Quasiconformal deformation of the chordal Loewner driving function and first variation of the Loewner energy

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Abstract

We derive the variational formula of the Loewner driving function of a simple chord under infinitesimal quasiconformal deformations with Beltrami coefficients supported away from the chord. As an application, we obtain the first variation of the Loewner energy of a Jordan curve, defined as the Dirichlet energy of the driving function of the curve. This result gives another explanation of the identity between the Loewner energy and the universal Liouville action introduced by Takhtajan and Teo, which has the same variational formula. We also deduce the variation of the total mass of $\mathrm{SLE}_{8/3}$ loops touching the Jordan curve under quasiconformal deformations.

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1 Introduction

One hundred years ago, Loewner introduced [Loe23] a method to encode a simple planar curve by a family of uniformizing maps (called the Loewner chain) which satisfies a differential equation driven by a real-valued function. This method has become a powerful tool in geometric function theory. It was instrumental in the proof of Bieberbach conjecture by De Branges [dB85] (which was also the original motivation of Loewner) and was revived around 2000 as a fundamental building block in the definition of the Schramm–Loewner Evolution [Sch00]. On the other hand, quasiconformal mappings are one of the fundamental concepts in geometric function theory and Teichmüller theory, thus we find it natural to investigate the interplay between quasiconformal maps and the Loewner transform. We mention that analytic properties of the Loewner driving function have been investigated in, e.g., [RS05, MR05, Lin05, LT16, FS17, RW21].

Our first result shows how quasiconformal deformations of the ambient domain $\mathbb{H} = \{z\colon \operatorname{Im}(z) > 0\}$ affect the driving function of a simple chord in $\mathbb{H}$ connecting 0 to ∞.

Theorem 1.1. *Let η be a simple chord in $(\mathbb{H}; 0, \infty)$ and $\nu \in L^\infty(\mathbb{H})$ be an infinitesimal Beltrami differential whose support is compact and disjoint from η. For $\varepsilon \in \mathbb{R}$ such that $\|\varepsilon\nu\|_\infty < 1$, let $\psi^{\varepsilon\nu}$ be the unique quasiconformal self-map of $\mathbb{H}$ with Beltrami coefficient $\varepsilon\nu$ such that $\psi^{\varepsilon\nu}(0) = 0$ and $\psi^{\varepsilon\nu}(z) - z = O(1)$ as $z \to \infty$. Then,*

$$\frac{\partial \lambda_t^{\varepsilon\nu}}{\partial \varepsilon}\bigg|_{\varepsilon=0} = -\frac{2}{\pi}\operatorname{Re}\int_{\mathbb{H}} \nu(z)\left(\frac{g_t'(z)^2}{g_t(z)-\lambda_t} - \frac{1}{z}\right)\mathrm{d}^2 z \tag{1.1}$$

and

$$\frac{\partial a_t^{\varepsilon\nu}}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \frac{1}{\pi}\operatorname{Re}\int_{\mathbb{H}} \nu(z)(g_t'(z)^2 - 1)\mathrm{d}^2 z \tag{1.2}$$

where $\lambda_\cdot$ is the driving function of η, $g_\cdot$ is the Loewner chain of η, and $\psi^{\varepsilon\nu}(\eta)$ has capacity parametrization $2a_\cdot^{\varepsilon\nu}$ and driving function $\lambda_\cdot^{\varepsilon\nu}$.

Our proof relies on the simple but crucial observation that the Loewner driving function and the capacity parametrization of the curve can be expressed by the pre-Schwarzian and Schwarzian derivatives of well-chosen maps, respectively. See Section 2.

We extend our considerations to the Loewner driving function $\mathbb{R} \to \mathbb{R}, t \mapsto \lambda_t$ associated with a Jordan curve $\gamma \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The loop driving function was described in [Wan19b] and can be thought as a consistent family of chordal Loewner driving functions. See Section 3.1 for the precise definition. We point out that for a given Jordan curve, there are a few choices we make to define the driving function:

- the orientation of γ;
- a point on γ called the *root*, which we denote by $\gamma(-\infty) = \gamma(\infty)$;
- another point on γ, which we call $\gamma(0)$;
- a conformal map $H_0 : \hat{\mathbb{C}} \smallsetminus \gamma(-\infty, 0) \to \mathbb{C} \smallsetminus \mathbb{R}_+$, such that $H_0(\gamma(0)) = 0$ and $H_0(\gamma(\infty)) = \infty$, where $\gamma(-\infty, 0)$ denotes the part of γ going from the root to $\gamma(0)$ following the orientation of the curve.

Then, there is a well-defined continuous parametrization of γ by $[-\infty, \infty]$ (i.e., the capacity parametrization). If the orientation and the root are fixed, different choices of $\gamma(0)$ and H_0 result in changes to the driving function of the form

$$\tilde{\lambda}_t = c\left(\lambda_{c^{-2}(t+t_0)} - \lambda_{c^{-2}t_0}\right)$$

for some $c > 0$ and $t_0 \in \mathbb{R}$. Such transformations do not change the Dirichlet energy of λ. Rather surprisingly, the Dirichlet energy does not depend on the root or the orientation either, as shown in [Wan19a, RW21]. These symmetries are explained by the following theorem.

Theorem 1.2 (See [Wan19b])**.** *The Loewner energy of γ, defined as*

$$I^L(\gamma) = \frac{1}{2}\int_{-\infty}^{\infty} |\dot{\lambda}_t|^2 \,\mathrm{d}t \tag{1.3}$$

equals $1/\pi$ times the universal Liouville action $\mathbf{S}$ *introduced by Takhtajan and Teo in* [TT06]*, defined as*

$$\mathbf{S}(\gamma) := \int_{\mathbb{D}} \left|\frac{f''}{f'}(z)\right|^2 \mathrm{d}^2 z + \int_{\mathbb{D}^*} \left|\frac{g''}{g'}(z)\right|^2 \mathrm{d}^2 z - 4\pi \log\left|\frac{f'(0)}{g'(\infty)}\right|. \tag{1.4}$$

Here, $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ are conformal maps such that $g(\infty) = \infty$, Ω and Ω^* are respectively the bounded and unbounded connected components of $\mathbb{C} \setminus \gamma$, and $g'(\infty) = \lim_{z\to\infty} g'(z)$. If γ passes through ∞, we replace γ by $A(\gamma)$ where A is any Möbius map sending γ to a bounded curve.

Remark 1.3. Although it may not be so apparent from (1.4), it follows immediately from (1.3) that I^L is invariant under Möbius transformations of $\hat{\mathbb{C}}$. A Jordan curve for which $\mathbf{S}$ is finite is called a *Weil–Petersson quasicircle.*

Using Theorem 1.1, we obtain the following first variation formula of the Loewner energy. This formula coincides with that of the universal Liouville action $\mathbf{S}$ in [TT06], thus giving another explanation of the identity $I^L = \mathbf{S}/\pi$. This variational formula was crucial in [TT06] to show that $\mathbf{S}$ is a Kähler potential of the Weil–Petersson Teichmüller space.

Theorem 1.4. *Let $\mu \in L^\infty(\mathbb{C})$ be an infinitesimal Beltrami differential with compact support in $\hat{\mathbb{C}} \setminus \gamma$. For $\varepsilon \in \mathbb{R}$ with $\|\varepsilon\mu\|_\infty < 1$, let $\omega^{\varepsilon\mu} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be any quasiconformal mapping with Beltrami coefficient $\varepsilon\mu$. Let $\gamma^{\varepsilon\mu} = \omega^{\varepsilon\mu}(\gamma)$. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} I^L(\gamma^{\varepsilon\mu}) = -\frac{4}{\pi}\,\mathrm{Re}\left[\int_{\Omega} \mu(z)\mathscr{S}[f^{-1}](z)\,\mathrm{d}^2 z + \int_{\Omega^*} \mu(z)\mathscr{S}[g^{-1}](z)\,\mathrm{d}^2 z\right] \tag{1.5}$$

where $\mathscr{S}[\varphi] = \varphi'''/\varphi' - (3/2)(\varphi''/\varphi')^2$ is the Schwarzian derivative and $\mathrm{d}^2 z$ is the Euclidean area measure.

In the language of conformal field theory, this theorem states that the holomorphic stress-energy tensor of the Loewner energy is given by a multiple of the Schwarzian derivative of the uniformizing maps of the complementary components of the curve.

Remark 1.5. The Loewner energy of $\gamma^{\varepsilon\mu}$ does not depend on the choice of solution $\omega^{\varepsilon\mu}$ to the Beltrami equation, as they only differ by post-composition by Möbius transformations of $\hat{\mathbb{C}}$. We also note that in [TT06] the Beltrami coefficient is supported on one side of the curve, say, Ω, and μ is an L^2-harmonic Beltrami differential on Ω. Here, we allow the support of μ to be on both sides of γ but disjoint from γ.

As the support of μ is away from γ, there exists an annulus A containing γ such that $A \cap \operatorname{supp}(\mu) = \varnothing$. In particular, $\omega^{\varepsilon\mu}$ is conformal in A. In [Wan21, Thm. 4.1], the second author showed that the change of the Loewner energy under a conformal map in the neighborhood of the curve could be expressed in terms of the $\mathrm{SLE}_{8/3}$ loop measure introduced in [Wer08], which is the induced measure obtained by taking the outer boundary of a loop under Brownian loop measure [LSW03, LW04]. Combining this with Theorem 1.4, we immediately obtain the following variational formula for the $\mathrm{SLE}_{8/3}$ loop measure.

