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ON THE LOEWNER ENERGY OF SIMPLE PLANAR CURVES

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He has made everything beautiful in its time.
He has also set eternity in the human heart;
Yet no one can fathom what God has done from beginning to end.

Ecclesiastes 3:11

To my mom 海燕

Abstract

In this thesis, we introduce and study the Loewner energy for simple planar curves and relate this quantity to ideas and concepts coming from random conformal geometry, geometric function theory and Teichmüller theory.

One motivation for the definition of the Loewner energy for chords connecting two boundary points of a simply connected domain is that it arises from the large deviations of Schramm-Loewner evolution (SLE). This provides a probabilistic interpretation of the Loewner energy that allows us to prove its reversibility for any deterministic chord.

We generalize the chordal Loewner energy to simple loops on the Riemann sphere and show that the loop energy has a remarkable number of symmetries. We further derive an equivalent characterization of the loop energy using zeta-regularized determinants of Laplacians. This then identifies it with a Kähler potential, introduced by Takhtajan and Teo, of the Weil-Petersson metric on the universal Teichmüller space. In relation to determinants of Laplacians, we derive another measure-theoretic description of the Loewner energy using the Brownian loop measure.

We also establish the correspondence between the Hölder regularity of a planar curve and its Loewner driving function. The correspondence is an important tool in our analysis but is also of independent interest.

Résumé

Dans cette thèse, nous introduisons et étudions l'énergie de Loewner pour les courbes planaires simples et relions cette quantité à des concepts issus de la géométrie conforme aléatoire, de la théorie géométrique des fonctions et de la théorie de Teichmüller.

Une des motivations de la définition de l'énergie de Loewner pour les cordes reliant deux points au bord d'un domaine simplement connexe tient au fait qu'elle résulte des grandes déviations de l'évolution de Schramm-Loewner (SLE). Ceci fournit une interprétation probabiliste de l'énergie de Loewner qui nous permet de prouver sa réversibilité pour toute corde déterministe.

Nous généralisons l'énergie de Loewner à des lacets simples sur la sphère de Riemann et montrons qu'elle présente un nombre remarquable de symétries. Nous obtenons en outre une caractérisation équivalente de l'énergie des lacets à l'aide de déterminants zeta-régularisés des laplaciens. Ceci l'identifie ensuite avec un potentiel kählérien introduit par Takhtajan et Teo de la métrique de Weil-Petersson sur l'espace de Teichmüller universel. En ce qui concerne les déterminants des laplaciens, nous obtenons une autre description de l'énergie de Loewner à l'aide de la mesure des lacets browniennes.

Nous établissons également la correspondance entre la régularité höldérienne d'une courbe planaire et sa fonction de Loewner. La correspondance est un outil important dans notre analyse, mais présente également un intérêt indépendant.

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Contents

Chapter 1

Introduction

The main object of study of the present thesis is the Loewner energy of simple planar curves. In the case of a simple curve joining two boundary points of a simply connected domain, this is simply defined to be the Dirichlet energy of the Loewner driving function of this curve (see Section 1.1). Chapters 2 to 5 of this thesis correspond respectively to the four papers [Wan16, RW17, Wan18a, Wan18b]. In the present introduction, we do briefly present some of the main results that are derived in those chapters, and we provide the necessary background to properly state them (to clarify what our own contributions are, we will label all the previously known results by letters and our new results by numbers). They can be listed as follows:

- We show the invariance of the Loewner energy under reversibility of the path (stated as Theorem 1.2 in Section 1.2 of this introduction). The idea of the proof is to use the interpretation of the Loewner energy via the large deviations of SLE (Theorem 1.1), together with the known reversibility result for SLE curves.
- We define the Loewner energy of simple loops and show their root-invariance (Theorem 1.3, also in Section 1.2 of this introductory chapter). This result, as well as the reversibility of the Loewner energy, does strongly suggest that there must exist other equivalent ways to express or describe the Loewner energy, which is what the results stated in the next bullet points will provide.
- We provide a formula that relates the Loewner energy of a smooth loop to zeta-regularized determinants of Laplacians (Theorem 1.4, in Section 1.3 of the introduction).
- We show that the Loewner energy of an arbitrary simple loop can be written as a renormalized total mass of Brownian loops attached

to the loop (Theorem 1.5 in Section 1.4).

- We prove that finite energy loops are quasicircles (Theorem 1.6 in Section 1.5 of this introduction), and confirm the conjecture that a $C^{\alpha+1/2}$ curve is driven by a C^{α} Loewner driving function for the case where $\alpha \in (1/2, 3/2]$ (Theorem 1.7 in Section 1.5).
- We prove the equivalence between having finite loop energy and being a Weil-Petersson quasicircle, and show that the Loewner energy is equal to the universal Liouville action which is a Kähler potential for the Weil-Petersson metric on the Weil-Petersson Teichmüller space (Theorem 1.9 in Section 1.6 – this section is longer as we need to recall more background material).

1.1 Definition of Loewner energy

1.1.1 Chordal Loewner energy

Loewner's theory on univalent functions [Loe23] is a powerful tool to study planar shapes in complex analysis and geometric function theory. For instance, it is used in the proof of the Bieberbach conjecture by de Branges [dBr85], and has also more recently received a lot of attention after Schramm's [Sch00] construction of random conformally invariant fractal curves. These curves are now termed Schramm-Loewner Evolution (SLE) and describe scaling limits of interfaces in many statistical mechanics lattice models when conformal symmetries are present.

Let us briefly recall the **Loewner transform** of a continuous simple chord γ from 0 to infinity in the open upper half-plane $\mathbb{H}$ (denoted by $(\mathbb{H}, 0, \infty)$).

- We parametrize the curve such that the conformal map (i.e. biholomorphic function) g_t from $\mathbb{H} \setminus \gamma[0,t]$ onto $\mathbb{H}$ with $g_t(z) = z + o(1)$ as $z \to \infty$ in fact satisfies $g_t(z) = z + 2t/z + o(1/z)$. In other words, the *capacity* of $\gamma[0,t]$ is t.
- One can extend g_t by continuity to the boundary point γ_t. The real-valued function $t \mapsto W(t) := g_t(\gamma_t)$ is referred to as the *driving function* of γ. Moreover, W is continuous and $W(0) = 0$.

- For $s > 0$, the driving function of the curve $g_s(\gamma[s,\infty)) - W(s)$ is given by $t \mapsto W(s+t) - W(s)$.
- For $\lambda > 0$, the driving function of $\lambda\gamma$ is $\tilde{W}(t) = \lambda W(t/\lambda^2)$.
- The function $t \mapsto W(t)$ does fully characterize the curve. In fact, for $z \in \mathbb{H} \setminus \gamma[0,t]$, the flow $s \mapsto g_s(z)$ is well defined for $s \le t$, and satisfies
$$\partial_s g_s(z) = 2/(g_s(z) - W(s)), \quad g_0(z) = z. \tag{1.1}$$
It is easy to see that $\gamma[0,t]$ is exactly the set of points z such that the ordinary differential equation (1.1) (with initial condition z) hits the singularity before time t.

When we start with an arbitrary continuous driving function W, we obtain an increasing family of compact hulls $(K_t)_{t\ge 0}$, where K_t is set of points z such that (1.1) hits the singularity before time t. From the last bullet point, if W arises from a simple curve, then $K_t = \gamma[0,t]$, see Section 2.2.1 for more details.

The **Loewner energy** (we will also often just say “the energy”) of a simple curve γ in $\mathbb{H}$ with from 0 targeting at ∞ is defined to be the Dirichlet energy of its driving function:

$$I(\gamma) := I_T(\gamma) := \frac{1}{2}\int_0^T (W(t)')^2 dt = \frac{1}{2}\int_0^T \left(\frac{d(g_t(\gamma_t))}{dt}\right)^2 dt$$

where T is the total capacity of γ which is finite if and only if γ does not make all the way to ∞. The energy does not need to be finite, but for a certain class of sufficiently regular curves γ, it will be. Conversely, as we will explain later on, any real-valued function W with finite energy does necessarily generate a simple curve γ (see Section 2.2.2 for details).

For a simple curve γ from 0 to ∞ in the upper half-plane driven by the function W, the energy satisfies the following *scaling property*:

$$I(\gamma) = I(\lambda\gamma), \quad \forall\lambda > 0$$

because the driving function of $\lambda\gamma$ is $\tilde{W}(t) = \lambda W(t/\lambda^2)$, and

$$\frac{1}{2}\int_0^\infty \tilde{W}'(t)^2 dt = \frac{1}{2}\int_0^\infty \left(W'(\lambda^{-2}t)\right)^2 \lambda^{-2} dt = \frac{1}{2}\int_0^\infty W'(s)^2 ds.$$

Thus, the energy is invariant under any conformal equivalence preserving $(\mathbb{H}, 0, \infty)$. So one can define the Loewner energy of a curve from a boundary point a to another boundary point b in a simply connected domain D, by applying any conformal map from (D, a, b) to $(\mathbb{H}, 0, \infty)$. We use $I_{D,a,b}$ to indicate the energy in a different domain than $(\mathbb{H}, 0, \infty)$.

Let us finally remark that a curve with 0 energy is driven by the constant 0 function, and is therefore the imaginary axis. This is (in view of conformal invariance) the hyperbolic geodesic between 0 and ∞. The energy of a curve from a to b in D can therefore be viewed as a way to measure how much a Jordan curve differs from the hyperbolic geodesic between a and b in D.

1.1.2 Loop Loewner energy

We also define the Loewner energy for Jordan curves (simple loops) via a limiting procedure. Let $\gamma : [0,1] \to \mathbb{C}$ be an oriented Jordan curve with the marked point $\gamma(0) = \gamma(1)$. For every $\varepsilon > 0$, $\gamma[\varepsilon, 1]$ is a chord connecting $\gamma(\varepsilon)$ to $\gamma(1)$ in the simply connected domain $\hat{\mathbb{C}} \setminus \gamma[0, \varepsilon]$, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \simeq S^2$ is the Riemann sphere. The **rooted loop Loewner energy** is

$$I^L(\gamma, \gamma(0)) := \lim_{\varepsilon \to 0} I_{\hat{\mathbb{C}} \setminus \gamma[0,\varepsilon], \gamma(\varepsilon), \gamma(0)}(\gamma[\varepsilon, 1]).$$

The loop energy generalizes the chordal energy. In fact, assume that $\gamma = \eta \cup \mathbb{R}_+$ is parametrized such that $\gamma[0, 1/2] = \mathbb{R}_+$ and $\gamma[1/2, 1] = \eta$, then from the additivity of chordal energy,

$$\begin{aligned} I^L(\gamma, \infty) =& I_{\mathbb{C}\setminus\mathbb{R}_+,0,\infty}(\eta) + \lim_{\varepsilon\to 0} I_{\hat{\mathbb{C}}\setminus\gamma[0,\varepsilon],\gamma(\varepsilon),\gamma(0)}(\gamma[\varepsilon, 1/2]) \\ =& I_{\mathbb{C}\setminus\mathbb{R}_+,0,\infty}(\eta), \end{aligned}$$

since $\gamma[\varepsilon, 1/2]$ is contained in the hyperbolic geodesic between $\gamma(\varepsilon)$ and $\gamma(0)$ in $\hat{\mathbb{C}} \setminus \gamma[0, \varepsilon]$ for all $0 < \varepsilon < 1/2$.

1.2 Reversibility and root-invariance via SLE

Now we explain briefly the relation between SLE and the Loewner energy, which led to the proof of reversibility and root-invariance of the Loewner energy.

1.2.1 Background on SLE

The **chordal Schramm-Loewner evolution** of parameter κ, denoted by SLE_κ, is the random (non self-intersecting) curve tracing out the hulls $(K_t)_{t\geq 0}$ generated by $\sqrt{\kappa}B$ via the Loewner transform, where B is the standard Brownian motion and $\kappa \geq 0$.

The SLEs have attracted much attention during the last 20 years, as they are the first construction of random self-avoiding paths which also play a central role in the emerging field of two dimensional random conformal geometry. SLEs describe the interfaces in the scaling limit of various statistical mechanics models:

- $\mathrm{SLE}_2 \leftrightarrow$ Loop-erased random walk [LSW04];
- $\mathrm{SLE}_{8/3} \leftrightarrow$ Self-avoiding walk (conjecture);
- $\mathrm{SLE}_3 \leftrightarrow$ Critical Ising model interface [Smi10];
- $\mathrm{SLE}_4 \leftrightarrow$ Level line of the Gaussian free field [SS09];
- $\mathrm{SLE}_6 \leftrightarrow$ Critical independent percolation interface [Smi01];
- $\mathrm{SLE}_8 \leftrightarrow$ Contour line of uniform spanning tree [LSW04] ;
- $\cdots$

We now very briefly review the definition and relevant properties of chordal SLE. For further SLE background, readers can also refer to [Law08], [Wer04].

SLEs are the unique processes on hulls generated by continuous functions satisfying *scaling-invariance* and *domain Markov property*. i.e. for $\lambda > 0$, the law is invariant under the scaling transformation

$$(K_t)_{t\geq 0} \mapsto (K_t^\lambda := \lambda K_{\lambda^{-2}t})_{t\geq 0}$$

and for all $s \in [0,\infty)$, if one defines $K_t^{(s)} = g_s(K_{s+t}) - W_s$, where (K_s) is driven by W, then $(K_t^{(s)})_{t\geq 0}$ has the same distribution as $(K_t)_{t\geq 0}$ and is independent of $\mathscr{F}_s = \sigma(W_r : r \leq s)$.

The scaling-invariance makes it possible to also define SLE_κ in other simply connected domains (just take the image of SLE in the upper half-plane via some conformal map from $(\mathbb{H}, 0, \infty)$ onto (D, a, b) to define SLE from a to b in D).

Theorem A ([RS05])**.** For $\kappa \in [0,4]$, SLE_κ is almost surely a simple curve $(\gamma_t)_{t\geq 0}$ starting at 0. Moreover, $|\gamma_t| \to \infty$ as $t \to \infty$ almost surely.

Given that SLE_κ curves arise as scaling limits of lattice based models from statistical physics, it was natural to conjecture that chordal SLE is reversible i.e. that the time-reversal of a chordal SLE_κ from a to b in D is a chordal SLE_κ from b to a in D (note that here, this means an identity in distribution between two random curves). Proving this has turned out to be a challenge that resisted for some years, but was settled by Zhan [Zha08] in the case of the simple curves $\kappa \in [0,4]$, via rather non-trivial couplings of both ends of the path (see also Dubédat's commutation relations [Dub07], and Miller and Sheffield's approach based on the Gaussian Free Field [MS16a, MS16b, MS16c] that also provides a proof of reversibility for the non-simple case where $\kappa \in (4,8]$).

Theorem B ([Zha08])**.** For $\kappa \in [0,4]$, the distribution of the trace of SLE_κ in $(\mathbb{H}, 0, \infty)$ coincides with that of its image under $z \to -1/z$.

Let us also recall the standard large deviation theorem for Brownian paths.

