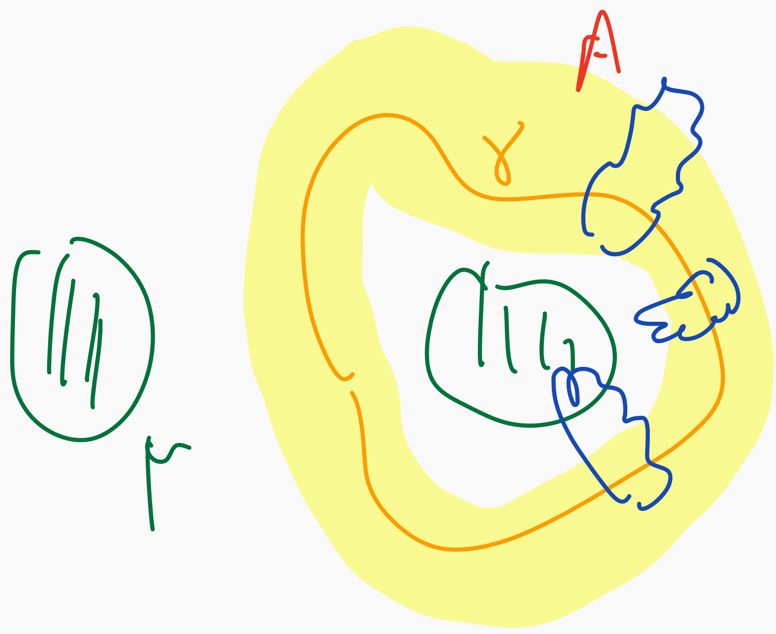

Thanks!