Corollary 1.6. *For every annulus A containing γ such that $A \cap \operatorname{supp}(\mu) = \varnothing$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathscr{W}(\gamma^{\varepsilon\mu}, \omega^{\varepsilon\mu}(A)^c; \hat{\mathbb{C}}) = \frac{1}{3\pi} \operatorname{Re}\left[\int_\Omega \mu(z)\mathscr{S}[f^{-1}](z)\,\mathrm{d}^2 z + \int_{\Omega^*} \mu(z)\mathscr{S}[g^{-1}](z)\,\mathrm{d}^2 z\right] \tag{1.6}$$

where $\mathscr{W}(\gamma^{\varepsilon\mu}, \omega^{\varepsilon\mu}(A)^c; \hat{\mathbb{C}})$ denotes the total mass of loops intersecting both $\gamma^{\varepsilon\mu}$ and the complement of $\omega^{\varepsilon\mu}(A)$ under the $\mathrm{SLE}_{8/3}$ loop measure on $\hat{\mathbb{C}}$.

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2 Deformation of chords in the half-plane

2.1 Variation of the chordal Loewner driving function

Let $\eta : (0, \infty) \to \mathbb{H}$ be a simple chord from 0 to ∞. Assume that η is parameterized by its half-plane capacity, and let g_t be the corresponding *Loewner chain*; i.e., $g_t : \mathbb{H} \smallsetminus \eta(0, t] \to \mathbb{H}$ is the uniformizing map satisfying

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty.$$

The function $[0, \infty) \to \mathbb{R}$, $t \mapsto \lambda_t := g_t(\eta(t))$ is called the *driving function* of the curve η.

We consider deformations of η under quasiconformal self-maps of $\mathbb{H}$. Let $\nu \in L^\infty(\mathbb{H})$ be a complex-valued function with a compact support in $\mathbb{H} \smallsetminus \eta$. The measurable Riemann mapping theorem states that for $\varepsilon \in \mathbb{R}$ with $|\varepsilon| < 1/\|\nu\|_\infty$, there exists a unique quasiconformal self-map $\psi^{\varepsilon\nu} : \mathbb{H} \to \mathbb{H}$ which solves the Beltrami equation

$$\partial_{\bar{z}}\psi^{\varepsilon\nu} = (\varepsilon\nu)\partial_z\psi^{\varepsilon\nu} \tag{2.1}$$

and has $\psi^{\varepsilon\nu}(0) = 0$ and $\psi^{\varepsilon\nu}(z) - z = O(1)$ as $z \to \infty$. We adopt the following notations, which are illustrated in Figure 1.

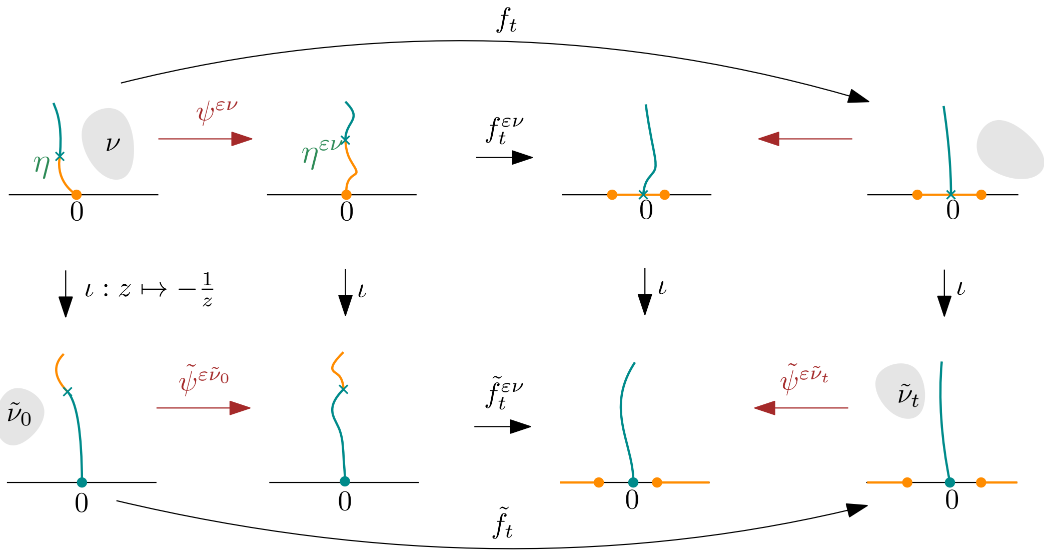

Figure 1: A commutative diagram illustrating the quasiconformal maps and related Loewner chains in Section 2.1. The gray shaded areas denote the support of the Beltrami differentials. The arrows in red are quasiconformal maps, whereas those in black are conformal maps.

- Denote the deformed chord by $\eta^{\varepsilon\nu} := \psi^{\varepsilon\nu} \circ \eta$.
- Let $g_t^{\varepsilon\nu} : \mathbb{H} \smallsetminus \eta^{\varepsilon\nu}[0,t] \to \mathbb{H}$ be the Loewner chain associated with the deformed curve $\eta^{\varepsilon\nu}[0,t]$.
- Denote the driving function of $\eta^{\varepsilon\nu}$ by $\lambda_t^{\varepsilon\nu} := g_t^{\varepsilon\nu}(\eta^{\varepsilon\nu}(t))$.
- Note that $\eta^{\varepsilon\nu}$ is not necessarily parameterized by its half-plane capacity. Denote the half-plane capacity of $\eta^{\varepsilon\nu}[0,t]$ by $2a_t^{\varepsilon\nu}$, so that $g_t^{\varepsilon\nu}(z) = z + 2a_t^{\varepsilon\nu} z^{-1} + O(z^{-2})$ as $z \to \infty$.

The goal of this section is to show Theorem 1.1. For this, we first express $\lambda_t^{\varepsilon\nu}$ and $a_t^{\varepsilon\nu}$ in terms of the pre-Schwarzian and Schwarzian derivatives of appropriately conjugated Loewner chain (Lemma 2.1). We then find the first variations of these derivatives using the measurable Riemann mapping theorem (Proposition 2.2).

Consider the *centered Loewner chain*

$$f_t(z) := g_t(z) - \lambda_t, \tag{2.2}$$

which satisfies $f_t(0) = 0$ and $f_t(z) = z + O(1)$ as $z \to 0$. Let $\iota(z) := -1/z$ be the inversion map. Define the *inverted Loewner chain*

$$\tilde{f}_t(z) := \iota \circ f_t \circ \iota(z) = -\frac{1}{f_t(-1/z)} = -\frac{1}{g_t(-1/z) - \lambda_t}. \tag{2.3}$$

Then, $\tilde{f}_t : \mathbb{H} \smallsetminus (\iota \circ \eta(0,t]) \to \mathbb{H}$ is the uniformizing map with normalization $\tilde{f}_t(0) = 0$, $\tilde{f}_t'(0) = 1$, and $\tilde{f}_t(\iota \circ \eta(t)) = \infty$. Combining the expansion of g_t at ∞ with (2.3), we see that as $z \to 0$,

$$\tilde{f}_t(z) = -\frac{1}{-z^{-1} - \lambda_t - 2tz + O(z^2)} = z - \lambda_t z^2 + (\lambda_t^2 - 2t)z^3 + O(z^4). \tag{2.4}$$

Similarly, define

$$f_t^{\varepsilon\nu}(z) := g_t^{\varepsilon\nu}(z) - \lambda_t^{\varepsilon\nu} \quad \text{and} \quad \tilde{f}_t^{\varepsilon\nu}(z) := \iota \circ f_t^{\varepsilon\nu} \circ \iota(z) = -\frac{1}{g_t^{\varepsilon\nu}(-1/z) - \lambda_t^{\varepsilon\nu}}. \tag{2.5}$$

A calculation analogous to (2.4) using the series expansion of $g_t^{\varepsilon\nu}$ at ∞ leads to

$$\tilde{f}_t^{\varepsilon\nu}(z) = z - \lambda_t^{\varepsilon\nu} z^2 + ((\lambda_t^{\varepsilon\nu})^2 - 2a_t^{\varepsilon\nu})z^3 + O(z^4) \quad \text{as} \;\; z \to 0. \tag{2.6}$$

For a conformal map φ, we define the *pre-Schwarzian* (or called non-linearity) and the *Schwarzian derivatives* of f by, respectively,

$$\mathscr{N}\varphi = \frac{\varphi''}{\varphi'} \quad \text{and} \quad \mathscr{S}\varphi = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2. \tag{2.7}$$

By the Schwarz reflection principle, $\tilde{f}_t$ and $\tilde{f}_t^{\varepsilon\nu}$ extend respectively to conformal maps on $\mathbb{C} \smallsetminus \iota(\eta(0,t] \cup \overline{\eta(0,t]})$ and $\mathbb{C} \smallsetminus \iota(\eta^{\varepsilon\nu}(0,t] \cup \overline{\eta^{\varepsilon\nu}(0,t]})$, where $\bar{\cdot}$ denotes the complex conjugate. In particular, they are conformal in some neighborhood of 0.