Let $T > 0$ and $C_0([0,T])$ be the set of continuous function W on $[0,T]$ with $W(0) = 0$, endowed with the L^∞ norm. Let

$$I_T(W) := \frac{1}{2}\int_0^T \left(\frac{dW(t)}{dt}\right)^2 dt$$

denote the Dirichlet energy of W on $[0,T]$. The set of functions $W \in C_0([0,T])$ such that $I_T(W)$ is finite, is the *Cameron-Martin* space of the Brownian motion B. We have also $I_T(W) < \infty$ if and only if the law of $W + B$ is absolutely continuous with respect to B. One can thus think of the Cameron-Martin space as the "skeleton" of Brownian motion. Schilder's theorem below is in this flavor.

Theorem C (Schilder's theorem)**.** The law of $\sqrt{\kappa}B$ on the finite interval $[0,T]$ satisfies the large deviation principle with good rate function I_T as κ approaches 0. More precisely, for any closed set F and any open set O of $C_0([0,T])$,

$$\varlimsup_{\kappa\to 0} \kappa \log \mathrm{P}(\sqrt{\kappa}B \in F) \le -\inf_{W\in F} I_T(W),$$

$$\varliminf_{\kappa\to 0} \kappa \log \mathrm{P}(\sqrt{\kappa}B \in O) \ge -\inf_{W\in O} I_T(W);$$

I_T is lower semi-continuous and

$$\{W \in C_0([0,T]) \mid I_T(W) \le c\}$$

is compact for every $c \ge 0$.

In particular, for $C \subset C_0([0,T])$, such that the infimums of the Dirichlet energies coincide on its interior and its closure, we have

$$\lim_{\kappa\to 0} \kappa \log \mathrm{P}(\sqrt{\kappa}B \in C) = -\inf_{W\in C} I_T(W).$$

Heuristically,

$$\text{“}\mathrm{P}(\sqrt{\kappa}B \in C) \approx \exp\left(-\inf_{W\in C} I_T(W)/\kappa\right).\text{”} \tag{1.2}$$

It is also worth mentioning that I_T defines a Hilbert norm on the Cameron-Martin space, which also characterizes the Wiener measure from the abstract Wiener space formalism.

1.2.2 Our results

Energy reversibility

Applying the Loewner transform naively in (1.2), we get for O_ε a small neighborhood of γ,

$$\text{“}\mathrm{P}(\mathrm{SLE}_\kappa \in O_\varepsilon) \approx \exp\left(-\inf_{\gamma'\in O_\varepsilon} I_T(\gamma')/\kappa\right).\text{”}$$

Hoping that the energy of γ' is close to γ as ε is small, then we have

$$\text{“}\mathrm{P}(\mathrm{SLE}_\kappa \in O_\varepsilon) \approx \exp\left(-I_T(\gamma)/\kappa\right).\text{”}$$

However, the topology on driving functions induces a complicated topology on the generated hulls (that is non-geometric). The precise statement of our large deviation result for SLE is as follows. We say that a *point configuration* (L, R), where L and R are finite subsets of $\mathbb{H}$, is compatible with a chord in $(\mathbb{H}, 0, \infty)$, if all the points in L (resp. R) are in the left (resp. right) connected component of $\mathbb{H} \setminus \gamma$. Here, γ is naturally oriented from 0 to ∞.

Theorem 1.1. *The energy of γ equals to the supremum of*

$$\lim_{\kappa\to 0} -\kappa \log\left(\mathrm{P}\left(\mathrm{SLE}_\kappa \text{ is compatible with } (L,R)\right)\right)$$

over all point configurations (L, R) that are compatible with γ.

We deduce the following result using the reversibility of SLE.

Theorem 1.2 (Energy reversibility)**.**

$$I_{D,a,b}(\gamma) = I_{D,b,a}(\gamma),$$

where γ is a chord in D connecting a and b.

See Chapter 2 for more details and proofs.

Root invariance

Although the chordal Loewner equation (1.1) in the half-plane is very simple, it turns out that $\mathbb{H}$ might not be the most natural domain to consider. In fact, the square of it, namely $\mathbb{C} \setminus \mathbb{R}_+$, is perhaps more natural for the following reasons:

- For regular enough curves in $\mathbb{C} \setminus \mathbb{R}_+$, the capacity parametrization is comparable with the arclength parametrization.
- The chordal energy of a chord η in $(\mathbb{C} \setminus \mathbb{R}_+, 0, \infty)$ is also the loop energy rooted at ∞ of $\eta \cup \mathbb{R}_+$. Theorem 1.3 below shows that viewing the Loewner energy in the loop setting in fact provides many more symmetries.

Note also that the loop energy does not depend on the orientation of γ nor on any increasing reparametrization fixing $\gamma(0)$. The former fact follows from the chordal energy reversibility, which can be used to show that $\tilde{\gamma}(t) := \gamma(1-t)$ has the same energy as γ. But it depends a priori on the root $\gamma(0)$ where the limit is taken. However, jointly with Steffen Rohde, we proved the following result.

Theorem 1.3. *The loop energy does not depend on the root.*

This result shows that the loop Loewner energy is a Möbius invariant quantity on the set of unparametrized Jordan curves, which attains its minimum 0 only on circles. Intuitively, the loop energy measures how much γ differs from a circle.

Our proof uses the reversibility of Loewner energy, sometimes implicitly so that we never specify the orientation of loops/arcs and alter freely the orientation. As the reversibility was proved using an interpretation via SLE_{0+}, our proof of Theorem 1.3 is not purely deterministic. See Chapter 3 for details.

However, Theorem 1.3 indicates that the role of the boundary $\mathbb{R}_+$ in the chordal setting $(\mathbb{C}\setminus\mathbb{R}_+, 0, \infty)$ is in fact not different from the chord itself and suggests that loop energy has to be a more fundamental quantity. Indeed, we derive an expression of the loop energy using zeta-regularized determinants of Laplacians which will explain the parametrization independence of the Loewner energy.

1.3 Relation to zeta-regularized determinants

1.3.1 Background

The zeta-regularization of operators was introduced by Ray and Singer [RaS71] and are used by physicists (e.g. Hawking [Haw77]) to make sense of quadratic path integrals. The determinants of Laplacians on Riemann surfaces also plays a crucial role in Polyakov's quantum theory of strings [Pol81]. Polyakov and Alvarez studied the variation of the functional integral under conformal changes of metric, for surfaces with or without boundary [Alv83], resp. [Pol81]. This is known as the Polyakov-Alvarez conformal anomaly formula (Theorem 4.21) and is one of the main tools in our proof. Osgood, Phillips and Sarnak [OPS88] showed that such variation is realized by the zeta-regularized determinants of Laplacians.

Let us review the definition of zeta-regularized determinants of Laplacians [RaS71]: Let Δ be the (the positive) Laplace-Beltrami operator on a compact Riemannian surface (D, g) with smooth boundary and Dirichlet boundary condition. As the name indicates, the zeta-regularized determinant $\det_\zeta(\Delta)$ is defined through a zeta function:

$$\begin{aligned}\zeta_\Delta(s) =& \sum_{j=1}^{\infty} \lambda_j^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{j=1}^{\infty} e^{-t\lambda_j} \,\mathrm{d}t \\ =& \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}(e^{-t\Delta}) \,\mathrm{d}t,\end{aligned}$$

where $0 < \lambda_1 \leq \lambda_2 \cdots$ is the discrete spectrum of Δ and $\Gamma(\cdot)$ is the Gamma function. From Weyl's law, λ_i grows linearly and ζ_Δ is therefore analytic in $\{\mathrm{Re}(s) > 1\}$. One extends ζ_Δ meromorphically to $\mathbb{C}$. The log of the zeta-regularized determinant of Δ is defined as

$$\log \det_\zeta(\Delta) := -\zeta_\Delta'(0).$$

The terminology "determinant" comes from the fact that

$$-\zeta_\Delta'(s) = \sum_{j=1}^{\infty} \log(\lambda_j) \lambda_j^{-s},$$

so that if we formally take $s = 0$, we get

$$\text{“} -\zeta_\Delta'(0) = \log \prod_{j=1}^{\infty} \lambda_j = \log \det(\Delta).\text{”}$$

When (M, g) is compact surface without boundary, Δ has a one dimensional kernel, and its regularized determinant $\det_\zeta'(\Delta)$ is defined similarly by considering only the non-zero spectrum.

1.3.2 Our results

Now we state the expression of Loewner energy using determinants of Laplacians which is also reminiscent of the partition function formulation of the SLE/Gaussian free field coupling by Dubédat [Dub09]. The set-up is the following:

- $g_0(z) = 4(1 + |z|^2)^{-2} \,\mathrm{d}z^2$ is the round metric on $\mathbb{C} \cup \{\infty\} \simeq S^2$;

- γ is a C^∞ smooth Jordan curve on S^2;
- D_1 and D_2 are two connected components $S^2 \setminus \gamma$;
- $\Delta_g(D_i)$ is the positive Laplace-Beltrami operator with Dirichlet boundary condition on D_i.

We introduce

$$\begin{aligned}\mathscr{H}(\gamma, g) := &\log \det{}'_\zeta \Delta_g(S^2) - \log \mathrm{Area}_g(S^2) \\ &- \log \det{}_\zeta \Delta_g(D_1) - \log \det{}_\zeta \Delta_g(D_2).\end{aligned}$$

Theorem 1.4. *If $g = e^{2\varphi} g_0$ is a metric conformally equivalent to the round metric g_0 on S^2, then:*

1. $\mathscr{H}(\cdot, g) = \mathscr{H}(\cdot, g_0)$
2. *Let γ be a smooth Jordan curve on S^2. We have the identity*

$$\begin{aligned}I^L(\gamma, \gamma(0)) &= 12\mathscr{H}(\gamma, g) - 12\mathscr{H}(S^1, g) \\ &= 12 \log \frac{\det_\zeta(\Delta_g(\mathbb{D}_1))\det_\zeta(\Delta_g(\mathbb{D}_2))}{\det_\zeta(\Delta_g(D_1))\det_\zeta(\Delta_g(D_2))},\end{aligned}$$

where $\mathbb{D}_1$ and $\mathbb{D}_2$ are two connected components of the complement of S^1.

Let us make some remarks:

- Clearly, 2 implies that circles minimize $\mathscr{H}(\cdot, g)$ among C^∞ smooth Jordan curves, which was also proven by Burghelea, et al. [BFK94] using a variation formula for $\det_\zeta(\Delta)$.
- It is not surprising that the Loewner energy has an expression which depends only on the conformal class of the metric, as we already knew that it is Möbius invariant. On the other hand it must depend on the conformal class, as otherwise circles play no particular role and cannot always be the minimizer.
- Theorem 1.4 gives rise to many identities by choosing different background metrics in the conformal class of the round metric. Indeed, it will allow us to relate the Loewner energy to the Weil-Petersson class of quasicircles.

- The conformal class of the metric is encoded in the measure on Brownian paths (when we forget the time parametrization) on the surface. In fact, the key is that the log determinant of the Laplacian has an interpretation in terms of the total mass of Brownian loops contained in the surface. We indeed derive an expression of the Loewner energy using the measure of Brownian loops attached to the curve.

1.4 Interpretation via Brownian loop measure

1.4.1 Background

Brownian loop measure on the complex plane $\mathbb{C}$ and its subdomains was introduced by Lawler and Werner in [LW04] and its definition can be immediately extended to a general Riemannian surface M (with or without boundary) in the following way.

Let $x \in M$, $t > 0$, consider the sub-probability measure $\mathbb{W}_x^t$ on the path of Brownian motion (running at speed 2, the diffusion of the infinitesimal generator the Laplace-Beltrami operator Δ_M) on M started from x and on the time interval $[0, t]$, which is killed if it hits the boundary of M. The measures $\mathbb{W}_{x \to y}^t$ on paths from x to y are obtained from the disintegration of $\mathbb{W}_x^t$ according to its endpoint y:

$$\mathbb{W}_x^t = \int_M \mathbb{W}_{x \to y}^t \, \mathrm{dvol}(y).$$

Define the Brownian loop measure on M to be

$$\mu_M^{loop} := \int_0^\infty \frac{\mathrm{d}t}{t} \int_M \mathbb{W}_{x \to x}^t \, \mathrm{dvol}(x).$$

Since the starting points of Brownian paths coincide with their endpoints, it defines a measure on the set of unrooted loops if one forgets the starting point and the time-parametrization (so that we distinguish loops only by their trace).

Now we explain how this measure is related to the determinant of the Laplacian: if we compute formally, the total mass of μ_M^{loop} is given by

$$\text{“}\left|\mu_M^{loop}\right| = \int_0^\infty \frac{\mathrm{d}t}{t} \int_M p_t(x,x)\,\mathrm{dvol}(x) = \int_0^\infty t^{-1}\,\mathrm{Tr}\,e^{-t\Delta}\,\mathrm{d}t.\text{”}$$

On the other hand, $1/\Gamma(s)$ is holomorphic and has the expansion near 0 as

$$1/\Gamma(s) = s + O(s^2).$$

Therefore for any holomorphic function f,

$$\left(\frac{f(s)}{\Gamma(s)}\right)'\Big|_{s=0} = f(0).$$

Take formally $f(s) = \int_0^\infty t^{s-1}\,\mathrm{Tr}(e^{-t\Delta})\,\mathrm{d}t$, we have

$$\text{“} - \log\det{}_\zeta(\Delta) = \zeta'_\Delta(0) = \int_0^\infty t^{-1}\,\mathrm{Tr}(e^{-t\Delta})\,\mathrm{d}t = \left|\mu_M^{loop}\right|\text{”}. \tag{1.3}$$

The Brownian loop measure satisfies the following two remarkable properties

- *(Restriction property)* If $M' \subset M$, then

$$d\mu_{M'}^{loop}(\delta) = 1_{\delta\in M'} d\mu_M^{loop}(\delta).$$

- *(Conformal invariance)* Let $M_1 = (M,g)$ and $M_2 = (M, e^{2\sigma}g)$ be two conformally equivalent Riemann surface, where $\sigma \in C^\infty(M,\mathbb{R})$. Then

$$\mu_{M_1}^{loop} = \mu_{M_2}^{loop}.$$

Notice that the total mass (under the Brownian loop measure) of loops contained in $\mathbb{C}$ is infinite (in fact, that for all positive R, both the mass of loops of diameter greater than R and the mass of loops of diameter smaller than R are infinite), which can be viewed as a consequence of its scale-invariance (or of the fact that the integral of $1/t$ diverges both at infinity and at 0). However, when $D \subset \mathbb{C}$ is a proper subset of $\mathbb{C}$ with non-polar boundary, and K_1, K_2 are two disjoint compact subsets of D, the total mass (under the Brownian loop measure) of the set of loops that do stay in D and intersect both K_1 and K_2 is finite (staying in D in some sense removes most large loops, and intersecting both K_1 and K_2 prevents the loops for being too small). We will denote this finite mass by

$$\mathscr{B}(K_1, K_2; D) := \mu_D^{loop}(\{\delta; \delta \cap K_1 \neq \emptyset, \delta \cap K_2 \neq \emptyset\}).$$

Werner's measure on simple (self-avoiding) loops in the complex plane defined in [Wer08] is simply the image of $\mu_{\mathbb{C}}^{loop}$ under the map that associates to a (Brownian) loop its outer boundary (i.e., the boundary of the unbounded connected component of its complement). As shown in [Wer08], this measure turns out to be invariant under the map $z \mapsto 1/z$ (and more generally under any conformal automorphism of the Riemann sphere), which in turn makes it possible to define this measure $\mu_{W,M}^{loop}$ on any Riemann surface M, in such a way that the above restriction and conformal invariance properties still hold. In fact, it is shown in [Wer08] that this is the unique (up to a multiplicative constant) family of measures on self-avoiding loops satisfying both the restriction property and the conformal invariance property. For other characterizations (via a restriction-type formula, or as a measure on SLE loops) of Werner's measure and its properties (it is supported on $\mathrm{SLE}_{8/3}$-type loops which have fractal dimension 4/3), see [Wer08].