Lemma 2.1. *Consider $\tilde{f}_t$ and $\tilde{f}_t^{\varepsilon\nu}$ as conformal maps extended by reflection to a neighborhood of 0. Then,*

$$\lambda_t = -\frac{1}{2}\mathscr{N}\tilde{f}_t(0), \quad \lambda_t^{\varepsilon\nu} = -\frac{1}{2}\mathscr{N}\tilde{f}_t^{\varepsilon\nu}(0), \quad \textit{and} \quad a_t^{\varepsilon\nu} = -\frac{1}{12}\mathscr{S}\tilde{f}_t^{\varepsilon\nu}(0). \tag{2.8}$$

Proof. The lemma follows from inspecting the coefficients of (2.4) and (2.6). □

Let $\varepsilon\tilde{\nu}_t$ be the Beltrami differential corresponding to the quasiconformal map

$$\tilde{\psi}^{\varepsilon\tilde{\nu}_t} := \tilde{f}_t^{\varepsilon\nu} \circ \iota \circ \psi^{\varepsilon\nu} \circ \iota \circ \tilde{f}_t^{-1} = \iota \circ f_t^{\varepsilon\nu} \circ \psi^{\varepsilon\nu} \circ f_t^{-1} \circ \iota. \tag{2.9}$$

In particular, $\tilde{\psi}^{\varepsilon\tilde{\nu}_0} = \iota \circ \psi^{\varepsilon\nu} \circ \iota$ is the quasiconformal map which deforms $\iota \circ \eta$ to $\iota \circ \eta^{\varepsilon\nu}$. Note that $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}$ is conformal in a neighborhood of 0 and satisfies $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) = 0$, $(\tilde{\psi}^{\varepsilon\tilde{\nu}_t})'(0) = 1$, and $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(\infty) = \infty$.

Proposition 2.2. *Let $\tilde{\nu}_t$ be the Beltrami differential defined above. Then,*

$$\begin{aligned}
\left.\frac{\partial \lambda_t^{\varepsilon\nu}}{\partial \varepsilon}\right|_{\varepsilon=0} &= \frac{2}{\pi}\operatorname{Re}\int_{\mathbb{H}} \frac{\tilde{\nu}_t(z) - \tilde{\nu}_0(z)}{z^3}\,\mathrm{d}^2 z \\
&= \frac{2}{\pi}\operatorname{Re}\int_{\mathbb{H}} \tilde{\nu}_0(z)\left(\frac{\tilde{f}_t'(z)^2}{\tilde{f}_t(z)^3} - \frac{1}{z^3}\right)\mathrm{d}^2 z \\
&= -\frac{2}{\pi}\operatorname{Re}\int_{\mathbb{H}} \nu(z)\left(\frac{f_t'(z)^2}{f_t(z)} - \frac{1}{z}\right)\mathrm{d}^2 z
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
\left.\frac{\partial a_t^{\varepsilon\nu}}{\partial t}\right|_{\varepsilon=0} &= \frac{1}{\pi}\operatorname{Re}\int_{\mathbb{H}} \frac{\tilde{\nu}_t(z) - \tilde{\nu}_0(z)}{z^4}\,\mathrm{d}^2 z \\
&= \frac{1}{\pi}\operatorname{Re}\int_{\mathbb{H}} \tilde{\nu}_0(z)\left(\frac{\tilde{f}_t'(z)^2}{\tilde{f}_t(z)^4} - \frac{1}{z^4}\right)\mathrm{d}^2 z \\
&= \frac{1}{\pi}\operatorname{Re}\int_{\mathbb{H}} \nu(z)(f_t'(z)^2 - 1)\,\mathrm{d}^2 z.
\end{aligned} \tag{2.11}$$

Proof of Theorem 1.1. It suffices to substitute $f_t(z) = g_t(z) - \lambda_t$ in Proposition 2.2. $\square$

Proof of Proposition 2.2. We can extend $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}$ to a quasiconformal self-map of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by reflecting it with respect to the real axis. The Beltrami differential for this extension of $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}$ is $\varepsilon\hat{\nu}_t$ where

$$\hat{\nu}_t(z) := \begin{cases} \tilde{\nu}_t(z) & \text{if } z \in \mathbb{H}, \\ 0 & \text{if } z \in \mathbb{R}, \\ \overline{\tilde{\nu}_t(\overline{z})} & \text{if } z \in \mathbb{H}^*. \end{cases} \tag{2.12}$$

Then, by the measurable Riemann mapping theorem,

$$\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(\zeta) = \zeta - \frac{\varepsilon}{\pi}\int_{\mathbb{C}} \hat{\nu}_t(z)\left(\frac{1}{z-\zeta} - \frac{1}{z} - \frac{\zeta}{z^2}\right) \mathrm{d}^2 z + o(\varepsilon) \tag{2.13}$$

locally uniformly in $\zeta \in \mathbb{C}$ as $\varepsilon \to 0$. Moreover, since ∂_ε commutes with ∂_ζ when applied to $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}$ and $\hat{\nu}_t$ has a compact support in $\mathbb{C} \setminus \{0\}$, we have

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \mathscr{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) = -\frac{2}{\pi}\int_{\mathbb{C}} \frac{\hat{\nu}_t(z)}{z^3}\,\mathrm{d}^2 z = -\frac{4}{\pi}\operatorname{Re}\int_{\mathbb{H}} \frac{\varepsilon\tilde{\nu}_t(z)}{z^3}\mathrm{d}^2 z, \tag{2.14}$$

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \mathscr{S}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) = -\frac{6}{\pi}\int_{\mathbb{C}} \frac{\hat{\nu}_t(z)}{z^4}\,\mathrm{d}^2 z = -\frac{12}{\pi}\operatorname{Re}\int_{\mathbb{H}} \frac{\varepsilon\tilde{\nu}_t(z)}{z^4}\,\mathrm{d}^2 z. \tag{2.15}$$

The chain rules for the pre-Schwarzian and Schwarzian derivatives are

$$\mathscr{N}(f \circ g) = ((\mathscr{N} f) \circ g)g' + \mathscr{N} g \quad \text{ and } \quad \mathscr{S}[f \circ g] = ((\mathscr{S} f) \circ g)(g')^2 + \mathscr{S} g.$$

Since $\tilde{f}_t^{\varepsilon\nu} = \tilde{\psi}^{\varepsilon\tilde{\nu}_t} \circ \tilde{f}_t \circ (\tilde{\psi}^{\varepsilon\tilde{\nu}_0})^{-1}$ and $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(z)$, $\tilde{f}_t(z)$, and $\tilde{\psi}^{\varepsilon\tilde{\nu}_0}(z)$ all behave as $z + o(z)$ as $z \to 0$, we have from (2.8) that

$$-2\lambda_t^{\varepsilon\mu} = \mathscr{N}\tilde{f}_t^{\varepsilon\mu}(0) = \mathscr{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) + \mathscr{N}\tilde{f}_t(0) - \mathscr{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_0}(0) = \mathscr{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) - 2\lambda_t - \mathscr{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_0}(0), \tag{2.16}$$

$$-12a_t^{\varepsilon\mu} = \mathscr{S}\tilde{f}_t^{\varepsilon\mu}(0) = \mathscr{S}\psi_t^{\varepsilon\nu_t}(0) + \mathscr{S}\tilde{f}_t(0) - \mathscr{S}\psi^{\varepsilon\nu_0}(0) = \mathscr{S}\psi_t^{\varepsilon\nu_t}(0) - 12t - \mathscr{S}\psi^{\varepsilon\nu_0}(0). \tag{2.17}$$

Combining these with (2.14) and (2.15), we obtain the first equalities in (2.10) and (2.11).

Observe that $\tilde{\psi}^{\varepsilon\tilde{\nu}_t} = \tilde{f}_t^{\varepsilon\nu} \circ \tilde{\psi}^{\varepsilon\tilde{\nu}_0} \circ \tilde{f}_t^{-1}$, where $\tilde{f}_t^{\varepsilon\nu}$ and $\tilde{f}_t^{-1}$ are conformal maps. Hence, by the composition rule for Beltrami differentials,

$$\tilde{\nu}_t(\tilde{f}_t(z)) = \tilde{\nu}_0(z)\frac{\tilde{f}_t'(z)^2}{|\tilde{f}_t'(z)|^2}. \tag{2.18}$$

Substituting (2.18) into (2.14) and (2.15), we have that

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \mathscr{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) = -\frac{4}{\pi}\operatorname{Re}\int_{\mathbb{H}} \tilde{\nu}_0(z)\frac{\tilde{f}_t'(z)^2}{\tilde{f}_t(z)^3}\,\mathrm{d}^2 z, \tag{2.19}$$

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \mathscr{S}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) = -\frac{12}{\pi}\operatorname{Re}\int_{\mathbb{H}} \tilde{\nu}_0(z)\frac{\tilde{f}_t'(z)^2}{\tilde{f}_t(z)^4}\,\mathrm{d}^2 z. \tag{2.20}$$

We thus have the second equalities in (2.10) and (2.11).

Finally, recall that $\tilde{\psi}^{\varepsilon\tilde{\nu}_0} = \iota \circ \psi^{\varepsilon\nu} \circ \iota$. Since the inversion map $\iota(\zeta) = -1/\zeta$ is conformal,

$$\tilde{\nu}_0(-1/\zeta) = \nu(\zeta)\frac{|\zeta|^4}{\zeta^4}. \tag{2.21}$$

Recall that $\tilde{f}_t(z) = -1/f_t(-1/z)$, and hence $\tilde{f}_t'(z) = f_t'(-1/z)/(zf_t(-1/z))^2$. Substituting $z = -1/\zeta$ in (2.19) and (2.20), we obtain the final equalities in (2.10) and (2.11). □

2.2 Variation of chordal Loewner energy

Let $\eta : (0,\infty) \to \mathbb{H}$ be a chord from 0 to ∞ parametrized by half-plane capacity (i.e. $a_t \equiv 1$) and $t \mapsto \lambda_t$ be its Loewner driving function. The *Loewner energy* of η (resp. partial Loewner energy of η up to time $T > 0$) is

$$I^L(\eta) = \frac{1}{2}\int_0^\infty \dot{\lambda}_t^2 \,\mathrm{d}t \qquad \text{resp.} \qquad I^L(\eta[0,T]) = \frac{1}{2}\int_0^T \dot{\lambda}_t^2 \,\mathrm{d}t$$

if λ is absolutely continuous and $\dot{\lambda}$ is its almost everywhere defined derivative with respect to t. We set $I^L(\eta) = \infty$ if λ is not absolutely continuous. We also define the Loewner energy of a chord η in a simply connected domain D connecting two distinct prime ends a, b as

$$I^L_{D;a,b}(\eta) := I^L(\varphi(\eta))$$

where φ is any conformal map $D \to \mathbb{H}$ with $\varphi(a) = 0$ and $\varphi(b) = \infty$. The partial Loewner energies in $(D; a, b)$ are defined similarly.

When λ_t is absolutely continuous, we can compute the first variations of $\dot{\lambda}_t^{\varepsilon\nu}$ and $\dot{a}_t^{\varepsilon\nu}$.

Proposition 2.3. *For all $\varepsilon \in (-1/\|\nu\|_\infty, 1/\|\nu\|_\infty)$, the variations $\lambda_t^{\varepsilon\nu} - \lambda_t$ and $a_t^{\varepsilon\nu}$ are continuously differentiable in t. Furthermore, if λ_t is absolutely continuous, then $\lambda_t^{\varepsilon\nu}$ is also absolutely continuous in t, and for almost every t,*

$$\frac{\partial \dot{\lambda}_t^{\varepsilon\nu}}{\partial\varepsilon}\bigg|_{\varepsilon=0} = \frac{1}{\pi}\operatorname{Re}\int_{\mathbb{H}} \nu(z)\left(12\frac{f_t'(z)^2}{f_t(z)^3} - 2\dot{\lambda}_t\frac{f_t'(z)^2}{f_t(z)^2}\right)\mathrm{d}^2 z \tag{2.22}$$

and

$$\frac{\partial \dot{a}_t^{\varepsilon\nu}}{\partial\varepsilon}\bigg|_{\varepsilon=0} = -\frac{4}{\pi}\operatorname{Re}\int_{\mathbb{H}} \nu(z)\frac{f_t'(z)^2}{f_t(z)^2}\,\mathrm{d}^2 z. \tag{2.23}$$

Proof. From the Loewner equation $\partial_t g_t(z) = 2/(g_t(z) - \lambda_t)$, we have

$$\partial_t(f_t(z) + \lambda_t) = \frac{2}{f_t(z)} \quad \text{and} \quad \partial_t f_t'(z) = -\frac{2f_t'(z)}{f_t(z)^2}.$$

Recalling $\tilde{f}_t(z) = -1/f_t(-1/z)$, it follows that $\tilde{f}_t'(z)$ is continuously differentiable in t. From (2.12) and (2.18), we see that $(\varepsilon, t) \mapsto \varepsilon\hat{\nu}_t$ is continuously differentiable. Then, $\lambda_t^{\varepsilon\nu} - \lambda_t = -\frac{1}{2}(\mathscr{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) - \mathscr{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_0}(0))$ and $a_t^{\varepsilon\nu} - t = -\frac{1}{12}(\mathscr{S}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) - \mathscr{S}\tilde{\psi}^{\varepsilon\tilde{\nu}_0}(0))$ are continuously differentiable in the same variables [AB60].

We can check directly from the integral representations of $\partial_\varepsilon \mathscr{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0)$ and $\partial_\varepsilon \mathscr{S}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0)$ that they are continuously differentiable in t. If λ_t is absolutely continuous, then we have from (2.10) that for almost every t,

$$\begin{aligned}\frac{\partial \dot{\lambda}_t^{\varepsilon\nu}}{\partial\varepsilon}\bigg|_{\varepsilon=0} &= -\frac{1}{2}\frac{\partial^2(\mathscr{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0))}{\partial\varepsilon\partial t}\bigg|_{\varepsilon=0} = -\frac{1}{2}\frac{\partial^2(\mathscr{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0))}{\partial t\partial\varepsilon}\bigg|_{\varepsilon=0} \\ &= -\frac{2}{\pi}\operatorname{Re}\int_{\mathbb{H}}\nu(z)\partial_t\left(\frac{f_t'(z)^2}{f_t(z)}\right)\mathrm{d}^2z.\end{aligned}$$

Similarly, (2.11) implies

$$\frac{\partial \dot{a}_t^{\varepsilon\nu}}{\partial\varepsilon}\bigg|_{\varepsilon=0} = \frac{1}{\pi}\operatorname{Re}\int_{\mathbb{H}}\nu(z)\,\partial_t(f_t'(z)^2)\,\mathrm{d}^2z.$$

Using the formulas for $\partial_t f_t$ and $\partial_t f_t'$ above, we have

$$\partial_t(f_t'(z)^2) = -4\frac{f_t'(z)^2}{f_t(z)^2} \quad \text{and} \quad \partial_t\left(\frac{f_t'(z)^2}{f_t(z)}\right) = -6\frac{f_t'(z)^2}{f_t(z)^3} + \dot{\lambda}_t\frac{f_t'(z)^2}{f_t(z)^2}.$$

This completes the proof. □

Proposition 2.3 allows us to compute the first variation of the chordal Loewner energy for a finite portion of the curve $\gamma^{\varepsilon\nu}$.

Corollary 2.4. *Let $T \in (0,\infty)$. Suppose λ_t is absolutely continuous on $[0,T]$ and $\dot{\lambda}_t \in L^2([0,T])$. Then,*

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} I^L(\eta^{\varepsilon\nu}[0,T]) = \frac{12}{\pi}\operatorname{Re}\int_{\mathbb{H}}\nu(z)\int_0^T \dot{\lambda}_t\frac{f_t'(z)^2}{f_t(z)^3}\,\mathrm{d}t\,\mathrm{d}^2z. \tag{2.24}$$

Proof. From Proposition 2.3, we see that $(\dot{\lambda}_t^{\varepsilon\nu})^2/\dot{a}_t^{\varepsilon\nu}$ is integrable on $[0,T]$ whenever $|\varepsilon| < 1/\|\nu\|_\infty$. Moreover, we obtain that for almost every $t \in [0,T]$,

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}\frac{(\dot{\lambda}_t^{\varepsilon\nu})^2}{\dot{a}_t^{\varepsilon\nu}} = 2\dot{\lambda}_t\frac{\partial\dot{\lambda}_t^{\varepsilon\nu}}{\partial\varepsilon}\bigg|_{\varepsilon=0} - \dot{\lambda}_t^2\frac{\partial\dot{a}_t^{\varepsilon\nu}}{\partial\varepsilon}\bigg|_{\varepsilon=0} = \frac{24\dot{\lambda}_t}{\pi}\operatorname{Re}\int_{\mathbb{H}}\nu(z)\frac{f_t'(z)^2}{f_t(z)^3}\,\mathrm{d}^2z.$$

Since ν is compactly supported in $\mathbb{H}\smallsetminus\eta$, the integral on the right-hand side is continuous in t and hence bounded on $[0,T]$. Using the Leibniz integral rule, we conclude that

$$\begin{aligned}I^L(\eta^{\varepsilon\nu}[0,T]) &= \frac{1}{2}\int_0^T\left|\frac{\mathrm{d}\lambda_t^{\varepsilon\nu}}{\mathrm{d}a_t^{\varepsilon\nu}}\right|^2\mathrm{d}a_t^{\varepsilon\nu} = \frac{1}{2}\int_0^T\frac{(\dot{\lambda}_t^{\varepsilon\nu})^2}{\dot{a}_t^{\varepsilon\nu}}\,\mathrm{d}t \\ &= I^L(\eta[0,T]) + \varepsilon\int_0^T\frac{12\dot{\lambda}_t}{\pi}\operatorname{Re}\left[\int_{\mathbb{H}}\nu(z)\frac{f_t'(z)^2}{f_t(z)^3}\mathrm{d}^2z\right]\mathrm{d}t + o(\varepsilon)\end{aligned}$$

as $\varepsilon \to 0$. □

3 Deformation of Weil–Petersson quasicircles

In this section, we prove that (2.24) holds when $T = \infty$ and give an alternative expression for the integral $\int_0^\infty \dot\lambda_t (f_t'(z)^2/f_t(z)^3)\,\mathrm{d}t$ by conjugating the Loewner chain by $z \mapsto z^2$. This simple operation generalizes the chordal Loewner driving function to the loop driving function and simplifies the integrand to a Schwarzian derivative which leads to the proof of Theorem 1.4.

Convention. Let $\mathbb{R}_+ = [0,\infty)$. For $\sqrt{z}$ or $z^{1/2}$, we take the branch of the complex square root whose image is in $\mathbb{H} \cup \mathbb{R}_+$.

3.1 Loop driving function

We first recall the definition of the Loewner driving function for a Jordan curve. See Figure 2 for an illustration for the maps used in Section 3.