One feature that makes Werner's measure convenient to work with on Riemann surfaces is that if we consider two disjoint compact sets $K_1, K_2 \subset \mathbb{C}$, then the total mass of loops that intersect both K_1 and K_2 is finite (see [NW11, Lem. 4]):

$$\mathscr{W}(K_1, K_2; \mathbb{C}) := \mu_{W,\mathbb{C}}^{loop}(\{\delta; \delta \cap K_1 \neq \emptyset, \delta \cap K_2 \neq \emptyset\}) < \infty.$$

This contrasts with the fact that the total mass (for the Brownian loop measure) of loops that intersect both K_1 and K_2 is infinite, due to the many very large Brownian loops that intersect both K_1 and K_2

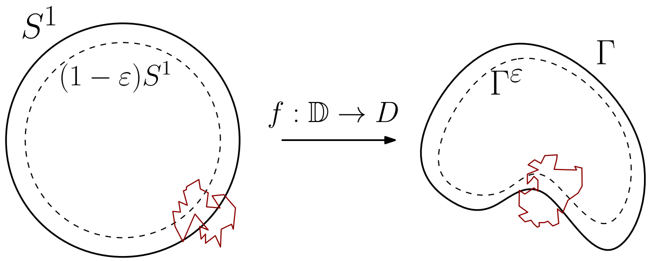

Figure 1.1: A self-avoiding loop touching both the circle (or Γ) and the equi-potential.

(but the outer boundary of these large loops tends not to intersect K_1 or K_2, which explains why $\mathscr{W}(K_1, K_2; \mathbb{C})$ is finite). This feature was also instrumental in the proof of the conformal invariance of simple Conformal Loop Ensembles on the Riemann sphere by Kemppainen and Werner in [KW16].

1.4.2 Our results

The identity in Theorem 1.4 and the interpretation from (1.3) suggest that in some sense

$$\text{“}I^L(\gamma) = 12\left(\left|\mu^{loop}_{D_1}\right| + \left|\mu^{loop}_{D_2}\right| - \left|\mu^{loop}_{\mathbb{D}_1}\right| - \left|\mu^{loop}_{\mathbb{D}_2}\right|\right)$$
$$= 12\mu^{loop}_{S^2}(\{\delta; \delta \cap S^1 \neq \emptyset\}) - 12\mu^{loop}_{S^2}(\{\delta; \delta \cap \gamma \neq \emptyset\}).\text{”}$$

However, as we pointed out above, both terms on the right-hand side diverge. On the other hand, we can still make sense of the identity for all finite energy Jordan curves as a renormalization of Brownian loop measure in the following way.

Let γ be a Jordan curve in $\mathbb{C}$, D the bounded connected component of $\mathbb{C} \setminus \gamma$ and f a conformal map from the unit disk $\mathbb{D}$ to D. For $1 > \varepsilon > 0$, let $S^{(1-\varepsilon)}$ denote the circle of radius $1-\varepsilon$, centered at 0, and $\gamma^{(1-\varepsilon)} := f(S^{(1-\varepsilon)})$ the equi-potential (see Figure 1.1).

Theorem 1.5 (Theorem 5.7). *We have*

$$\begin{aligned} I^L(\gamma) &= \lim_{\varepsilon\to 0} I^L(\gamma^{1-\varepsilon}) \\ &= \lim_{\varepsilon\to 0} 12\mathscr{W}(S^1, S^{(1-\varepsilon)}; \mathbb{C}) - 12\mathscr{W}(\gamma, \gamma^{(1-\varepsilon)}; \mathbb{C}). \end{aligned}$$

In fact, we also show in [VW19] that the Loewner energy of $\gamma^{(1-\varepsilon)}$ is monotone in ε. As $\varepsilon \to 0$, the energy increases to the energy of γ. In the other direction $\varepsilon \to 1$, the energy decreases to 0. This is expected as f is conformal, therefore sends a small circle around 0 to a curve that is asymptotically a circle.

1.5 Loewner regularity correspondence

1.5.1 Background on quasiconformal mappings

We will see later that finite energy curves are images of segments (quasichords) and finite energy loops are images of circles (quasicircles) by quasiconformal maps. Moreover, the universal Teichmüller space that is closely related to the Loewner energy is based on quasiconformal maps. This section briefly recalls the minimal facts about quasiconformal maps, we refer interested readers to [Ahl06, Leh12] and references therein for more details.

Heuristically, conformal mappings in the plane send small circles (asymptotically) to circles. By analogy with this observation, an orientation preserving homeomorphism is called quasiconformal if it sends small circles centers at z to ellipses, whose eccentricities are bounded by a constant which does not depend upon the point z.

To make the heuristic precise, recall for an orientation preserving diffeomorphism $f : D \subset \mathbb{C} \to \mathbb{C}$, the complex derivatives are defined as

$$\partial f = \frac{1}{2}(\partial_x f - i\partial_y f), \quad \overline{\partial} f = \frac{1}{2}(\partial_x f + i\partial_y f),$$

and the derivative $\partial_\theta f$ in the direction θ:

$$\partial_\theta f(z) = \lim_{r\to 0} \frac{f(z+re^{i\theta}) - f(z)}{re^{i\theta}} = \partial f + \overline{\partial} f \cdot e^{-2i\theta}.$$

It follows that

$$\max_\theta |\partial_\theta f(z)| = |\partial f| + \left|\overline{\partial} f\right|, \quad \min_\theta |\partial_\theta f(z)| = |\partial f| - \left|\overline{\partial} f\right| > 0.$$

The quantity $|\partial f| - \left|\overline{\partial} f\right|$ is positive since the Jacobian

$$J_f = |\partial f|^2 - \left|\overline{\partial} f\right|^2$$

is positive (by definition) for an orientation preserving diffeomorphism.

In fact, the definition of a general quasiconformal map does not need the function to be a diffeomorphism. We only require the following analytic condition on the function: A continuous function u defined on D is said to be *absolutely continuous on lines (ACL)* if for any rectangle $\{x+iy : a \le x \le b, c \le y \le d\} \subset D$, u is absolutely continuous on almost all horizontal and vertical lines in the rectangle. The ACL property implies that u is differentiable a.e. and in particular, that the angular derivatives $\partial_\theta u$ are defined for every θ and a.e. $z \in D$.

Let $K \ge 1$. An orientation preserving homeomorphism f of a domain D is K**-quasiconformal** if

1. f is ACL in D;
2. $\max_\theta |\partial_\theta f(z)| \le K \min_\theta |\partial_\theta f(z)|$ a.e. in D.

In other words,

$$\frac{\max_\theta |\partial_\theta f(z)|}{\min_\theta |\partial_\theta f(z)|} = \frac{|\partial f(z)| + \left|\overline{\partial} f(z)\right|}{|\partial f(z)| - \left|\overline{\partial} f(z)\right|} \le K.$$

It can be rewritten in the form

$$\left|\overline{\partial} f(z)\right| \le \frac{K-1}{K+1} |\partial f(z)|.$$

Being 1-quasiconformal is equivalent to being conformal, since $\overline{\partial} f \equiv 0$, which is the Cauchy-Riemann equation. The *complex dilation* of f:

$$\mu_f(z) := \frac{\overline{\partial} f(z)}{\partial f(z)},$$

is defined everywhere and is measurable, and satisfies

$$\|\mu\|_\infty := \operatorname*{ess\,sup}_{z \in D} |\mu(z)| \le \frac{K-1}{K+1} < 1.$$

Theorem D. If $\mu \in L^\infty(D, \mathbb{C})$ satisfies $\|\mu\|_\infty < 1$, the *Beltrami equation*:

$$\overline{\partial} f(z) = \mu(z) \partial f(z), \quad \text{a.e. } z \in D, \tag{1.4}$$

has a quasiconformal solution (which is moreover ACL). If f, g are two quasiconformal solutions to (1.4), then $f \circ g^{-1}$ and $g \circ f^{-1}$ are conformal.

In fact, the complex dilation of the composition of two quasiconformal maps is given by

$$\mu_{f \circ g^{-1}}(g(z)) = \frac{\mu_f(z) - \mu_g(z)}{1 - \mu_f(z)\overline{\mu_g(z)}} \left[\frac{\partial g(z)}{|\partial g(z)|} \right]^2 \in \mathbb{D}. \tag{1.5}$$

We see that the family of quasiconformal homeomorphisms of D is closed under composition and taking inverse, so is therefore a group.

A quasiconformal mapping between two Jordan domains (simply connected domain bounded by Jordan curves) can always be extended to a homeomorphism of their closure. Therefore it makes sense to talk about the boundary values of such a mapping.

An orientation preserving homeomorphism $\varphi : S^1 \to S^1$ is *quasisymmetric* if there exists $M \ge 1$, such that for all $\theta \in [0, 2\pi]$ and $t \in (0, \pi)$,

$$\frac{1}{M} \le \left| \frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})} \right| \le M.$$

Quasisymmetric homeomorphisms of S^1 form a group, denoted by

$\mathrm{QS}(S^1)$.

Theorem E (Beurling-Ahlfors theorem)**.** The boundary value of a K-quasiconformal homeomorphism $\mathbb{D} \to \mathbb{D}$ is in $\mathrm{QS}(S^1)$. Conversely, every quasisymmetric homeomorphism can be extended to a quasiconformal homeomorphism of $\mathbb{D}$ that is continuous on $\overline{\mathbb{D}}$.

A (K-)**quasicircle** is the image of a circle by a (K-)quasiconformal map from $\hat{\mathbb{C}}$ to itself. A (K-)**quasichord** in $(\mathbb{D}, -1, 1)$ is the image of sub-interval containing -1 or 1 of $[-1, 1]$ by a (K-)quasiconformal map from $\mathbb{D}$ to itself while fixing $-1, 1$. In another domain (D, a, b) quasichords are defined via its conformal equivalence to $(\mathbb{D}, -1, 1)$.

1.5.2 Background on regularity correspondence

We will explain how the regularity of a planar curve is related to the Hölder regularity of its Loewner driving function, which in turn shows that the Loewner transform provides a powerful tool to study the regularity of the curve.

Recall that for $0 < \beta \le 1$, $W : [0, T] \to \mathbb{R}$ is β-Hölder continuous if there exists a constant c, such that

$$|W(t) - W(s)| \le c\,|t - s|^{\beta}, \quad \forall 0 \le s, t \le T.$$

We define its C^{β}-norm $\|f\|_{C^{\beta}}$ to be the smallest such constant c. For $\alpha > 0$, the Hölder space C^{α} consists of functions in C^k whose k-th derivative is β-Hölder continuous, where $k = \lfloor \alpha \rfloor$, $\beta = \alpha - \lfloor \alpha \rfloor$.

Quasichords can be characterized by their Loewner driving function from the work of Lind, Marshall and Rohde.

Theorem F ([MR05, Lin05])**.** If the curve γ in $(\mathbb{H}, 0, \infty)$ generated by $W : [0, T] \to \mathbb{R}$ is a quasichord, then W is 1/2-Hölder continuous. Conversely, if $\|W\|_{C^{1/2}} < 4$, then W generates a K-quasichord, where K only depends on $\|W\|_{C^{1/2}}$.

The upper-bound of the norm is sharp, in the sense that there are examples of driving function with $C^{1/2}$-norm equal to 4, whose Loewner chain is not traced out by a curve.

It is also natural to ask: What can we say about the relation between the regularity of the driving function and the regularity of the curve, when the driving function has Hölder regularity strictly larger than 1/2? Prior to our work, only one direction was well understood. Slightly imprecisely, the following results state that C^α driving functions generate $C^{\alpha+1/2}$ curves for $\alpha > 1/2$, where C^α is understood with the usual convention as $C^{n,\beta}$, where n is the integer part of α and $\beta = \alpha - n$ (see Section 3.3.1).

Theorem G. ([Won14]) If $\beta \in (0, 1/2]$ and $W \in C^{0,1/2+\beta}([0, T])$, then the Loewner curve η in $\mathbb{H}$ generated by W is a simple curve of class $C^{1,\beta}$ when reparametrized as $t \mapsto \eta(t^2)$. If $W \in C^{1,\beta}$, the curve is in $C^{1,\beta+1/2}$ (weakly $C^{1,1}$ when $\beta = 1/2$).

Theorem H ([Won14, LT16])**.** If $\alpha > 3/2$ and $W \in C^\alpha$, then W generates a simple curve of class $C^{\alpha+1/2}$ if $\alpha + 1/2 \notin \mathbb{N}$, and in the Zygmund class $\Lambda_*^{\alpha-1/2}$ otherwise.

The Zygmund class $\Lambda_*^{\alpha-1/2}$ contains the class $C^{\alpha+1/2}$. In the other direction, one can ask about the regularity of the driving function given the regularity of the curve. Here Earle and Epstein proved the following result using a local quasiconformal variation near the tip of the curve:

Theorem I ([EE01])**.** If $n \in \mathbb{Z}$, $n \geq 2$ and $\eta \in C^n$, then its driving function is C^{n-1} on the half-open interval $(0, T]$.

They stated the result in the radial setting, but using a change of coordinate it is not hard to see that the regularity of the driving function remains the same in the chordal case. Their result precedes the work of Wong, Lind and Tran, which in turn supports the natural (stronger) conjecture that $C^{\alpha+1/2}$ curves should have C^α driving functions when $\alpha > 1/2$. The conjecture for $1/2 < \alpha \leq 3/2$ is in fact proved in the thesis that we discuss below.

1.5.3 Our results

Notice that if W has finite Dirichlet energy, then it is 1/2-Hölder continuous since

$$\begin{aligned}
&|W(t_1) - W(t_2)| \\
&\le \int_{t_1}^{t_2} |\dot{W}| \,\mathrm{d}t \le (t_2 - t_1)^{1/2} \left(\int_{t_1}^{t_2} |\dot{W}|^2 \,\mathrm{d}t \right)^{1/2} \\
&\le (t_2 - t_1)^{1/2} (2I(W))^{1/2}.
\end{aligned}$$

By cutting the interval into small enough pieces, we can make sure that on each piece, the $C^{1/2}$-norm of W is as small as we wish, and obtain that finite energy driving functions generate quasichords in (D, a, b). It is also true for the loop energy.

Theorem 1.6. *Finite energy chords and loops are necessarily K-quasichords and K-quasicircles, respectively, where K is bounded by a constant depending only on the value of the energy.*

See Proposition 2.3, 3.9. Quasicircles may have Hausdorff dimension of any value in $[1, 2)$. However, finite energy curves are in fact rectifiable ([FS17], Thm. 2). This shows that finite energy loops form a much smaller class than the quasicircles. This is consistent with the fact that their driving function has local $C^{1/2}$-norm arbitrarily close to 0.

From the purely analytical point of view on the Loewner transform, $C^{1/2}$ functions are the roughest driving functions for which we know that the family of hulls generated arise from a curve (called the *trace* of the Loewner chain), under the condition of small Hölder norm. In the case of SLEs, that are generated by Brownian motions which are only in $C^{1/2-\varepsilon}$, the trace exists almost surely [RS05]. However, we do not know what analytic properties of the Brownian path that guarantee the trace existence actually are. It is still an open question to characterize the precise family of driving functions that generate a simple curve.