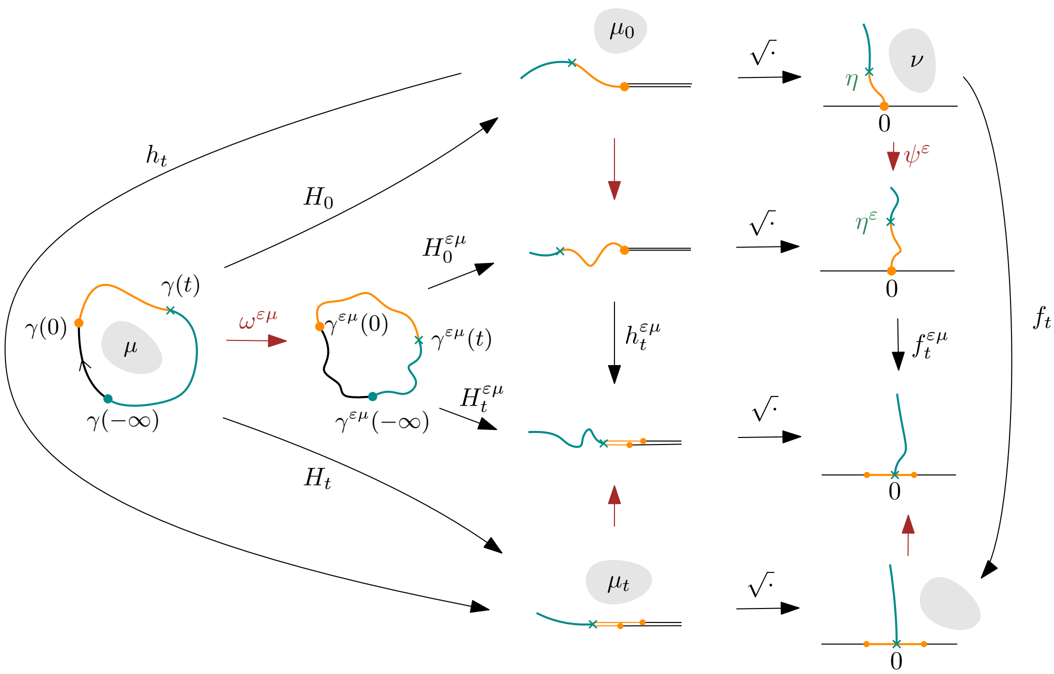

Figure 2: A commutative diagram illustrating the quasiconformal maps and related conformal mapping-out functions in Section 3. The gray shaded areas denote the support of the Beltrami differentials. The arrows in red are quasiconformal maps, whereas those in black are conformal maps.

Let $\gamma : [-\infty,\infty] \to \hat{\mathbb{C}}$ be a Jordan curve. We choose a family of uniformizing maps $H_t : \hat{\mathbb{C}} \smallsetminus \gamma[-\infty,t] \to \mathbb{C} \smallsetminus \mathbb{R}_+$ such that $H_t(\gamma(t)) = 0$ and $H_t(\infty) = \infty$. Note that each H_t is unique up to a real multiplicative factor. We fix H_0, then normalize H_t for $t \neq 0$ by setting $H_t \circ H_0^{-1}(z) = z + o(z)$ as $z \to \infty$. For $t \in \mathbb{R}$, define

$$h_t = H_t \circ H_0^{-1} \quad \text{and} \quad f_t(z) = \sqrt{h_t(z^2)}. \tag{3.1}$$

Then, $f_t(z) = z + O(1)$ as $z \to \infty$. Define λ_t and a_t from the expansion

$$f_t(z) = z - \lambda_t + \frac{2a_t}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty. \tag{3.2}$$

Assume that $a_t = t$ for every $t \in \mathbb{R}$. We call $(\lambda_t)_{t\in\mathbb{R}}$ the driving function for the chain $(f_t)_{t\in\mathbb{R}}$ as well as for $(h_t)_{t\in\mathbb{R}}$ and for γ.

Remark 3.1. To explain the role of λ_t and a_t, let us consider the chain $(f_t)_{t\geq 0}$ first. Let $\eta = \sqrt{H_0 \circ \gamma(\cdot)}$, which is a simple chord in $\mathbb{H}$ from 0 to ∞. The conformal map h_t takes $H_0(\hat{\mathbb{C}} \smallsetminus \gamma[-\infty, t])$ onto $\mathbb{C} \smallsetminus \mathbb{R}_+$, hence f_t is a conformal map from $\mathbb{H} \smallsetminus \eta[0,t]$ onto $\mathbb{H}$. Then, $(f_t)_{t\geq 0}$ is the centered Loewner chain for the curve η, and (3.2) simply means that $2a_t$ is the half-plane capacity of $\eta[0,t]$ and $(\lambda_t)_{t\geq 0}$ is the corresponding driving function. Now, take any $s \in \mathbb{R}$. By the same reasoning as above, $(f_{s+t} \circ f_s^{-1})_{t\geq 0}$ is the centered Loewner chain for the curve $t \mapsto \sqrt{H_s \circ \gamma(s+t)}$. Moreover, (3.2) implies

$$f_{s+t} \circ f_s^{-1}(z) = z - (\lambda_{s+t} - \lambda_s) + \frac{2(a_{s+t} - a_s)}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty$$

for $s \in \mathbb{R}$ and $t \geq 0$. Thus, $\lambda_{s+t} - \lambda_s$ and $2(a_{s+t} - a_s)$ are respectively the driving function and the half-plane capacity corresponding to the chain $(f_{s+t} \circ f_s^{-1})_{t\geq 0}$. In particular, the assumption $a_t = t$ for all $t \in \mathbb{R}$ means that the chain $(f_{s+t} \circ f_s^{-1})_{t\geq 0}$ is in capacity parameterization.

Remark 3.2. The loop driving function generalizes the chordal Loewner driving function. If η is a simple chord in $(\mathbb{H}; 0, \infty)$ with driving function $\lambda : \mathbb{R}_+ \to \mathbb{R}$, then the Jordan curve $\gamma := (\eta(\cdot))^2 \cup \mathbb{R}_+$ with the same orientation as η (from 0 to ∞), root ∞, $\gamma(0) = 0$, and $H_0(z) = z$, has the driving function $(\tilde{\lambda})_{t\in\mathbb{R}}$ where $\tilde{\lambda}_t = \lambda_t$ if $t \geq 0$ and $\tilde{\lambda}_t = 0$ otherwise.

Definition 3.3. The *Loewner energy* of a Jordan curve γ is

$$I^L(\gamma) = \frac{1}{2}\int_{-\infty}^{\infty} \dot{\lambda}_t^2 \, \mathrm{d}t,$$

where $(\lambda_t)_{t\in\mathbb{R}}$ is the driving function of γ described above. Theorem 1.2 shows that this energy does not depend on the parametrization of the curve (but we will not use this fact in our proof).

Remark 3.4. From Remark 3.1, for all $s < t$, the partial chordal Loewner energy of $\gamma[s,t]$ in the simply connected domain $\hat{\mathbb{C}} \smallsetminus \gamma[-\infty, s]$ with prime ends $\gamma(s)$ and $\gamma(-\infty)$ is given by

$$I^L_{\hat{\mathbb{C}} \smallsetminus \gamma[-\infty,s]}(\gamma[s,t]) = \frac{1}{2}\int_s^t \dot{\lambda}_r^2 \, \mathrm{d}r$$

where the slit domain $\hat{\mathbb{C}} \smallsetminus \gamma[-\infty, s]$ is always understood with marked prime ends being the two ends of $\gamma[-\infty, s]$.

3.2 Variation of Loewner energy for a part of the quasicircle

We now consider deformations of the Jordan curve γ. Let $\mu \in L^\infty(\hat{\mathbb{C}})$ be a complex-valued function with compact support in $\hat{\mathbb{C}} \smallsetminus \gamma$. For $\varepsilon \in \mathbb{R}$ with $\|\varepsilon\mu\|_\infty < 1$, let $\omega^{\varepsilon\mu} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a quasiconformal homeomorphism which satisfies the Beltrami equation

$$\partial_{\bar{z}}\omega^{\varepsilon\mu} = (\varepsilon\mu)\partial_z\omega^{\varepsilon\mu}.$$

Denote the deformation of γ under the quasiconformal map $\omega^{\varepsilon\mu}$ as

$$\gamma^{\varepsilon\mu} = \omega^{\varepsilon\mu} \circ \gamma.$$

Let $H_t^{\varepsilon\mu} : \hat{\mathbb{C}} \smallsetminus \gamma^{\varepsilon\mu}[-\infty, t] \to \mathbb{C} \smallsetminus \mathbb{R}_+$ be the uniformizing map with the normalizations $H_t^{\varepsilon\mu} \circ H_t^{-1}(0) = 0$ and $H_t^{\varepsilon\mu} \circ H_t^{-1} = z + O(1)$ as $z \to \infty$. Note that this implies $\gamma^{\varepsilon\mu}(t) = 0$ and $\gamma^{\varepsilon\mu}(\infty) = \infty$, as well as $H_t^{\varepsilon\mu} \circ (H_0^{\varepsilon\mu})^{-1} = z + o(z)$ as $z \to \infty$.
We define similarly the chains $(h_t^{\varepsilon\mu})_{t\in\mathbb{R}}$ and $(f_t^{\varepsilon\mu})_{t\in\mathbb{R}}$, the driving function $(\lambda_t^{\varepsilon\mu})_{t\in\mathbb{R}}$, and the capacity function $(a_t^{\varepsilon\mu})_{t\in\mathbb{R}}$. That is,

$$h_t^{\varepsilon\mu} = H_t^{\varepsilon\mu} \circ (H_0^{\varepsilon\mu})^{-1} \quad \text{and} \quad f_t^{\varepsilon\mu} = \sqrt{h_t^{\varepsilon\mu}(z^2)}.$$

We define $\lambda_t^{\varepsilon\mu}$ and $a_t^{\varepsilon\mu}$ from the expansion

$$f_t^{\varepsilon\mu}(z) = z - \lambda_t^{\varepsilon\mu} + \frac{2a_t^{\varepsilon\mu}}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty.$$

Remark 3.5. The map $\omega^{\varepsilon\mu}$, and hence the Jordan curve $\gamma^{\varepsilon\mu}$, is unique only up to a post-composition by some Möbius transformation. The choice of $\omega^{\varepsilon\mu}$ does not affect our analysis, because the first step in it is always to apply the appropriately normalized uniformizing map $H_t^{\varepsilon\mu}$ from $\hat{\mathbb{C}} \smallsetminus \gamma^{\varepsilon\mu}[-\infty, t]$ onto $\mathbb{C} \smallsetminus \mathbb{R}_+$.

In this section, we translate Corollary 2.4 into an analogous formula for the Weil–Peterson curve γ and its deformation $\gamma^{\varepsilon\mu}$. The following is the main result.

Proposition 3.6. *Suppose $s < t$ and $I^L_{\hat{\mathbb{C}}\smallsetminus\gamma[-\infty,s]}(\gamma[s,t]) < \infty$. Then,*

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} I^L_{\hat{\mathbb{C}}\smallsetminus\gamma^{\varepsilon\mu}[-\infty,s]}(\gamma^{\varepsilon\mu}[s,t]) = -\frac{4}{\pi}\,\mathrm{Re}\int_{\mathbb{C}\smallsetminus\gamma} \mu(z)(\mathscr{S}H_t(z) - \mathscr{S}H_s(z))\,\mathrm{d}^2z. \tag{3.3}$$

The following lemma is a straightforward calculation used in the proof of Proposition 3.6.

Lemma 3.7. *Suppose λ_t is absolutely continuous. Then, for each $z \in H_0(\hat{\mathbb{C}} \smallsetminus \gamma)$, the Schwarzian $\mathscr{S}h_t(z)$ is absolutely continuous in t. Moreover, for almost every t,*

$$\frac{\partial}{\partial t}\mathscr{S}h_t(z) = -\frac{3h_t'(z)^2}{4h_t(z)^{5/2}}\dot{\lambda}_t. \tag{3.4}$$

Proof. From the relation

$$\mathscr{S}h_{t+u}(z) = \mathscr{S}[h_{t+u} \circ h_t^{-1}](h_t(z)) \cdot h_t'(z)^2 + \mathscr{S}h_t(z),$$

we deduce

$$\frac{\partial}{\partial t}\mathscr{S}h_t(z) = \frac{\partial}{\partial u}\Big|_{u=0}(\mathscr{S}h_{t+u}(z) - \mathscr{S}h_t(z)) = h_t'(z)^2 \cdot \frac{\partial}{\partial u}\Big|_{u=0}\mathscr{S}[h_{t+u}\circ h_t^{-1}](h_t(z)).$$

Hence, it suffices to show that

$$\frac{\partial}{\partial u}\mathscr{S}[h_{t+u}\circ h_t^{-1}](h_t(z))\Big|_{u=0} = -\frac{3\dot\lambda_t}{4h_t(z)^{5/2}}. \tag{3.5}$$

To see this, note that $\tilde f_u := f_{t+u}\circ f_t^{-1}$ solves the Loewner equation

$$\partial_u \tilde f_u(z) = \frac{2}{\tilde f_u(z)} - \dot\lambda_{t+u}.$$

(See Remark 3.1.) Since $\tilde h_u(z) := h_{t+u}\circ h_t^{-1}(z) = \tilde f_u(\sqrt{z})^2$, we have

$$\partial_u \tilde h_u(z) = 2\tilde f_u(\sqrt{z})\partial_u \tilde f_u(\sqrt{z}) = 4 - 2\dot\lambda_{t+u}\tilde h_u(z)^{1/2}.$$

Then, because $\tilde h_0(z) = z$,

$$\frac{\partial(\mathscr{S}\tilde h_u(z))}{\partial u}\Big|_{u=0} = (\partial_u \tilde h_u(z)|_{u=0})''' = (4 - 2\dot\lambda_t z^{1/2})''' = -\frac{3\dot\lambda_t}{4z^{5/2}}. \tag{3.6}$$

Replacing z in (3.6) with $h_t(z)$, we obtain (3.5). This completes the proof. □

Proof of Proposition 3.6. Let us first consider the case $s=0$. Letting $\eta(t) = \sqrt{H_0\circ\gamma(t)}$ and $\eta^\varepsilon(t) = \sqrt{H_0^{\varepsilon\mu}\circ\gamma^{\varepsilon\mu}(t)} = \sqrt{H_0^{\varepsilon\mu}\circ\omega^{\varepsilon\mu}\circ\gamma(t)}$, we have $\eta^\varepsilon = \psi^\varepsilon\circ\eta$ where $\psi^\varepsilon(z) = \sqrt{(H_0^{\varepsilon\mu}\circ\omega^{\varepsilon\mu}\circ H_0^{-1})(z^2)}$. Let $\varepsilon\nu$ be the Beltrami differential corresponding to the quasiconformal map $\psi^\varepsilon : \mathbb{H}\to\mathbb{H}$. Let $\varepsilon\mu_0$ denote the Beltrami differential of $H_0^{\varepsilon\mu}\circ\omega^{\varepsilon\mu}\circ H_0^{-1}$. Then, $\nu(\zeta) = \mu_0(\zeta^2)(|\zeta|^2/\zeta^2)$.

Substituting this ν into (2.24) and letting $\zeta = \sqrt{z}$, since $f_t(\zeta) = \sqrt{h_t(z)}$ and $f_t'(\zeta)/\zeta = h_t'(z)/\sqrt{h_t(z)}$, we get

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}I^L(\eta^\varepsilon[0,T])\Big|_{\varepsilon=0} &= \frac{12}{\pi}\,\mathrm{Re}\int_{\mathbb{H}}\nu(\zeta)\int_0^T \dot\lambda_t\frac{f_t'(\zeta)^2}{f_t(\zeta)^3}\mathrm{d}t\,\mathrm{d}^2\zeta\\ &= \frac{3}{\pi}\,\mathrm{Re}\int_{\mathbb{C}\smallsetminus\mathbb{R}_+}\mu_0(z)\int_0^T \dot\lambda_t\frac{h_t'(z)^2}{h_t(z)^{5/2}}\mathrm{d}t\,\mathrm{d}^2 z.\end{aligned}$$

Applying Lemma 3.7, we have

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}I^L(\eta^\varepsilon[0,T])\Big|_{\varepsilon=0} &= -\frac{4}{\pi}\,\mathrm{Re}\int_{\mathbb{C}\smallsetminus\mathbb{R}_+}\mu_0(z)\int_0^T \frac{\partial(\mathscr{S}h_t(z))}{\partial t}\,\mathrm{d}t\,\mathrm{d}^2 z\\ &= -\frac{4}{\pi}\,\mathrm{Re}\int_{\mathbb{C}\smallsetminus\mathbb{R}_+}\mu_0(z)\mathscr{S}h_T(z)\,\mathrm{d}^2 z.\end{aligned}$$

Recalling our definition of $\varepsilon\mu_0$, we have $\mu_0(H_0(z)) = \mu(z)(H_0'(z)^2/|H_0'(z)|^2)$. Moreover, from $H_T = h_T\circ H_0$, we have

$$\mathscr{S}h_T(H_0(z))\cdot H_0'(z)^2 = \mathscr{S}H_T(z) - \mathscr{S}H_0(z).$$

Hence,

$$\int_{\mathbb{C}\smallsetminus\mathbb{R}_+}\varepsilon\mu_0(z)\mathscr{S}h_T(z)\,\mathrm{d}^2 z = \int_{\mathbb{C}\smallsetminus\gamma}\varepsilon\mu(z)(\mathscr{S}H_T(z) - \mathscr{S}H_0(z))\,\mathrm{d}^2 z.$$

Therefore, the case $s=0$ holds. In fact, this implies (3.3) for any $s\in\mathbb{R}$ because the parameterization of γ is arbitrary up to translations as discussed in Remark 3.1. □

3.3 Variation of the loop Loewner energy

The goal of this section is to prove Theorem 1.4. Let $H_\infty : \hat{\mathbb{C}} \smallsetminus \gamma \to \mathbb{C} \smallsetminus \mathbb{R}$ be any conformal map which maps $\Omega \to \mathbb{H}$ and $\Omega^* \to \mathbb{H}^*$, respectively. Note that the map H_∞ restricted to Ω coincides with f^{-1} (as in Theorem 1.4) post-composed by a Möbius transformation, so $\mathscr{S} H_\infty|_\Omega = \mathscr{S}[f^{-1}]$. Similarly, $\mathscr{S} H_\infty|_{\Omega^*} = \mathscr{S}[g^{-1}]$. In view of Remark 3.4 and Proposition 3.6, it suffices to show that as $s \to -\infty$ and $t \to \infty$, we have

$$\int_{\mathbb{C}\smallsetminus\gamma} \mu(z)(\mathscr{S} H_t(z) - \mathscr{S} H_s(z))\, \mathrm{d}^2 z \to \int_{\mathbb{C}\smallsetminus\gamma} \mu(z)\mathscr{S} H_\infty(z)\, \mathrm{d}^2 z, \tag{3.7}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_s^t \left|\frac{\mathrm{d}\lambda_r^{\varepsilon\mu}}{\mathrm{d}a_r^{\varepsilon\mu}}\right|^2 \mathrm{d}a_r^{\varepsilon\mu} \to \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{-\infty}^\infty \left|\frac{\mathrm{d}\lambda_r^{\varepsilon\mu}}{\mathrm{d}a_r^{\varepsilon\mu}}\right|^2 \mathrm{d}a_r^{\varepsilon\mu}. \tag{3.8}$$

For this, we need a few lemmas.