In the higher regularity case, we also prove the converse of Theorem G. That is when the curve is in $C^{\alpha+1/2}$ where $1/2 < \alpha \le 3/2$, the conjecture mentioned at the end of Section 1.5.2 is true.

We say that an (arc-length parametrized) simple arc $\gamma : [0, S] \to \mathbb{C} \setminus \mathbb{R}_{>0}$ of regularity at least C^1 is *tangentially attached* to $\mathbb{R}_+$ if $\gamma(0) = 0$, and the right-derivative $\gamma'(0) = -1$. We use the square root function $\sqrt{\cdot}$ on $\mathbb{C} \setminus \mathbb{R}_+$ that takes values in $\mathbb{H}$. Let T be the half-plane capacity of $\sqrt{\gamma[0, S]}$.

Theorem 1.7. *Let $0 < \beta \le 1$, and γ be a $C^{1,\beta}$ simple arc tangentially attached to $\mathbb{R}_+$. The driving function W of $\sqrt{\gamma}$ has the following regularity on the closed interval $[0, T]$:*

- *$C^{0,\beta+1/2}$ if $0 < \beta < 1/2$;*
- *weakly $C^{0,1}$, if $\beta = 1/2$;*
- *$C^{1,\beta-1/2}$ with $\dot{W}_0 = 0$, if $1/2 < \beta < 1$;*
- *weakly $C^{1,1/2}$, if $\beta = 1$.*

The respective norm of W is bounded by a function of the local regularity $\|\gamma\|_{1,\beta}$ and constants associated with the global geometry of γ.

The *weak regularity* stands for a logarithmic correction term in the modulus of continuity (see Section 3.3.1). Examples of curves with bottle-necks easily show that the C^α norm of the driving function cannot be bounded solely in terms of the local behavior of γ. See Section 3.3 for more details.

We believe that the proof for $\alpha > 3/2$ is in the same spirit as for α in $(1/2, 3/2]$, but with more technicality. Since the focus of this thesis is on the Loewner energy, and the regularity was sufficient for the following application to the Loewner energy, we did not treat the question in full generality. And we were able to obtain the following sufficient condition for a loop to have finite energy:

Corollary 1.8. *If γ is a $C^{3/2+\varepsilon}$ Jordan curve for some $\varepsilon > 0$, then $I^L(\gamma) < \infty$.*

This $C^{3/2+\varepsilon}$ regularity is also reminiscent, see e.g. [Fi10], of the regularity of Weil-Petersson quasicircles that we will discuss in the next section.

1.6 Relation to Teichmüller theory

1.6.1 Background on conformal welding

Another way to view the family of Jordan curves is by their welding functions. More precisely, when γ is a Jordan curve, $\varphi : S^1 \to S^1$ is said to be a **welding function** of γ, if

$$\varphi = g^{-1} \circ f \text{ restricted to } S^1 = \partial\mathbb{D} = \partial\mathbb{D}^*.$$

Here, f is a conformal map from $\mathbb{D}$ to the inside of γ, and g from $\mathbb{D}^*$ to the outside of γ (where we often fix $g(\infty) = \infty$). It makes sense to talk about the boundary values of f and g, since by Carathéodory's theorem, f and g extend to homeomorphisms of the closure of $\mathbb{D}$ and $\mathbb{D}^*$.

The *conformal welding problem* is a classical (and still partially open) question in geometric function theory: if we are given a sense preserving homeomorphism $\varphi : S^1 \to S^1$ (that fixes ± 1 and $-i$), is it possible to find a Jordan curve $\gamma \subset \mathbb{C}$, such that φ is a welding function of γ (see Figure 1.2)?

Again, the welding function of quasicircles (they are quasisymmetric homeomorphisms of S^1 by the theorem below) are well-understood analytically. In general, however, we do not know how to characterize the family of homeomorphisms on S^1 that do admit a welding curve. The uniqueness of solution (if it exists), is implied by the conformal removability of the curve.

As finite energy loops are quasicircles (Theorem 1.6), we are content with staying in the realm of quasiconformal geometry.

Theorem J. The welding function of a quasicircle is quasisymmetric. Conversely, if φ is a quasisymmetric homeomorphism of S^1, then there exists a unique (up to Möbius transformation) solution γ to the welding problem. Moreover, γ is a quasicircle.

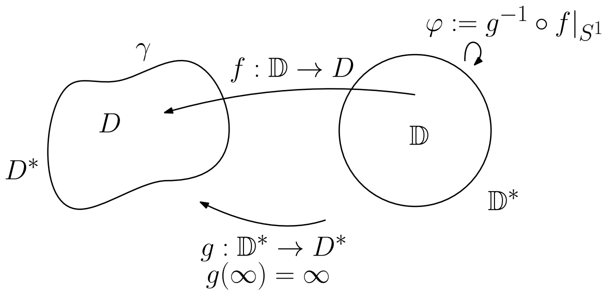

Figure 1.2: Conformal maps f and g associated to φ.

1.6.2 Background on the universal Teichmüller space

The universal Teichmüller space $T(1)$ is an infinite dimensional complex Banach manifold. We will present two (equivalent) ways to model it. They provide different views on the same object: sometimes one is easier to manipulate while studying certain questions, and sometimes it is the other (we will comment on this later).

Model A: Let $\text{Möb}(S^1) \simeq \text{PSU}(1,1) \simeq \text{PSL}(2,\mathbb{R})$ denote the group of Möbius transformations of S^1. The *universal Teichmüller space* is

$$T(1) := \text{Möb}(S^1) \setminus \text{QS}(S^1) \simeq \{\varphi \in \text{QS}(S^1), \varphi \text{ fixes } \pm 1, -i\}.$$

Model B: We denote the space of Beltrami differentials in the unit disk by

$$B(\mathbb{D}^*) = \{\mu \in L^\infty(\mathbb{D}^*, \mathbb{C}) \mid \|\mu\|_\infty < 1\}.$$

For $\mu \in B(\mathbb{D}^*)$, we extend it to a Beltrami differential in $\mathbb{C}$ by setting

$$\hat{\mu}\left(\frac{1}{\bar{z}}\right) := \overline{\mu(z)}\frac{z^2}{\bar{z}^2}, \quad \text{for } z \in \mathbb{D}^*.$$

From Theorem D, the Beltrami equation

$$\bar{\partial}\omega_\mu(z) = \hat{\mu}(z)\partial\omega_\mu(z) \tag{1.6}$$

has a unique solution such that ω_μ fixes $\pm 1, -i$. The choice of the extension of μ implies that $\omega_\mu(1/\overline{z}) = 1/\overline{\omega_\mu(z)}$ therefore ω_μ maps S^1 to S^1. The universal Teichmüller space is defined as the equivalence classes

$$T(1) := B(\mathbb{D}^*)/\sim,$$

where $\mu \sim \nu$ if $\omega_\mu|_{S^1} = \omega_\nu|_{S^1}$.

Model B is identified with the Model A, by the map

$$\beta : B(\mathbb{D}^*)/\sim \to \text{Möb}(S^1)\setminus \text{QS}(S^1), \quad [\mu] \mapsto \omega_\mu|_{S^1}.$$

For another insight into the definition, consider ω^μ the solution to the Beltrami equation with $\tilde{\mu} = \mu$ in $\mathbb{D}^*$ and 0 in $\mathbb{D}$ fixing $\pm 1, -i$. Then ω^μ is conformal from $\mathbb{D}$ to some Jordan domain in $\hat{\mathbb{C}}$. In fact, we have

$$\mu \sim \nu \Leftrightarrow \omega^\mu|_{\mathbb{D}} = \omega^\nu|_{\mathbb{D}}.$$

To see this, since the complex dilation of ω_μ coincides with ω^μ on $\mathbb{D}^*$, $\omega^\mu \circ \omega_\mu^{-1}$ is conformal from $\mathbb{D}^*$ to the outside of $\gamma = \omega^\mu(S^1)$, and the welding function of γ is given by

$$(\omega^\mu \circ \omega_\mu^{-1})^{-1} \circ \omega^\mu|_{S^1} = \omega_\mu|_{S^1}.$$

To describe the complex structure of $T(1)$, Model B is more suitable since the space $B(\mathbb{D}^*) \subset L^\infty(\mathbb{D}^*, \mathbb{C})$ is naturally a complex Banach space and it induces the complex structure on $T(1)$ by the following theorem.

Theorem K ([Na88, Sect. 3.4])**.** There exists a complex structure on $T(1)$ such that the projection $B(\mathbb{D}^*) \to T(1)$ is holomorphic.

The complex local charts on $T(1)$ are in fact provided by Bers' coordinates, defined using Schwarzian derivatives of $\omega^\mu|_{\mathbb{D}}$. Here, the term *holomorphic* stands for infinitely dimensional holomorphy: Let E and F be two complex Banach space, and $U \subset E$ a non-empty open subset in E. A mapping $f : U \to F$ is holomorphic if and only if it is continuous and the *complex Gateaux derivative* $d_x f(\lambda)$ defined as

$$d_x f(\lambda) = \lim_{t\to 0, t\in\mathbb{C}} \left\{ \frac{f(x+t\lambda) - f(x)}{t} \right\} \in F$$

exists for all $(x, \lambda) \in U \times E$.

The space $B(\mathbb{D}^*)$ is also endowed with a complete metric

$$d_h(\mu, \nu) := \operatorname{ess\,sup}_{z\in\mathbb{D}^*} \mathrm{h}(\mu(z), \nu(z))$$

where $\mathrm{h}(\cdot,\cdot)$ is the hyperbolic distance on the unit disk. In fact, $d_h(\mu,\nu) = d_h(\mu * \nu^{-1}, 0)$, where $\mu * \nu^{-1}$ is the complex dilation of $\omega_\mu \circ \omega_\nu^{-1}$.

The *Teichmüller distance* on $T(1)$ is induced by the metric on $B(\mathbb{D}^*)$, given by

$$d_T([\mu],[\nu]) := \min_{\mu\in[\mu],\nu\in[\nu]} d_h(\mu,\nu).$$

On the other hand, the group structure of $T(1)$ can be seen easily from Model A (by composition of homeomorphisms). We denote in the sequel the neutral element as $[id]$ in Model A, and $[0]$ in Model B as it corresponds to the 0 complex dilation. It is also worth mentioning, that $T(1)$ endowed with the Teichmüller distance, is not a topological group (the composition $T(1) \times T(1) \to T(1)$ is not continuous, see [Leh12, III. 3.4]).

1.6.3 Background on the Weil-Petersson class

Let $\mathrm{Diff}(S^1)$ be the group of orientation preserving C^∞ diffeomorphism of S^1. We have the inclusions

$$\text{Möb}(S^1) \subset \mathrm{Diff}(S^1) \subset \mathrm{QS}(S^1).$$

The quotient $\mathscr{S} := \text{Möb}(S^1) \setminus \mathrm{Diff}(S^1)$ is the *smooth part* of the universal Teichmüller space (using Model A). The group $\mathrm{Diff}(S^1)$ occurs in string theory as the space of reparametrizations of the closed string.

It was shown by Bowick and Rajeev [BR87b] that $\mathscr{S}$ carries a unique structure of infinite dimensional, holomorphically homogeneous Kähler manifolds whose symplectic form also arises from Kirillov-Kostant-Souriau's construction on coadjoint orbits.

Nag and Verjovsky showed that this physically motivated Kähler structure on $\mathscr{S}$ coincides with the Kähler structure induced from the universal Teichmüller space. By *Kähler structure* we mean three compatible structures: That are symplectic, Riemannian and complex structures. As it would be impossible to give an introduction to the infinite dimensional Kähler geometry here, we are content with explicitly showing what it means in this context. The space $\mathscr{S}$ is *homogeneous* Kähler manifolds means that these structures are invariant under the right $\mathrm{Diff}(S^1)$-group action on $\mathscr{S}$, hence we only need to specify the structures at the origin $[id] \in \mathscr{S}$.

The tangent space at $[id]$ consists of smooth vector fields $\mathrm{Vect}(S^1)$ on S^1 with Fourier expansion:

$$v = \sum_{n \neq \pm 1, 0} v_n e_n := \sum_{n \neq \pm 1, 0} v_n e^{in\theta} \frac{\partial}{\partial \theta} \quad \text{satisfying } \overline{v}_n = v_{-n}.$$

The *almost complex structure* $J : \mathrm{Vect}(S^1) \to \mathrm{Vect}(S^1)$ (such that $J^2 = -I$) induced from the complex structure (Theorem K) is given by the Hilbert transform [NV90]:

$$Jv = -i \sum_{n=2}^{\infty} v_n e_n + i \sum_{-\infty}^{-2} v_n e_n.$$

The family $\{e_n := e^{in\theta} \partial / \partial \theta\}_{n \neq \pm 1, 0}$ generates the complexification of $\mathrm{Vect}(S^1)$:

$$\mathrm{Vect}^{\mathbb{C}}(S^1) = \{ \sum_{n \neq \pm 1, 0} u_n e_n \mid u_n \in \mathbb{C} \},$$

with the Lie bracket

$$[e_m, e_n] = i(n - m) e_{n+m}.$$

Theorem L ([BR87b])**.** There is a unique homogeneous Kähler form (up to scaling) on $\mathrm{Vect}(S^1)$, which is given by

$$\omega(u,v) = -\omega(v,u) = 2\alpha \operatorname{Im}\left(\sum_{m=2}^{\infty}(m^3-m)u_m\overline{v_m}\right),$$

for some $\alpha \in \mathbb{R}_+$.

The *Kähler metric* g on $\mathrm{Vect}(S^1)$ is compatible with ω and J. That is,

$$\begin{aligned} g(u,v) =&\omega(u,Jv) = 2\alpha \operatorname{Im}\left(\sum_{m=2}^{\infty} i(m^3-m)u_m\overline{v_m}\right) \\ =&2\alpha \operatorname{Re}\left(\sum_{m=2}^{\infty}(m^3-m)u_m\overline{v_m}\right). \end{aligned}$$

Notice that the inner product g is also well-defined when u, v are only $H^{3/2}$-smooth vector fields.

Now we translate the metric into the Model B. First, the holomorphic tangent space $T_{[0]}T(1)$ is identified with

$$\Omega^{-1,1}(\mathbb{D}^*) = \{\mu \in L^\infty(\mathbb{D}^*) \mid \mu = \rho^{-1}\overline{\phi},\, \phi \text{ holomorphic}\}$$

where $\rho(z) = 1/(1-|z|^2)^2$. And the kernel of the differential of the projection $B(\mathbb{D}^*) \to T(1)$ at 0 is given by $\mathscr{N}(\mathbb{D}^*)$ defined as

$$\left\{\mu \in L^\infty(\mathbb{D}^*) \mid \int_{\mathbb{D}^*} \mu\phi \, dz^2 = 0,\, \forall \phi \in L^1(\mathbb{D}^*) \text{ holomorphic}\right\}.$$

We say that $\mu \in \Omega^{-1,1}(\mathbb{D}^*)$ corresponds to u a vector field on S^1 vanishing at $\pm 1, -i$ if

$$u(e^{i\theta}) = \partial_t|_{t=0}\omega_{t\mu}(e^{i\theta}),$$

where we recall the definition of $\omega_{t\mu}$ in (1.6).