Lemma 3.8. *Suppose $\gamma(\infty) = \infty$. Then, $\mathscr{S} H_s \to 0$ locally uniformly as $s \to -\infty$.*

Proof. Given any $R > 0$, there exists a large negative s_R such that for $s \le s_R$, we have $\gamma[-\infty, s] \cap \{z : |z| < R\} = \varnothing$. Then, H_s is conformal on $R\mathbb{D}$. By the Nehari bound, $|\mathscr{S} H_s(z)| \le 6/(R(1 - \frac{|z|^2}{R^2}))^2$ for every $z \in R\mathbb{D}$. The right-hand side of the inequality tends uniformly to 0 as $R \to \infty$ on any compact subset of $\mathbb{C}$. ☐

Lemma 3.9. *Suppose $\gamma(\infty) = \infty$. Then, $\mathscr{S} H_t \to \mathscr{S} H_\infty$ locally uniformly on $\mathbb{C} \smallsetminus \gamma$ as $t \to \infty$.*

Proof. Choose either component of $\mathbb{C} \smallsetminus \gamma$ and call it U_∞. Let us denote by $\gamma^U(s)$ the prime end of $\gamma(s)$ as viewed from U_∞.

Let $\gamma_t = \gamma[-\infty, t]$ and denote by $\gamma_t^U := \bigcup_{s\in(-\infty,t)} \gamma^U(s)$ the prime ends of γ_t accessible from U_∞. Let Γ_t be the hyperbolic geodesic in $\mathbb{C} \smallsetminus \gamma_t$ connecting $\gamma(t)$ and $\gamma(\infty)$. Let U_t be the component of $\mathbb{C} \smallsetminus (\gamma_t \cup \Gamma_t)$ such that the prime ends of γ_t as viewed from U_t comprise γ_t^U. Observe that if $\mathrm{hm}(z, D; \cdot)$ is the harmonic measure on the domain D as viewed from $z \in D$, then $z \in U_t$ if and only if $\mathrm{hm}(z, \mathbb{C} \smallsetminus \gamma_t; \gamma_t^U) > 1/2$.

We claim that if $z \in U_\infty$, then $z \in \bigcup_{T\ge0} \bigcap_{t\ge T} U_t$. Suppose $z \in U_\infty$. Choose a sufficiently small constant $a \in (0, 1)$ such that $\mathrm{hm}(0, \mathbb{D} \smallsetminus [a, 1], [a, 1]) > 1/2$. Since $\gamma(\infty) = \infty$, for all sufficiently large $t > 0$, we can find $R > 0$ such that $\gamma_t \cap B_{aR}(z) \neq \varnothing$ but $\gamma(t, \infty) \cap B_R(z) = \varnothing$. Then,

$$\mathrm{hm}(z, \mathbb{C} \smallsetminus \gamma_t; \gamma_t^U) \ge \mathrm{hm}(z, B_R(z) \smallsetminus \gamma_t; \gamma_t^U) = \mathrm{hm}(z, B_R(z) \smallsetminus \gamma_t; \gamma_t).$$

By the Beurling projection theorem,

$$\mathrm{hm}(z, B_R(z) \smallsetminus \gamma_t, \gamma_t) \ge \mathrm{hm}(0, R\mathbb{D} \smallsetminus [aR, R]; [aR, R]) = \mathrm{hm}(0, \mathbb{D} \smallsetminus [a, 1], [a, 1]) > 1/2.$$

Note that H_t is a conformal map which sends $\mathbb{C} \smallsetminus (\gamma_t \cup \Gamma_t)$ onto $\mathbb{C} \smallsetminus \mathbb{R}$. Then, by the Carathéodory kernel theorem, H_t post-composed with an appropriate Möbius transformation converges locally uniformly on U_∞ to H_∞ as $t \to \infty$. Consequently, $\mathscr{S} H_t \to \mathscr{S} H_\infty$ locally uniformly on U_∞ as $t \to \infty$. An analogous argument applies to the other component of $\hat{\mathbb{C}} \smallsetminus \gamma$. ☐

Proof of Theorem 1.4. Given $\gamma(\infty) = \infty$, the limit (3.7) follows by applying Lemmas 3.8 and 3.9. Otherwise, choose a Möbius map $A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ so that $A(\infty) = \gamma(\infty)$. If $H_\infty : \hat{\mathbb{C}} \smallsetminus \gamma \to \mathbb{C} \smallsetminus \mathbb{R}$ is a conformal map as in the theorem statement, then $H_\infty \circ A$ is a conformal map from $\hat{\mathbb{C}} \smallsetminus \gamma$ onto $\mathbb{C} \smallsetminus \mathbb{R}$ with $\mathscr{S}[H \circ A] = \mathscr{S}H \cdot (A')^2$. The pullback $A^*\mu$ of μ under φ satisfies $\mu(A(z)) = (A^*\mu)(z)(A'(z))^2/|A'(z)|^2$. Letting $\zeta = A(z)$, we have

$$\int_{\hat{\mathbb{C}} \smallsetminus \gamma} \mu(\zeta) \mathscr{S} H_\infty(\zeta) \, \mathrm{d}^2\zeta = \int_{\hat{\mathbb{C}} \smallsetminus A^{-1} \circ \gamma} A^*\mu(z) \mathscr{S}[H \circ A](z) \, \mathrm{d}^2 z,$$

so we can consider the curve $A^{-1} \circ \gamma$ instead.

To show (3.8), it suffices to prove that we can switch between the integral over $t \in (-\infty, \infty)$ and the derivative in ε. To this end, we prove that the following integral is absolutely convergent:

$$\int_{-\infty}^{\infty} \left| \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \frac{(\dot{\lambda}_t^{\varepsilon\nu})^2}{\dot{a}_t^{\varepsilon\nu}} \right| \mathrm{d}t < \infty. \tag{3.9}$$

It follows from (2.24) that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \frac{(\dot{\lambda}_t^{\varepsilon\nu})^2}{\dot{a}_t^{\varepsilon\nu}} = \dot{\lambda}_t \int_{\mathbb{H}} \nu(z) \frac{f_t'(z)^2}{f_t(z)^3} \, \mathrm{d}^2 z.$$

From Lemma A.1, the assumption that $\int_{-\infty}^{\infty} \dot{\lambda}_t^2 \, \mathrm{d}t < \infty$, and the Cauchy–Schwarz inequality, it follows that $\int_{-\infty}^{\infty} |\dot{\lambda}_t \frac{f_t'(z)^2}{f_t(z)^3}| \, \mathrm{d}t$ is finite and locally uniform in z. Since ν is compactly supported and $\|\nu\|_\infty < \infty$, this implies (3.9) and hence (3.8). This proves Theorem 1.4 as explained at the beginning of the section. □

A Appendix: Large time behavior of Loewner chain

The following lemma is used in the proof of Theorem 1.4 to switch the order between the derivative and the integral in (3.8). Note that if $\lambda_t = 0$ for all t, then $f_t(z) = \sqrt{z^2 + 4t}$. In particular, $f_t(z)^2/(4t) \to 1$ and $2|t|^{1/2}|f_t'(z)| \to 1$ as $t \to \pm\infty$.