Theorem M ([NV90]). Assume that $\mu, \nu \in \Omega^{-1,1}(\mathbb{D}^*)$ corresponds to C^∞-smooth vector fields u, v respectively. Then the Hermitian form

$$g(u,v) - i\omega(u,v) = 2\alpha \sum_{m=2}^{\infty} (m^3 - m)\overline{u_m} v_m$$

is also given by

$$\frac{12\alpha}{\pi^2} \int_{\mathbb{D}^*} \int_{\mathbb{D}^*} \frac{\mu(z)\overline{\nu(\zeta)}}{(1 - z\overline{\zeta})^4} dz^2 d\zeta^2 = \frac{4\alpha}{\pi} \int_{\mathbb{D}^*} \mu(z)\overline{\nu}(z)\rho(z) dz^2.$$

The inner product on $\Omega^{-1,1}(\mathbb{D}^*)$

$$\langle \mu, \nu \rangle_{\mathrm{WP}} := \int_{\mathbb{D}^*} \mu(z)\overline{\nu(z)}\rho(z) dz^2$$

is known as the *Weil-Petersson* metric on $T(1)$. However it is defined only on the subspace of $T_{[0]}T(1)$:

$$H^{-1,1}(\mathbb{D}^*) := \left\{ \mu \in \Omega^{-1,1}(\mathbb{D}^*) \mid \int_{\mathbb{D}^*} |\mu(z)|^2 \rho(z) dz^2 < \infty \right\}.$$

Transporting $H^{-1,1}(\mathbb{D}^*)$ to the holomorphic tangent space at other points of $T(1)$ by right-translation, defines a sub-bundle of the tangent bundle. We denote the integral submanifold of $T(1)$ by $T_0(1)$, which is now complete, and has the homogeneous infinite dimensional Kähler structure that extends that of the smooth part $\mathscr{S}$. See [TT06] for more details.

The preimage of $T_0(1)$ by the projection $\mathrm{QS}(S^1) \to T(1)$ is called *Weil-Petersson class* $\mathrm{WP}(S^1)$. We have

$$\begin{aligned} \mathscr{S} = \text{Möb}(S^1) \setminus \mathrm{Diff}(S^1) \subset T_0(1) &= \text{Möb}(S^1) \setminus \mathrm{WP}(S^1) \\ \subset T(1) &= \text{Möb}(S^1) \setminus \mathrm{QS}(S^1). \end{aligned}$$

Apart from the abstract construction, motivated by its natural and unique Kähler structure, the WP-class has many equivalent analytic characterizations, due to Nag, Sullivan, Cui, Takhtajan, Teo, Shen, etc. We gather these below:

Let $\varphi \in \mathrm{QS}(S^1)$, so it can be viewed as the welding function of a quasicircle. Let γ be a bounded quasicircle that solves the welding problem of φ, and let f and g be the associated conformal maps on $\mathbb{D}$ and $\mathbb{D}^*$. We also assume that $g(\infty) = \infty$, see Figure 1.2.

Theorem N (Equivalent characterization of WP-class)**.** A homeomorphism $\varphi \in \mathrm{WP}(S^1)$ if and only if one of the following equivalent conditions holds:

- $\int_{\mathbb{D}} |\nabla \operatorname{Re}(\log f'(z))|^2 \, \mathrm{d}z^2 = \int_{\mathbb{D}} |f''(z)/f'(z)|^2 \, \mathrm{d}z^2 < \infty$;
- $\int_{\mathbb{D}^*} |g''(z)/g'(z)|^2 \, \mathrm{d}z^2 < \infty$;
- $\int_{\mathbb{D}} |S(f)|^2 \rho^{-1}(z) \, \mathrm{d}z^2 < \infty$;
- $\int_{\mathbb{D}^*} |S(g)|^2 \rho^{-1}(z) \, \mathrm{d}z^2 < \infty$;
- φ has quasiconformal extension to $\mathbb{D}$, whose complex dilation $\mu = \partial_{\bar{z}}\varphi / \partial_z \varphi$ satisfies
$$\int_{\mathbb{D}} |\mu(z)|^2 \rho(z) \, \mathrm{d}z^2 < \infty;$$
- φ is absolutely continuous with respect to arc-length measure, and $\log \varphi'$ belongs to the Sobolev space $H^{1/2}(S^1)$;
- the Grunsky operator associated to f or g is Hilbert-Schmidt.

In the above, $\rho(z) \, \mathrm{d}z^2 = 1/(1 - |z|^2)^2 \, \mathrm{d}z^2$ is the hyperbolic metric on $\mathbb{D}$ or $\mathbb{D}^*$, and
$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$
is the Schwarzian derivative of f.

The Weil-Petersson metric is stronger than the Teichmüller metric. However, in contrast to $T(1)$ endowed with Teichmüller metric, the group operations are continuous on $T_0(1)$ with respect to the Weil-Petersson metric:

Theorem O ([TT06])**.** $T_0(1)$ endowed with the WP-metric is a topological group.

1.6.4 Background on Takhtajan-Teo's Liouville action

One important feature of the Kähler geometry is the existence of *Kähler potential* which can always be defined locally, and is such that given the complex structure, it determines the Kähler form/metric. More precisely, the Kähler form can be derived from the Kähler potential $\mathbf{S}$ as $i\partial\overline{\partial}\mathbf{S}$. Here, the operators

$$\partial = \sum_k \partial_{z_k} dz_k \;\text{ and }\; \overline{\partial} = \sum_k \overline{\partial}_{z_k} d\overline{z_k}$$

are Dolbeault operators, respectively, and $(z_k, \overline{z_k})$ are local complex coordinates.

Takhtajan and Teo defined a universal Liouville action on $T_0(1)$ and show it to be a (global) Kähler potential of the WP-metric. More precisely, if f and g are the conformal maps that solve the welding of $\varphi \in \mathrm{QS}(S^1)$ as in Figure 1.2, the **universal Liouville action** $\mathbf{S_1} : T_0(1) \to \mathbb{R}$ is defined by

$$\begin{aligned}\mathbf{S_1}([\varphi]) = &\int_{\mathbb{D}} \left|\frac{f''}{f'}(z)\right|^2 \mathrm{d}z^2 + \int_{\mathbb{D}^*} \left|\frac{g''}{g'}(z)\right|^2 \mathrm{d}z^2 \\ &+ 4\pi \log|f'(0)| - 4\pi \log|g'(\infty)|,\end{aligned}$$

where $g'(\infty) = \lim_{z\to\infty} g'(z)$. One may check that the definition does not depend on the representative of $[\varphi]$ and the choice of f and g as long as $g(\infty) = \infty$ and $g^{-1} \circ f = \varphi$.

Recall that $H^{-1,1}(\mathbb{D}^*) \simeq T_{[0]}T_0(1)$.

Theorem P ([TT06] Thm. 2.4.1)**.** For $[\varphi] \in T_0(1)$, $\mu, \nu \in H^{-1,1}(\mathbb{D}^*)$,

$$\partial_\mu \overline{\partial_\nu} \mathbf{S_1}([\varphi]) = \int_{\mathbb{D}} \mu(z)\overline{\nu}(z)\rho(z) dz^2,$$

where $\rho(z) = 1/(1-|z|^2)^2$ and we identify μ and ν with holomorphic vector fields on local charts of $T_0(1)$ using Bers' coordinates [TT06, p.9].

1.6.5 Our results and further directions

Now we can finally state our result. We say that γ is a **Weil-Petersson quasicircle** if its welding homeomorphism $g^{-1} \circ f$ is in $\mathrm{WP}(S^1)$. We establish the following link between finite energy loops and Weil-Petersson quasicircles in Section 4.8.

Theorem 1.9. *Let γ be a bounded simple loop in $\hat{\mathbb{C}}$, γ has finite Loewner energy if and only if γ is a Weil-Petersson quasicircle. Moreover,*

$$I^L(\gamma) = \mathbf{S_1}([g^{-1} \circ f])/\pi.$$

It thus provides another characterization of $T_0(1)$ and a new viewpoint on its Kähler potential (or alternatively a way to look at the Loewner energy). Again, the root-invariance and the reversibility of the loop energy can be viewed as a corollary of this result, because there is no more parametrization involved in the definition of $\mathbf{S_1}([g^{-1} \circ f])$.

We have seen that the Loewner transform relates the regularity of the curve to the regularity of the driving function. Theorem 1.9 shows moreover that the L^2-integrability of the complex dilation and that of the Schwarzian derivatives are also related to the L^2-integrability of the derivative of the driving function.

The above connection between Loewner theory and Teichmüller theory via the Loewner energy is still intriguing and mysterious to my humble understanding. However, since both theories have developed their own specific techniques during their rather long histories, it will be illuminating to reformulate objects of interest and tools on one side into the other's language, as well as to apply of the techniques of one to solve the other's problem. Here is a list of future directions that we may pursue.

- On the Loewner side, we are already convinced that the natural probabilistic model is SLE. What is the random model naturally associated to the Weil-Petersson class? Does it give an intrinsic description of SLE loop measure and imply the reversibility of SLEs?

- What are the (stochastic) gradient flows of Loewner energy and other dynamics on the loop spaces? Does the path-integral quantization of the Loewner energy give Malliavin's measure [Mal99, KS07]?
- We have seen that $T_0(1)$ is a topological group, how are the group operations reflected in the Loewner transform?
- How is the Kähler geometry on the Weil-Petersson Teichmüller space encoded in the Loewner driving function and why is it canonical from the latter point of view (since the Weil-Petersson metric is the unique homogeneous Kähler metric)? How is the geometric quantization of Weil-Petersson Teichmüller space related to its path-integral quantization?
- To what extent do geometrical properties of Loewner energy, for instance, the root-invariance, encrypt information on the random fractal model of SLE?
- How does Loewner energy generalize to other configurations of loops in higher genus surfaces, and how does it relate to Weil-Petersson metric on finite dimensional Teichmüller spaces?

These questions aim to provide a better understanding of the connection between finite energy loops and Weil-Petersson class, as well as the correspondence between their structures. In view of the fact that the probabilistic SLE model is hidden behind Loewner energy, and the universal Teichmüller space contains Teichmüller spaces of compact surfaces whose Kähler metrics are analogous to the one on $T_0(1)$, the correspondence also hints at a potential probabilistic approach to the quantization of Teichmüller spaces. This might also provide steps toward a new approach to a non-perturbative string theory.

Chapter 2

Energy reversibility via SLE

2.1 Introduction

This chapter corresponds to the paper [Wan16] and the main goal is to prove the energy reversibility:

Theorem 2.1. *The Loewner energy of the time-reversal $\hat{\eta}$ of a simple curve η from a to b in D is equal to the Loewner energy of η. That is, $I_{D,a,b}(\eta) = I_{D,b,a}(\hat{\eta})$.*

Our proof is based on the fact that the energy of the Loewner chain is the large deviation rate function of the SLE_κ driving function when $\kappa \to 0+$. Loosely speaking, for a given γ with finite energy, we want to relate $I(\gamma)$ to the decay as $\kappa \to 0$ of the probability for an SLE_κ to be in a certain neighborhood $B_\varepsilon(\gamma)$ of γ via a formula of the type

$$I(\gamma) = \lim_{\varepsilon \to 0} \left(\lim_{\kappa \to 0} -\kappa \ln \mathrm{P}[\mathrm{SLE}_\kappa \in \mathrm{B}_\varepsilon(\gamma)] \right).$$

The main point will be to prove this for a reversible notion of B_ε (so that a path is in the ε-neighborhood of γ if and only if its time-reversal is in the ε-neighborhood of the time-reversal of γ). Once this will be done, the reversibility of the energy of the Loewner chain will follow immediately from this expression and the reversibility of SLE_κ (Theorem B) for all small κ. Similar idea is outlined by Julien Dubédat in section 9.3 of [Dub07].

Usual neighborhoods are not so well-suited for our purpose: Taking the neighborhood of γ in the sense of some Hausdorff metric in the upper half-plane is not well-adapted to the large deviation framework for the driving function, and the L^∞ norm on the driving function does not define reversible neighborhoods. Our choice will be to fix a finite number of points on the left side and the right side of γ, and to consider the collection of driving functions whose Loewner transform passes on the same side of these points than γ does. This set of driving functions will play the role of $B_\varepsilon(\gamma)$ and the large deviation principle will apply well (the $\varepsilon \to 0$ limit will be replaced by letting the set of constraint points become dense in the upper half-plane). In order to properly apply the convergence, we will use some considerations about compactness of the set of K-quasichords, and the relation between finite energy chains and quasichords.

The chapter is structured as follows: We will first derive in Section 2.2 some facts on deterministic Loewner chains with finite energy and their regularity. In Section 2.3, we derive Theorem 2.1 using the strategy that we have just outlined, and we conclude in Section 2.4 establishing some connections with ideas from SLE restriction properties and SLE commutation relations.

2.2 Loewner energy and quasichords

2.2.1 Loewner transform

We say that a subset K of $\mathbb{H}$ is a *Compact* $\mathbb{H}$*-hull of half-plane capacity* hcap(K) *seen from* ∞, if (i) K is bounded, (ii) $H_K := \mathbb{H} \backslash K$ is simply connected, and (iii) the unique conformal transformation $g_K : H_K \to \mathbb{H}$ such that $g_K(z) = z + o(1)$ when $z \to \infty$ also satisfies

$$g_K(z) = z + \frac{\mathrm{hcap(K)}}{z} + o(\frac{1}{z}),$$

when $z \to \infty$. We will refer to g_K as the *mapping-out function* of K.

Let $\mathscr{K}$ denote the set of compact $\mathbb{H}$-hulls, endowed with the metric (the *Carathéodory topology*)

$$d_K(K_1, K_2) = d(g_{K_1}^{-1}, g_{K_2}^{-1}),$$

where the metric d generates the topology corresponding to uniform convergence on compact subsets of the upper half-plane. Recall that this topology generated by d_K differs from the Hausdorff distance topology. For instance, the arc $\{e^{i\theta}, \theta \in (0, \pi - \varepsilon]\}$ converges to the half-disc of radius 1 and center 0 for the Caratheodory topology, but not for the Hausdorff metric. Let $(K_t)_{t>0} \subset \mathscr{K}$ be an increasing family of compact $\mathbb{H}$-hulls for inclusion. For $s < t$, define $K_{s,t} = g_{K_s}(K_t \backslash K_s)$. We say that the sequence $(K_t)_{t>0}$ has *local growth* if $\operatorname{diam}(K_{t,t+h})$ converges to 0 uniformly on compacts in t as $h \to 0$ (where diam is the diameter for the Euclidean metric).

Let $\mathscr{L}_\infty$ denote the set of all increasing sequences of compact $\mathbb{H}$-hulls $(K_t)_{t\in[0,\infty)}$ satisfying the local growth, parameterized in the way such that $\operatorname{hcap}(K_t) = 2t$. We endow $\mathscr{L}_\infty$ with the topology of uniform convergence on compact time-intervals.

Now we describe the chordal Loewner transform, which associates each continuous real-valued function with a family in $\mathscr{L}_\infty$: When $\lambda \in C([0,\infty))$, consider the Loewner differential equation

$$\partial_t g_t(z) = 2/(g_t(z) - \lambda_t)$$

with initial condition $g_0(z) = z$. The *chordal Loewner chain in* $\mathbb{H}$ *driven by the function* λ (or *the Loewner transform of* λ), is the increasing family $(K_t)_{t>0}$ defined by

$$K_t = \{z \in \mathbb{H} \mid \tau(z) \le t\},$$

where $\tau(z)$ is the maximum survival time of the solution, i.e.

$$\tau(z) = \sup\{t \ge 0 \mid \inf_{0\le s\le t} |g_s(z) - \lambda_s| > 0\}.$$

It turns out that $(K_t)_{t>0}$ is indeed in $\mathscr{L}_\infty$. The family (g_t) is sometimes called the *Loewner flow* generates by λ. We will also use $f_t := g_t - \lambda_t$, referred to as the *centered Loewner flow*.

Note finally that the Loewner equation is also defined for $z \in \mathbb{R}$, and it is easy to see that the closure of K_t in $\overline{\mathbb{H}}$ satisfies $\overline{K_t} = \{z \in$

$\overline{\mathbb{H}} \mid \tau(z) \leq t\}$.

The following result describes the inverse of the Loewner transform and tells us that both procedures are continuous when we equip $C([0,\infty))$ with the topology of uniform convergence on compact intervals.

Theorem 2.2 (see [LSW01] 2.6)**.** *The Loewner transform L from $C([0,\infty))$ to $\mathscr{L}_\infty$ is a homeomorphism. The inverse transform is given by*

$$\lambda_t = \bigcap_{h>0} \overline{K_{t,t+h}}, \quad \forall t \geq 0$$

where $K_{s,s'} = g_s(K_{s'} \backslash K_s)$ for $s < s'$.

We will also mention radial Loewner chains that are defined in a similar way in the unit disc $\mathbb{D}$: The radial Loewner transform of $\xi \in C([0,\infty))$ is the $(K_t)_{t\geq 0}$ family given by the radial Loewner equation

$$\partial_t h_t(z) = -h_t(z)\frac{h_t(z) + e^{i\xi(t)}}{h_t(z) - e^{i\xi(t)}}$$

with initial condition $h_0(z) = z$ for all z in the closed unit disc, where K_t is defined as in the chordal case.

2.2.2 Chordal Loewner energy

Let $T \in [0,\infty)$ and let $C_0([0,T])$ be the set of continuous functions λ on $[0,T]$ with $\lambda(0) = 0$, endowed with the L^∞ norm. A function $\lambda \in C_0([0,T])$ is *absolutely continuous*, if there exists a function $h \in L^1([0,T])$, such that $\int_0^t h(s)ds = \lambda(t)$, $\forall t \geq 0$. In the sequel we write $\dot\lambda := h$ and λ is implicitly absolutely continuous whenever $\dot\lambda$ is considered. The *energy up to time T* of the Loewner chain γ driven by λ is defined as

$$I_T(\gamma) = I_T(\lambda) := \frac{1}{2}\int_0^T \dot\lambda(s)^2 \,\mathrm{d}s$$

for λ absolutely continuous and $I_T(\lambda) = \infty$ in all other cases. This energy has been recently introduced and used in the paper [FS17], with the goal of providing a rough path approach to some features of SLE. While this SLE goal is quite different from the present paper, it has some similarities "in spirit" with the present paper, as it tries to provide a more canonical (and less based on Itô stochastic calculus – therefore less directional) approach to SLE. Note that our project was developed independently of [FS17].

Let us list a few properties of this energy:

- The map $f \mapsto I_T(f)$ is lower-semicontinuous. Indeed, by elementary analysis, $I_T(\lambda)$ is the supremum over all finite partitions $\Pi = (0 = t_0 < t_1 < t_2 < \ldots < t_k = T)$ of

$$\sum_{t_{j-1}, t_j \in \Pi} \frac{|\lambda(t_j) - \lambda(t_{j-1})|^2}{t_j - t_{j-1}}.$$

  Hence, if λ_n is a sequence of functions converge uniformly on $[0, T]$ to λ, then

$$\begin{aligned}\liminf_{n\to\infty} I_T(\lambda_n) &\geq \sup_{\Pi} \liminf_{n\to\infty} \sum_{t_{j-1}, t_j \in \Pi} \frac{|\lambda_n(t_j) - \lambda_n(t_{j-1})|^2}{t_j - t_{j-1}} \\ &= I_T(\lambda).\end{aligned}$$

- The sets $\{\lambda \in C_0([0, T]) \mid I_T(\lambda) \leq c\}$ are compact for every $c \geq 0$ in $C_0([0, T])$. This follows from the fact that a bounded energy set is equicontinuous and thus relatively compact in $C_0([0, T])$ by the Arzelà-Ascoli Theorem.
- Similarly, we define *the energy* of γ driven by λ as $I(\lambda) := I_\infty(\lambda)$ for $\lambda \in C_0([0, \infty))$ (endowed with the topology of uniform convergence on compact intervals). We still have that I is lower semicontinuous, and the set of functions such that $I(\lambda) \leq c$ is compact in $C_0([0, \infty))$. Sometimes we will omit the subscript of I_T if there is no ambiguity, and for $\lambda \in C_0([0, T])$, we also define $I(\lambda) := I_T(\lambda) = I(\lambda(\cdot \wedge T))$.
- We write $H_T \subset C_0([0, T])$ for the set of finite I_T energy functions,

and similarly, $H \subset C_0([0,\infty))$ for finite I energy functions.

- For a simple curve γ from 0 to ∞ in the upper half-plane driven by the function λ, the energy satisfies obviously the following scaling property:
$$I(\gamma) = I(u\gamma), \quad \forall u > 0$$
because the driving function of $u\gamma$ is $\lambda^u(t) = u\lambda(t/u^2)$. Thus, the energy is invariant under conformal equivalence preserving $(\mathbb{H}, 0, \infty)$. So one can define the Loewner energy of a curve from a boundary point a to another boundary point b in a simply connected domain D, by applying any conformal map from (D, a, b) to $(\mathbb{H}, 0, \infty)$. We use $I_{D,a,b}$ to indicate the energy in a different domain than $(\mathbb{H}, 0, \infty)$.
- It is also straightforward to see that the Loewner energy satisfies the simple additivity property
$$I(\gamma) = I(f_s(\gamma[s,\infty))) + I_s(\gamma),$$
using the fact that the driving function of $f_t(\gamma[t,\infty))$ is $\lambda(s+t) - \lambda(s)$.
- If one is looking for a functional of a Loewner chain that satisfies additivity and conformal invariance, one is therefore looking for additive functionals of the driving functions with the right scaling property. There are of course several options (for instance one can add to I_T the integral of the absolute value of the second derivative of λ to the appropriate power). However, taking the integral of the square of the derivative is the most natural choice – and it does satisfy the reversibility property that we derive in the present paper.
- Let us finally remark that a Loewner chain with 0 energy is driven by the 0 function, hence is the imaginary axis in $\mathbb{H}$ which is the hyperbolic geodesic between 0 and ∞. In view of the conformal invariance, the energy of a curve from a to b in D can therefore be viewed as a way to measure how much does a curve differs from the hyperbolic geodesic from a to b in D (see also related comments in Section 2.4.1 for the conformal restriction properties).

It may seem at first that the definition of the Dirichlet energy is quite ad hoc and depends on our choice of uniformizing domain for the Loewner flow (the fact that one works in the upper half-plane). Let us briefly indicate that this is not the case. Suppose for instance that we are working in the simply connected domain D, and that ∂D is a smooth analytic arc in the neighborhood of b. It is then natural to parameterize the curve γ by its capacity in D as seen from b, i.e. by

$$\begin{aligned}\mathrm{cap}(K) := -\frac{1}{6}S\psi_K(b) &= -\frac{1}{6}\left[\frac{\psi_K'''(b)}{\psi_K'(b)} - \frac{3}{2}\left(\frac{\psi_K''(b)}{\psi_K'(b)}\right)^2\right] \\ &= -\frac{1}{6}\left[\psi_K'''(b) - \frac{3}{2}\psi_K''(b)^2\right],\end{aligned}$$

where $\psi_K : D \setminus K \to D$ is a conformal mapping such that $\psi_K(b) = b$ and $\psi_K'(b) = 1$. These derivatives are well defined due to the Schwarz reflection principle. It is not hard to see that there exists a conformal equivalence h from (D, a, b) to $(\mathbb{H}, 0, \infty)$ and $\alpha \in \mathbb{C}^*$ such that $\alpha^2\mathrm{cap}(\cdot)$ equals to the half-plane capacity of its image by h. For any continuous parametrization by $[0, T]$ of the oriented chord γ, let $\psi_t = \psi_{\gamma[0,t]}$ such that ψ_t maps moreover γ_t to a. Then the driving function defined as

$$\lambda(t) := \frac{\alpha}{2}\psi_t''(b)$$

corresponds to the driving function of the image by h of the curve in the half-plane parametrized by $\alpha^2\mathrm{cap}(\gamma[0,t])$. In other words, one can express the energy of the Loewner chain directly as the Dirichlet energy of $\psi_t''(b)/2$ viewed as function of $\mathrm{cap}(\gamma[0,t])$ which are quantities naturally and directly defined in D.

2.2.3 Quasichords

The following result will explain that a Loewner chain with finite energy is necessarily a quasichord:

Proposition 2.3. *For every $\lambda \in H$, there exists $K = K(I(\lambda))$, depending only on $I(\lambda)$, such that the trace of Loewner transform γ of*

λ is a K-quasichord, i.e. the image of $i\mathbb{R}_+$ by a K-quasiconformal mapping preserving $\mathbb{H}$, 0 and ∞.

The following corollary of Proposition 2.3 will be useful in our proof of reversibility of the energy. A quasiconformal map φ is said to be *compatible* with λ if it preserves $\mathbb{H}, 0$ and ∞, and if $\varphi(i\mathbb{R}_+) = \gamma$ and $|\varphi(i)| = 1$.

Corollary 2.4. *Let (λ_n) be a sequence of driving functions with energy bounded by $C < \infty$. Let φ_n be a $K(C)$-quasiconformal mapping compatible with λ_n, there exists $\lambda \in H$ with $I(\lambda) \leq C$, and φ compatible with λ, such that on a subsequence, λ_n and φ_n converge respectively (uniformly on compacts) to λ and φ.*

Proof. Since the family of $K(C)$-quasiconformal mapping preserving $\mathbb{H}$ is a normal family (cf. [Leh12] I.2.3), together with the compactness of $I^{-1}([0, C])$, we can extract a subsequence where λ_n and φ_n converge both and respectively to λ and φ. The limit φ is either a $K(C)$-quasiconformal map or a constant map into $\partial\mathbb{H}$. The latter is excluded because of the choice $|\varphi_n(i)| = 1$, and that the energy needed to touch a point near ± 1 is not bounded (we will see it in Proposition 2.10). Thus φ is $K(C)$-quasiconformal, and we have to show that the quasichord $\tilde{\gamma} := \varphi(i\mathbb{R}_+)$ is the Loewner transform of λ.

As explained in [LMR10], Theorem 4.1. and Lemma 4.2, the modulus of continuity of γ_n depends only on $K(C)$ when parameterized by capacity. The equicontinuity allows us to take a subsequence s.t. γ_n converges uniformly to $\tilde{\gamma}$ as a capacity-parameterized curve (which is also given by Theorem 2 in [FS17]). Together with the uniform convergence of λ_n to λ, elementary calculus allows us to conclude that $\tilde{\gamma}$ is the Loewner transform of λ. $\square$

Before proving the proposition, let us review some further general background material on Loewner chains: The Loewner chain generated by a continuous function has local growth, and there are simple examples of continuous driving functions such that the Loewner chain is not a simple curve as well as examples where it is not even generated by a continuous curve. For instance, Lind, Marshall and Rohde (see [LMR10]) exhibit a driving function with $\lambda(t) \sim 4\sqrt{1-t}$ when $t \to 1-$, that generates a Loewner chain with infinite spiral at time $1-$. An interesting subclass of simple curve is the quasiarc, which is the image of a segment under a quasiconformal mapping of $\mathbb{C}$. Marshall, Rohde [MR05] studied when a continuous function generates a quasiarc (as a radial Loewner chain). Lind [Lin05] derived the following sharp condition for a driving function to generate a quasislit half-plane in the chordal setting, i.e. the image of $\mathbb{H}\backslash[0, i]$ by a K-quasiconformal mapping fixing $\mathbb{H}$ and ∞, its complement in $\mathbb{H}$ is a quasiarc which is not tangent to the real line.

Let $T < \infty$, recall that $\mathrm{Lip}_{1/2}(0, T)$ is the space of Hölder continuous functions with exponent $1/2$, which consists of functions $\lambda(t)$ satisfying

$$|\lambda(s) - \lambda(t)| \leq c\,|s-t|^{1/2}, \quad \forall 0 \leq s, t \leq T$$

for some $c < \infty$, equipped with the norm $\|\lambda\|_{1/2,[0,T]}$ being the smallest c.

Theorem 2.5 ([Lin05])**.** *If the domain $\mathbb{H}\backslash\gamma[0, T]$ generated by λ is a quasislit half-plane, then $\lambda \in \mathrm{Lip}_{1/2}(0, T)$. Conversely, if $\lambda \in \mathrm{Lip}_{1/2}(0, T)$ with $\|\lambda\|_{1/2,[0,T]} < 4$, then λ generates a K-quasislit half-plane for some K depending only on $\|\lambda\|_{1/2,[0,T]}$.*

The constant 4 is sharp because of the spiral example mentioned before (where the local 1/2-Hölder norm is as close to 4 as one wishes). We remark that H injects into $\mathrm{Lip}_{1/2}([0, \infty))$ since

$$\begin{aligned} |\lambda(t_1) - \lambda(t_2)| &\leq \int_{t_1}^{t_2} \left|\dot\lambda\right| \mathrm{d}t \leq (t_2 - t_1)^{1/2} \left(\int_{t_1}^{t_2} \left|\dot\lambda\right|^2 \mathrm{d}t \right)^{1/2} \\ &\leq (t_2 - t_1)^{1/2} (2I(\lambda))^{1/2}. \end{aligned}$$

The proof in [Lin05] is based on the following lemma, which gives a necessary and sufficient condition for being a quasislit half-plane in terms of the conformal welding, and is similar to Lemma 2.2 in [MR05]. More precisely, let λ be the driving function of γ with $\|\lambda\|_{1/2,[0,T]} < 4$, and let f_T be the conformal mapping of $\mathbb{H}\backslash\gamma[0,T] \to \mathbb{H}$, such that $f_T(z)-z$ is bounded, and $f_T(\gamma_T)=0$. It is shown in [Lin05, Lemma 3] that f_T^{-1} extended to $\partial\mathbb{H}$, and that for $0 \leq t < T$, γ_t has exactly two preimages of different sign which give a pairing on a finite interval of $\mathbb{R}$. Then we extend the pairing to $f_T(x)$ and $f_T(-x)$. The conformal welding $\phi : \mathbb{R} \to \mathbb{R}$ is the decreasing function sending one point to its paired point.