Lemma A.1. *Let γ be a Jordan curve with $I^L(\gamma) < \infty$. Let H_t and f_t the uniformizing maps defined in Section 3.1. Then, as $t \to \pm\infty$, $f_t(z)^2/(4t) \to 1$ and $|f_t'(z)| = |t|^{-1/2+o(1)}$ locally uniformly for $z \in H_0(\hat{\mathbb{C}} \smallsetminus \gamma)$.*

Proof. Let us denote $f_t(z) = u_t + iv_t$, where $u_t \in \mathbb{R}$ and $v_t > 0$. The real and imaginary parts of the Loewner equation $\partial_t f_t(z) = 2/f_t(z) - \dot{\lambda}_t$ correspond to

$$\dot{u}_t = \frac{2u_t}{u_t^2 + v_t^2} - \dot{\lambda}_t \quad \text{and} \quad \dot{v}_t = -\frac{2v_t}{u_t^2 + v_t^2}. \tag{A.1}$$

Let us consider the $t \to +\infty$ limit first. Given $\varepsilon > 0$, since γ has finite Loewner energy, we can choose a large T_0 so that $\int_{T_0}^{\infty} \dot{\lambda}_t^2 \, \mathrm{d}t < \varepsilon$. Since

$$\frac{\mathrm{d}u_t^2}{\mathrm{d}t} = \frac{4u_t^2}{u_t^2 + v_t^2} - 2\dot{\lambda}_t u_t,$$

for $T_0 \le t \le T$, we have

$$\begin{aligned} u_t^2 &\le u_{T_0}^2 + 4\int_{T_0}^{T} \frac{u_t^2}{u_t^2+v_t^2}\,\mathrm{d}t + 2\int_{T_0}^{T} |\dot\lambda_t u_t|\,\mathrm{d}t \\ &\le u_{T_0}^2 + 4(T-T_0) + 2\Big(\int_{T_0}^{T} \dot\lambda_t^2\,\mathrm{d}t\Big)^{1/2}\Big(\int_{T_0}^{T} u_t^2\,\mathrm{d}t\Big)^{1/2} \\ &\le u_{T_0}^2 + 4(T-T_0) + 2\Big(\varepsilon(T-T_0)\sup_{t\in[T_0,T]} u_t^2\Big)^{1/2}. \end{aligned} \tag{A.2}$$

Hence,

$$\sup_{t\in[T_0,T]} u_t^2 \le u_{T_0}^2 + 4(T-T_0) + 2\Big(\varepsilon(T-T_0)\sup_{t\in[T_0,T]} u_t^2\Big)^{1/2}.$$

Completing the square, we obtain

$$\Big(\sup_{t\in[T_0,T]} u_t^2\Big)^{1/2} \le \big(u_{T_0}^2 + (4+\varepsilon)(T-T_0)\big)^{1/2} + \big(\varepsilon(T-T_0)\big)^{1/2}.$$

Then, there exists a $T_1 \ge T_0$ such that for all $T \ge T_1$,

$$\sup_{t\in[T_0,T]} u_t^2 \le u_{T_0}^2 + 4(1+\varepsilon)(T-T_0). \tag{A.3}$$

Now, from

$$\frac{\partial}{\partial t}\mathrm{Re}(f_t(z)^2) = \frac{\mathrm{d}(u_t^2 - v_t^2)}{\mathrm{d}t} = 4 - 2\dot\lambda_t u_t,$$

we have

$$\begin{aligned} \big|\mathrm{Re}(f_T(z)(z)^2) - 4T\big| &\le \mathrm{Re}(f_{T_0}(z)^2) + 4T_0 + 2\int_{T_0}^{T} |\dot\lambda_t u_t|\,\mathrm{d}t \\ &\le \mathrm{Re}(f_{T_0}(z)^2) + 4T_0 + 2\Big(\varepsilon(T-T_0)\sup_{t\in[T_0,T]} u_t^2\Big)^{1/2}. \end{aligned}$$

Substituting (A.3), we can find a $T_2 \ge T_1$ such that for all $T \ge T_2$,

$$\big|\mathrm{Re}(f_T(z)^2) - 4T\big| \le 5\sqrt{\varepsilon}T.$$

Since the choice of ε was arbitrary, we conclude $\mathrm{Re}(f_t(z)^2)/(4t) \to 1$ as $t \to \infty$.

As for $\mathrm{Im}(f_t(z)^2) = 2u_t v_t$, note that (A.1) implies v_t is monotonically decreasing. Hence, (A.3) implies $\mathrm{Im}(f_t(z)^2)/t \to 0$ as $t \to \infty$. Combining the limits of the real and imaginary parts, we obtain $f_t(z)^2/(4t) \to 1$ as $t \to \infty$. Moreover, since $f_{T_0}(z) = u_{T_0} + iv_{T_0}$ depends continuously on z whereas T_0 was chosen independently of z, the limit we proved converges uniformly on each compact subset of $H_0(\hat{\mathbb{C}} \smallsetminus \gamma)$.

Let us now consider the $t \to -\infty$ limit. This time, since the Loewner energy of γ is finite, we can find a large negative $\tilde T_0$ such that $\int_{-\infty}^{\tilde T_0} \dot\lambda_t^2\,\mathrm{d}t < \varepsilon$. Hence, (A.2) implies that there exists a $\tilde T_1 \le \tilde T_0$ such that for all $T \le \tilde T_1$,

$$\sup_{t\in[T,\tilde T_1]} u_t^2 \le u_{\tilde T_0}^2 + 4(1+\varepsilon)|T-T_0|.$$

Again, we have

$$\big|\mathrm{Re}(f_T(z)^2) - 4T\big| \le \mathrm{Re}(f_{\tilde{T}_0}^2) + 4|\tilde{T}_0| + 2\Big(\varepsilon|T-\tilde{T}_0| \sup_{t\in[T,\tilde{T}_0]} u_t^2\Big)^{1/2},$$

and choosing ε to be arbitrarily small, we obtain $\mathrm{Re}(f_t(z)^2)/(4t) \to 1$ as $t \to -\infty$.

For the imaginary part of $f_t(z)^2$, we consider

$$\frac{\partial}{\partial t}\mathrm{Im}(f_t(z)^2) = \frac{\mathrm{d}(2u_t v_t)}{\mathrm{d}t} = -2\dot{\lambda}_t v_t.$$

Since $\mathrm{Re}(f_t(z)^2) = u_t^2 - v_t^2$, we have that as $T \to -\infty$,

$$\sup_{t\in[T,\tilde{T}_0]} v_t^2 \le \sup_{t\in[T,\tilde{T}_0]} |\mathrm{Re}(f_t(z)^2)| + \sup_{t\in[T,\tilde{T}_0]} u_t^2 \le (8+o(1))|T|.$$

Hence, using the Cauchy–Schwarz inequality as above, we have

$$\big|\mathrm{Im}(f_T(z)^2)\big| \le \big|\mathrm{Im}(f_{\tilde{T}_0}(z)^2)\big| + 2\int_T^{\tilde{T}_0} |\dot{\lambda}_t v_t|\,\mathrm{d}t = o(T)$$

as $T \to -\infty$. Therefore, $f_t(z)^2/(4t) \to 1$ as $t \to -\infty$. Again, this limit converges uniformly on compact subsets of $\mathbb{H}$ since $f_{\tilde{T}_0}(z)$ depends continuously on z and $\tilde{T}_0$ can be chosen independently of z on a compact set.

Finally, to estimate $|f_t'(z)|$, consider the equation

$$\frac{\partial}{\partial t}\log|f_t'(z)| = \mathrm{Re}\Big(\frac{\partial_t f_t'(z)}{f_t'(z)}\Big) = -\mathrm{Re}\frac{2}{f_t(z)^2}.$$

As $t \to \pm\infty$, the right-hand side behaves as $(-1/2+o(1))t^{-1}$. We thus obtain $|f_t'(z)| = |t|^{-1/2+o(1)}$ as claimed. □

References

[AB60] Lars Ahlfors and Lipman Bers. Riemann's mapping theorem for variable metrics. *Ann. of Math. (2)*, 72:385–404, 1960.

[dB85] Louis de Branges. A proof of the Bieberbach conjecture. *Acta Math.*, 154(1-2):137–152, 1985.

[FS17] Peter K. Friz and Atul Shekhar. On the existence of SLE trace: finite energy drivers and non-constant κ. *Probab. Theory Related Fields*, 169(1-2):353–376, 2017.

[Lin05] Joan R. Lind. A sharp condition for the Loewner equation to generate slits. *Ann. Acad. Sci. Fenn. Math.*, 30(1):143–158, 2005.

[Loe23] Karl Loewner. Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I. *Math. Ann.*, 89(1-2):103–121, 1923.

[LSW03] Gregory Lawler, Oded Schramm, and Wendelin Werner. Conformal restriction: the chordal case. *J. Amer. Math. Soc.*, 16(4):917–955, 2003.

[LT16] Joan Lind and Huy Tran. Regularity of Loewner curves. *Indiana Univ. Math. J.*, 65(5):1675–1712, 2016.

[LW04] Gregory F. Lawler and Wendelin Werner. The Brownian loop soup. *Probab. Theory Related Fields*, 128(4):565–588, 2004.

[MR05] Donald E. Marshall and Steffen Rohde. The Loewner differential equation and slit mappings. *J. Amer. Math. Soc.*, 18(4):763–778, 2005.

[RS05] Steffen Rohde and Oded Schramm. Basic properties of SLE. *Ann. of Math. (2)*, 161(2):883–924, 2005.

[RW21] Steffen Rohde and Yilin Wang. The Loewner energy of loops and regularity of driving functions. *Int. Math. Res. Not. IMRN*, 2021(10):7715–7763, 2021.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[TT06] Leon A. Takhtajan and Lee-Peng Teo. Weil-Petersson metric on the universal Teichmüller space. *Mem. Amer. Math. Soc.*, 183(861):viii+119, 2006.

[Wan19a] Yilin Wang. The energy of a deterministic Loewner chain: reversibility and interpretation via SLE_{0+}. *J. Eur. Math. Soc. (JEMS)*, 21(7):1915–1941, 2019.

[Wan19b] Yilin Wang. Equivalent descriptions of the Loewner energy. *Invent. Math.*, 218(2):573–621, 2019.

[Wan21] Yilin Wang. A note on Loewner energy, conformal restriction and Werner's measure on self-avoiding loops. *Ann. Inst. Fourier (Grenoble)*, 71(4):1791–1805, 2021.

[Wer08] Wendelin Werner. The conformally invariant measure on self-avoiding loops. *J. Amer. Math. Soc.*, 21(1):137–169, 2008.