Lemma 2.6. $\mathbb{H}\backslash\gamma[0,T]$ *is a* K*-quasislit half-plane if and only if* ϕ *is well-defined as above, and there exists* $M > 0$ *depending only on* K*, such that*

(i) *for all* $x > 0$,
$$\frac{1}{M} \leq \frac{x}{-\phi(x)} \leq M,$$

(ii) *and for all* $0 \leq x < y < z$ *with* $y - x = z - y$,
$$\frac{1}{M} \leq \frac{\phi(x) - \phi(y)}{\phi(y) - \phi(z)} \leq M.$$

And if $\|\lambda\|_{1/2,T} < 4$*, the conformal welding of* ϕ *of* $\gamma[0,T]$ *satisfies both conditions, with* M *depending only on* $\|\lambda\|_{1/2,[0,T]}$.

Quasiconformal mappings fixing $\mathbb{H}$ can always be extended to a homeomorphism of the closure $\overline{\mathbb{H}}$, thus induces a homeomorphism on $\partial\mathbb{H}$.

Lemma 2.7 ([Leh12] section 5.1)**.** *There exists a function* $l(K)$ *such that the boundary value* h *of a* K*-quasiconformal mapping leaving* $\mathbb{H}$ *and* ∞ *invariant is always* $l(K)$*-quasisymmetric. i.e.*
$$\frac{1}{l(K)} \leq \frac{h(x) - h(y)}{h(y) - h(z)} \leq l(K),$$
for all $x < y < z$ *and* $y - x = z - y$.

And conversely,

Theorem 2.8 (Kenig-Jerison extension, [AIM08, Thm 5.8.1]). *There exists $K = K(k)$ such that every k-quasisymmetric function h on $\mathbb{R}$ with $h(0) = 0$ can be extended to a K-quasiconformal mapping with $i\mathbb{R}_+$ fixed.*

The condition $h(0) = 0$, and the extension fixes $i\mathbb{R}_+$ is in particular convenient for the following proof. The extension exists for arbitrary quasisymmetric h by translation. Now we are ready for proving Proposition 2.3.

Proof of Proposition 2.3. Let us first consider the case of driving functions λ such that $L := I(\lambda) < 8$ (therefore $\|\lambda\|_{1/2} < 4$). Thus for every $n \in \mathbb{N}^*$, there exists $t_n > 0$ and a $K(L)$-quasiconformal mapping φ_n, preserving $\mathbb{H}, 0$ and ∞ such that $\gamma_{[0,n]} = \varphi_n(i[0, t_n])$, and $\varphi_n(i) = \gamma_1$. The module of the ring domain $\mathbb{H}\backslash i[1, t_n]$ is bounded by $K(L)$ times the module of $\mathbb{H}\backslash\gamma[1, n]$ since the latter is the image by the $K(L)$-quasiconformal map φ_n. The module of $\mathbb{H}\backslash\gamma[1, n]$ converges to 0, because both boundary components have spherical diameters bounded away from 0 but has mutual spherical distance goes to 0. For the proof see [LV73], p. 34. Thus $t_n \to \infty$. The sequence φ_n has a locally uniformly convergent subsequence, which does not contract to a boundary point because of $\varphi_n(i) = \gamma_1$. Let φ denote the limit of this subsequence. It is a $K(L)$-quasiconformal mapping, and since for every $y > 0$, $\varphi_n(iy)$ is on γ for sufficiently large n we can conclude that $\gamma = \varphi(i\mathbb{R}_+)$.

The case of general driving functions λ is treated by concatenating several pieces with small energy. This idea is used in [LMR10], Thm 4.1. We repeat it here in order to see that the quasiconformal constant after concatenation is bounded by a constant which depends only on the constant of both parts of the quasichords.

Let β be a K_β-quasichord from 0 to ∞ in $\mathbb{H}$, φ_β a corresponding quasiconformal mapping. Let $\alpha := \gamma[0, T]$ a Loewner chain driven by λ with $I_T(\lambda) \le 4$, with the centered mapping-out function f_T. By Lemma 2.6, $\mathbb{H}\backslash\alpha$ is K_α-quasislit half-plane, where K_α depends only on $I_T(\alpha)$. It suffices to construct a quasiconformal mapping φ

compatible with $f_T^{-1}(\beta) \cup \alpha$, with constant depending only on K_α and K_β (Figure 2.1).

In fact, we only need to construct F quasiconformal in $\mathbb{H}$ which keeps track of the welding, i.e.

$$\varphi_\beta \circ F(-x) = \phi_\alpha \circ \varphi_\beta \circ F(x),$$

where ϕ_α is the conformal welding of α with the associated constant M_α as defined in Lemma 2.6. The mapping φ which makes the diagram commute can be extended by continuity to $\mathbb{H} \to \mathbb{H}$, and is the quasiconformal map that we are looking for. To construct F, first define F on $\mathbb{R}$ by

$$F(x) = \varphi_\beta^{-1} \circ \begin{cases} \varphi_\beta(x) & x \geq 0 \\ \phi_\alpha \circ \varphi_\beta(-x) & x < 0, \end{cases}$$

Since φ_β^{-1} is a K_β-quasiconformal mapping ∞ onto itself, it extends to a $l(K_\beta)$-quasisymmetric function on $\mathbb{R}$ (Lemma 2.7). Moreover, the mapping

$$\psi : x \mapsto \begin{cases} \varphi_\beta(x) & x \geq 0 \\ \phi_\alpha \circ \varphi_\beta(-x) & x < 0, \end{cases}$$

is K-quasisymmetric on each side of 0, where K depends only on K_α and K_β. Besides, the ratio of dilatations on both sides is controlled by M_α, i.e. for $x > 0$,

$$\frac{\psi(x)}{-\psi(-x)} = \frac{\varphi_\beta(x)}{-\phi_\alpha \circ \varphi_\beta(x)} \in \left[\frac{1}{M_\alpha}, M_\alpha\right]$$

thanks to Lemma 2.6.

We conclude that ψ, hence F, are K-quasisymmetric with K depending only on K_α and K_β. Using Theorem 2.8, we conclude that F can be extended to a quasiconformal mapping preserving $i\mathbb{R}_+$ with constant of quasiconformality depending only on K_α and K_β.

Cutting the driving function into small energy pieces, with 1/2-Lip

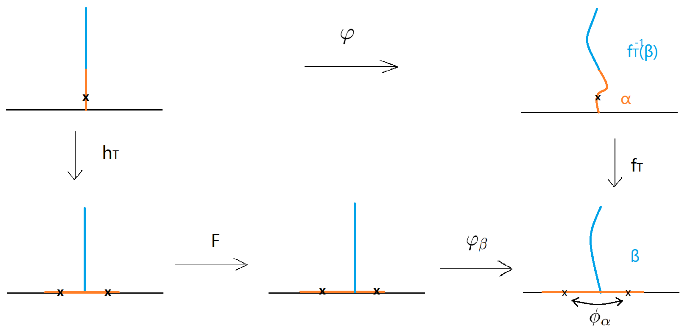

Figure 2.1: h_T and f_T are conformal mapping-out functions, φ_β is K_β-quasiconformal compatible with β, and F is constructed in the proof. The mapping φ which makes the diagram commute, then extended to $\mathbb{H}$, is quasiconformal and compatible with $f_T^{-1}(\beta) \cup \alpha$.

norm less than 4, allows us to conclude the proof. $\square$

2.3 Reversibility via point constraints

2.3.1 One-point constraint minimizing curve

When $\lambda \in C_0([0,\infty))$ generates a simple curve γ from 0 to ∞ in $\mathbb{H}$ (which is the case for SLE$_\kappa$ curves when $\kappa \leq 4$), then $\mathbb{H} \setminus \gamma$ has two connected components $H^+(\gamma)$ and $H^-(\gamma)$ having respectively 1 and -1 on the boundary, that we can loosely speaking call the right-hand side and the left-hand side of γ. We also say that γ is *to the right (resp. left) of a subset* $K \subset \mathbb{H}$, if $K \subset H^-(\gamma)$ (resp. $K \subset H^+(\gamma)$).

Theorem 2.9 ([Sch01]). *For a point* $z = x+iy \in \mathbb{H}$*, then for* $w = x/y$ *and* $\kappa \leq 4$*, we define* $h_\kappa(w)$ *to be the probability that the SLE*$_\kappa$ *trace* *passes to the right of* $\{z\}$. *Then,*

$$h_\kappa(w) = \frac{1}{2}\int_w^\infty (s^2+1)^{-4/\kappa}\,\mathrm{d}s \Big/ \int_0^\infty (s^2+1)^{-4/\kappa}\,\mathrm{d}s.$$

We now fix a point z in the upper half-plane with argument θ, and define

$$D(z) := \{\lambda \in C_0([0,\infty)) \mid \tau(z) < \infty \text{ or } \lim_{t\to\infty} \arg(f_t(z)) = \pi\},$$

where $(f_t)_{t\leq 0}$ is the centered flow driven by λ (here and in the sequel, we always choose the argument of a point in the upper half-plane to be in $(0,\pi)$).

By Lemma 3 in [Sch01], we see that $\lambda \in D(z)$ if and only if $z \notin H^+(\gamma)$. Hence the probability that SLE_κ is in $D(z)$ is $h_\kappa(w)$, where $w = \mathrm{Re}(z)/\mathrm{Im}(z)$ with the notation in Theorem 2.9. We also consider the function $F_\kappa(\cdot) := h_\kappa(\cot(\cdot))$. We study this probability in the small κ limit.

Proposition 2.10. *If* $\theta \in (0,\pi/2)$, *then as* $\kappa \to 0$, $-\kappa \ln F_\kappa(\theta)$ *converges to the infimum of* $I(\lambda)$ *over* $D(z)$. *Furthermore, this infimum is equal to* $-8\ln(\sin(\theta))$ *and there exists a unique function* λ *in* $D(z)$ *with this minimal energy.*

Proof. The first fact and the existence of minimizing curve follow from the large deviation principle. It is in fact a subcase of the multiple point constraint result that we will prove in Prop. 2.11, so we do not give the detailed argument here. In order to evaluate the value of this limit, we can use Theorem 2.9: Indeed, for $w = \cot(\theta) \geq 0$,

$$\kappa \ln(h_\kappa(w)) = \kappa \int_0^w \frac{h_\kappa'(s)}{h_\kappa(s)}\,\mathrm{d}s + \kappa \ln(h_\kappa(0)),$$

and it is straightforward (see appendix) to check that if one defines

$F(w) = 8w/(w^2+1)$, then the quantity

$$\varepsilon_\kappa(w) := -\kappa \frac{h'_\kappa(w)}{h_\kappa(w)} - F(w) = \frac{\kappa(w^2+1)^{-4/\kappa}}{\int_w^\infty (s^2+1)^{-4/\kappa}\,\mathrm{d}s} - F(w)$$

converges to 0 as $\kappa \to 0$, uniformly on $[0,\infty)$. Hence,

$$\lim_{\kappa\to 0} \kappa \ln(h_\kappa(w)) = \int_0^w \frac{-8s}{1+s^2} ds = -4\ln(w^2+1) = 8\ln(\sin(\theta)).$$

Let us now prove the uniqueness of the minimizer (we will in fact also construct it explicitly): Suppose that γ is a minimizing curve, denote the driving function of γ by λ, with centered flow (f_t). Let $z_t = f_t(z)$ and $w_t = \cot(\arg(z_t))$. It is easy to see that $\tau(z)$ is finite. Indeed, if this is not the case, then $w_t \to -\infty$ when $t \to \infty$ and by the continuity of $t \mapsto w_t$, there exists $\tau < \infty$ s.t. $w_\tau = 0$ and $z_\tau \in i\mathbb{R}_+$. But, λ must be constant after τ to maintain the minimal energy. It would imply that γ hits z, which contradicts the contrapositive assumption. Note that for $t < T$, the curve $f_t(\gamma[t,\infty))$ is also a minimizing curve corresponding to $D(z_t)$ and λ is constant after the time $\tau(z)$. Hence, by the previously derived result, we get that for all $t < \tau(z)$,

$$\frac{1}{2}\int_t^{\tau(z)} \dot\lambda(s)^2 ds = -8\ln(\sin(\arg(z_t))).$$

By differentiation with respect to t, we get that

$$(\dot\lambda_t)^2 = 16 \times \frac{\mathrm{Re}(z_t)}{\mathrm{Im}(z_t)} \times \partial_t(\arg(z_t)) \tag{2.1}$$

But by Loewner's equation:

$$\partial_t \arg(z_t) = \mathrm{Im}((\partial_t z_t)/z_t) = \mathrm{Im}(2/z_t^2 - \dot\lambda_t/z_t)$$

and a straightforward computation then shows that (2.1) can be rewritten as

$$(\dot\lambda_t - (8\,\mathrm{Re}(z_t)/|z_t|^2))^2 = 0. \tag{2.2}$$

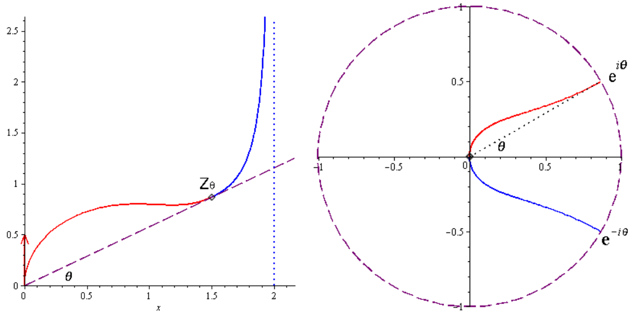

Figure 2.2: Sketch of the minimizing curve in $\mathbb{H}$ (passing to the right of $z = \rho e^{i\pi/6}$ and of the optimal curve in $\mathbb{D}$ from $e^{-i\pi/6}$ to $e^{i\pi/6}$ (passing to the left of 0).

This shows that if there exists a minimizer, then it is unique, as the previous equation describes uniquely λ up to the hitting time of z.

Conversely, if we define the driving function λ that solves this differential equation for all times before the (potentially infinite) hitting time τ of z, we have indeed a minimizer: One way to see that this λ generates a curve passing through z is to consider its the image in the unit disk $\mathbb{D}$ by applying the conformal mapping ψ_θ sending z to 0, 0 to $e^{i\theta}$, hence sending ∞ to $e^{-i\theta}$. After the change of domain, the equation (2.2) gives a simple characterization in the unit disc: The image of the minimizing curve γ under ψ_θ is symmetric with respect to the real axis. The part above the real axis can be viewed as a radial Loewner chain (in the radial time-parameterization) starting from $e^{i\theta}$ driven by the function $\xi(t)$ such that $\cos(\xi(t)) = e^{-t}\cos(\theta)$, hence hits 0 in the $t \to \infty$ limit. The corresponding time in the original half-plane parameterization is finite, and the energy of γ is indeed $-8\ln(\sin(\theta))$. $\square$

We list some remarks on the minimizer and its relation to SLE:

- Readers familiar with the $\mathrm{SLE}_\kappa(\rho)$ may want to interpret this first part of this curve in the radial characterization (from $e^{i\theta}$ until it hits the origin) as a radial $\mathrm{SLE}_0(2)$ curve starting at $e^{i\theta}$ with marked point at $e^{-i\theta}$. By coordinate change ([SW05]) when coming back to the half-plane setting, the minimizing curve before hitting z can be viewed as a chordal $\mathrm{SLE}_0(-8)$ starting at the origin, with inner marked point at z.

- The uniqueness of the minimizer γ also implies that γ is the limit of SLE_κ conditioned to be in $D(z)$, as $\kappa \to 0$. More precisely, if W_κ^z denotes the conditional law of SLE_κ in $D(z)$, then for all positive δ, the W_κ^z-probability that the curve is at $\mathscr{L}_\infty$ distance greater than δ of γ goes to 0 as $\kappa \to 0$.
- Note finally that Freidlin-Wentzell theory also provides a direct proof of the convergence of conditional SLE_κ to the minimizer γ on any compact time interval before the hitting time T of z by γ. Indeed, under the conditional law W_κ^z, the driving function X_t and the flow of z for $t < T$ are described by

$$(E_\kappa): \quad \begin{cases} dX_t = \sqrt{\kappa} d\beta_t + \dfrac{F(w_t)}{\mathrm{Im}(z_t)} dt + \dfrac{\varepsilon_\kappa(w_t)}{\mathrm{Im}(z_t)} dt \\ dz_t = \dfrac{2}{z_t} dt - dX_t \end{cases}$$

where β is a Brownian motion and

$$w_t = \mathrm{Re}(z_t)/\,\mathrm{Im}(z_t), \; F(w) = \frac{8w}{w^2+1} \wedge 0$$

and $\varepsilon_\kappa(\cdot)$ converges uniformly to 0 as $\kappa \to 0$. The minimizer $\gamma[0,T]$ is driven by the solution to the deterministic differential equation E_0. We can show that there exists a unique strong solution to E_κ up to time T with initial conditions $X_0 = 0, z_0 = z$. That solution to E_κ converges to the unperturbed one in probability as $\kappa \to 0$, similar to the Freidlin-Wentzell theorem (see [FSW12]).

2.3.2 Finite point constraints

Now we deal with the minimizing curve under finitely many point constraints. Let $\overline{z}$ denote a set of n labeled points

$$\overline{z} = \{(z_1, \varepsilon_1), \cdots, (z_n, \varepsilon_n)\} \in (\mathbb{H} \times \{-1, 1\})^n.$$

The label $\varepsilon_i = +1$ (resp. $\varepsilon_i = -1$) is interpreted as "right" and "left".

Similarly to the one point constraint case, we define the set of functions *compatible* with $\overline{z}$ as

$$D(\overline{z}) = \{\lambda \in C_0([0, \infty)) \mid \forall i, \tau_i < \infty \text{ or } \lim_{t\to\infty} \varepsilon_i w_i(t) = +\infty\},$$

where $\tau_i = \tau(z_i)$, $w_i(t) = \cot(\arg(f_t(z_i)))$ for $t < \tau_i$.

In the case where λ generates a simple curve γ from 0 to ∞, then $\lambda \in D(\overline{z})$ if and only if for every i, the point z_i is not in $H^{-\varepsilon_i}(\gamma)$.

Let $D^t(\overline{z}) \subset C_0([0, t])$ denote the set of functions such that $\lambda(\cdot \wedge t) \in D(\overline{z})$. We also identify $\lambda \in C_0([0, t])$ with $\lambda(\cdot \wedge t) \in C_0([0, \infty))$ in order to make sense of hitting times, note that the comparison between τ_i and t only depends on the restriction of the function to $[0, t]$, thus does not depend how is the function extended to $\mathbb{R}_+$. Note also that a function of $C_0([0, t])$ is in $D^t(\overline{z})$ if and only if for all $i \leq n$, either $\tau_i \leq t$ or $w_i(t)\varepsilon_i \geq 0$.

The main goal of this subsection is to prove the following result, that states that the probability of SLE_κ is in $D(\overline{z})$ decays exponentially as $\kappa \to 0$ with rate that is equal to the infimum of the energy in $D(\overline{z})$:

Proposition 2.11. *There exists a positive constant τ such that for all $t \geq \tau$,*

$$\lim_{\kappa\to 0} -\kappa \ln(W_\kappa(D(\overline{z}))) = \inf_{\lambda \in D(\overline{z})} I(\lambda) = \inf_{\lambda \in D^t(\overline{z})} I_t(\lambda). \tag{2.3}$$

Moreover, there exists a function λ in $D^\tau(\overline{z})$ such that $I(\lambda)$ is equal to this infimum of the energy over $D(\overline{z})$.

Note that in this multiple point case, the minimizing curve is not

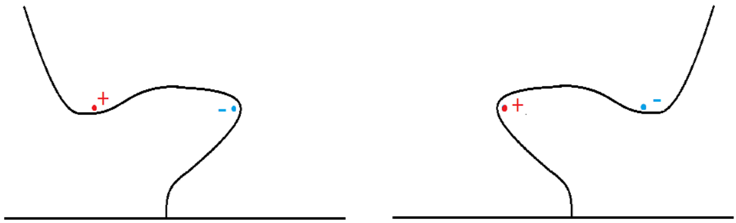

Figure 2.3: two curves compatible with the two-point constraint.

necessarily unique anymore. For instance, we can consider two different points z_1 and z_2 that are symmetric to each other with respect to the imaginary axis, the left one assigned with $\varepsilon_1 = (+1)$ and the right one assigned $\varepsilon_2 = (-1)$. Then, if there exists a minimizing curve in $D(\overline{z})$, the symmetric one with respect to the imaginary axis is necessarily a different curve, and is also minimizing the energy in $D(\overline{z})$ (see Figure 2.3).

It is useful to consider the sets $D^T(\overline{z})$, as it allows to reduce the study of the driving function λ to the truncated function $\lambda|_{[0,T]} \in C_0([0,T])$, for which one can use large deviations easily. Moreover, it also shows that it suffices to look at the truncated part of the driving function on a finite interval to decide whether it is a minimizer of $D(\overline{z})$.

By scale-invariance of the energy, it is enough to consider that case where all the points z_i are in $\mathbb{D} \cap \mathbb{H}$. For all $R > 0$, the hitting time $\tau_R = \inf\{t \geq 0 \mid |\gamma_t| = R\}$ is bounded. As half-plane capacity is non-decreasing, it follows that

$$2\tau_R = \mathrm{hcap}(\gamma[0,\tau_R]) \leq \mathrm{hcap}(R\mathbb{D} \cap \mathbb{H}) = R^2 \mathrm{hcap}(\mathbb{D} \cap \mathbb{H}).$$

This last half-plane capacity is in fact easily shown to be equal to 1, so that τ_R is anyway bounded by $R^2/2$.

We have seen that if γ has finite energy, then it is a simple curve. If

$\tau_i > \tau_R$, then by comparing the harmonic measure seen from z_i at both sides of $\gamma[0, \tau_R]$, we get that $\arg(f_{\tau_R}(z_i)) \in (0, \theta') \cup (\pi - \theta', \pi)$, for some $\theta' = \theta'(R)$ that tends to 0 as $R \to \infty$. But the energy needed after τ_R to change the sign of $w_i(t)$ will therefore be larger than $-8 \ln(\sin(\theta'))$ (which goes to infinity as $R \to \infty$).

But on the other hand, Lemma 2.12 will tell us that one has an *a priori* upper bound on the infimum of the energy in $D(\overline{z})$. Hence, we can conclude that there exists R such that for any function in $D(\overline{z})$ with energy smaller that $\inf_{\lambda \in D(\overline{z})} I(\lambda) + 1$, the sign of w_i can not change after $R^2/2$. This therefore implies that $\lambda(\cdot \wedge R^2/2)$ is also in $D(\overline{z})$.

Lemma 2.12. *There exists a simple curve γ with finite-energy that visits all n points $z_1, \ldots, z_n$.*

Proof. In the one-point constraint case, we have seen that the energy needed for touching a point in $\mathbb{H}$ is finite. It is then easy to iteratively use this to define a curve with finite energy that touches successively all constraint points (start with the part of the minimizing curve that hits z_1, and then continue in the complement of this first bit with the minimizing curve that hits z_i where z_i has the smallest index among points that have not been hit yet and so on). We choose the driving function to be constant after this curve has hit all the n points. $\square$

Let us now define the set $O^T(\overline{z})$ of driving functions λ in $C_0([0,T])$ such that for all $i \le n$, $T < \tau_i$ and the sign of the real part of $f_T(z_i)$ is ε_i. We can note that this set is open in $C_0([0,T])$. Indeed, if λ is in $O^T(\overline{z})$, then by inspecting the evolution of the n points under the Loewner flow, if we perturb only slightly the driving function (in the sense of the L^∞ metric on $[0,T]$), we will not change the fact that one stays in $O^T(\overline{z})$. Note of course that $O^T(\overline{z})$ is a subset of $D^T(\overline{z})$. The complement of $D^T(\overline{z})$ is the union of $O^T(z_i, -\varepsilon_i)$, so that $D^T(\overline{z})$ is closed in $C_0([0,T])$.

The following lemma will be useful in our proof of Proposition 2.11:

Lemma 2.13. *For every $t > 0$,*

$$\inf_{\lambda \in D^t(\overline{z})} I_t(\lambda) = \inf_{\lambda \in O^t(\overline{z})} I_t(\lambda).$$

In the proof of the lemma, we will do surgeries on driving functions of the following type: saying that the part $\lambda[t, t+\delta]$ is replaced by $a[0,\delta]$ where $a(0) = 0$, means that the new driving function $\tilde{\lambda}$ is defined as, $\tilde{\lambda}(s) = \lambda(s)$ for $s \in [0,t]$, $\tilde{\lambda}(t+s) - \tilde{\lambda}(t) = a(s)$ for $s \in [0,\delta]$, and $\tilde{\lambda}(s') - \tilde{\lambda}(s) = \lambda(s') - \lambda(s)$ for $s, s' \geq t + \delta$.

Our goal is to show that for any λ in $D^t(\overline{z})$ with finite energy L, and any $c > 0$, we can find a perturbed $\lambda_\varepsilon \in O^t(\overline{z})$ with energy $I(\lambda_\varepsilon) \leq L + c$, and $\|\lambda - \lambda_\varepsilon\|_\infty \leq c$.

Note that (just as in the previous argument showing that $O^t(\overline{z})$ is open), if γ does not go through any point z_i, then one can ensure by taking ε small enough that any such perturbation will not change on which side z_i ends. Hence, it suffices to prove the result in the case where γ in fact visits all the points $z_1, \dots, z_n$.

The idea is that when the flow starting from z_i is about to hit 0 (ie. when γ is about to hit z_i), one can replace a small portion of the driving function by some optimal curve (targeting at a well chosen point) to avoid a neighborhood of z_i, up to the time when the point z_i tends to the real line but away from 0, and therefore becomes very hard to be reached again. The modification on energy is controlled as well as the impact on other constraints point.

Proof of Lemma 2.13. Let us only explain how to proceed in the case with two constraint points z_1, z_2 and $\varepsilon_1 = \varepsilon_2 = -1$, as the other cases are treated similarly and require no extra idea. Assume that γ visits first z_1 and then z_2 (at times T_1 and T_2) and that $\varepsilon_1 = \varepsilon_2 = -1$.

For the simplicity of notation we write $I_{t,T}(\gamma) := I_T(\gamma) - I_t(\gamma)$. We use the notation $f(t)$ for the centered flow instead of f_t to avoid too heavy subscripts. As before, let $z(t) = f(t)(z_1)$ and $w(t) := \cot(\arg(z(t)))$.

Using a scaling we assume $|f(T_1)(z_2)| > 1$. For every $0 < \eta \ll 1$, there exists $\delta < \eta$, such that

$$I_{T_1-s,T_1+s}(\gamma) \leq \eta,\ |f(T_1-s)(z_1)| \leq \eta$$

and $|f(T_1 \pm s)(z_2)| \geq 1$ for all $s \leq \delta$.

For every $s \leq \delta$, we know from Section 2.3.1 that the minimal energy for a curve to hit a point with argument θ is

$$-8\ln(\sin(\theta)) = 4\ln(w^2+1)$$

where $w = \cot(\theta)$. Hence $w(T_1-s)^2 \leq \exp(\eta/4) - 1$. For every $\varepsilon > 0$, we choose a point $\tilde{z}_1$ with argument slightly smaller than $\arg(z(T_1-s))$, and close enough to $z(T_1-s)$, such that the minimizing curve driven by a, targeting at $\tilde{z}_1$ is to the right of a small neighborhood of $z(T_1-s)$, and such that $I(a) \leq \eta + \varepsilon$.

Let s_1 denote the hitting time of $\tilde{z}_1$ under the driving function a, then $s_1 \asymp s$. If we do the surgery $a[0,\infty)$ to the initial driving function from time T_1-s, the cotangent of the argument $\tilde{w}$ driven by the new function, satisfies that $\tilde{w}(T_1-s+s_1) < 0$ by construction. After T_1-s+s_1, the driving function is constant, so $\tilde{w}(t)$ shrinks very fast towards $-\infty$ under the flow driven by the 0 function which is $z \mapsto \sqrt{z^2+4t}$. When $|\tilde{w}(t)|$ reached the threshold of "the point need energy $2L$ to be hit by the rest of the curve", i.e. when $\tilde{w}(t)^2 \geq \exp(L/2) - 1$, let $\tilde{T_1}$ denote that time, then for any surgery made after time $\tilde{T_1}$, $\forall r \geq \tilde{T_1}$, $\tilde{w}(r) < 0$ if the energy $I_{\tilde{T_1},\infty}$ is smaller than $2L$. Thus we don't worry about the sign of $\tilde{w}(t)$ to change in the future if the energy on the remaining curve is not too much perturbed. Moreover the time $\tilde{T_1} \to T_1$ as $s \to 0$.

Now choose $s < \delta$, such that $\tilde{T_1} \leq T_1 + \delta$. We do the surgery of λ by replacing the part $\lambda([T_1-s,\tilde{T_1}])$ by $a([0,\tilde{T_1}-T_1+s])$. The energy of the total curve is increased by at most ε.

The new curve γ^1 driven by λ^1 does not necessarily pass through z_2. Write the new flow $(f^1(t))$, since we assumed $|f(T_1-s)(z_2)| \geq 1$, by Gronwall type argument, the impact of replacing a piece of length s_1+s_2 of driving function, will change $f(\tilde{T_1})(z_2)$ to $\tilde{z}_2 := f^1(\tilde{T_1})(z_2)$, with Euclidean distance less than

$$\begin{aligned}
&\int_0^{s_1+s_2} \left|\dot\lambda(T_1 - s + t) - \dot a(t)\right| \mathrm{d}t \\
&\le \sqrt{s_1+s_2}\sqrt{2I_{T_1-s,\tilde T_1}(\lambda) + 2I(a)} \\
&\le \sqrt{\delta}\sqrt{4\eta + 2\varepsilon},
\end{aligned}$$

which can be chosen to be arbitrarily small.

Now we apply the same argument to z_2. Since λ^1 and λ have same increments on $[\tilde T_1, \infty)$, $f^1(\tilde T_1)(\gamma^1[\tilde T_1,\infty))$ passes through $f(\tilde T_1)(z_2)$ at time $T_2 - \tilde T_1$. With arbitrairily small compromise in the energy, we can modify again the λ^1 to λ^2 such that the curve $f(\tilde T_1)(\gamma^1[\tilde T_1,\infty))$ passes to the right of a neighborhood of $f(\tilde T_1)(z_2)$, which contains $\tilde z_2$ by well choosing $\eta, \delta, \varepsilon$ in the previous step. $\square$

We are now finally ready for proving Proposition 2.11. Let T be a function $C_0([0,\infty)) \to \mathbb{R}_+$, we define the set of functions $\mathscr{D}^T(\overline{z})$ *compatible with* $\overline{z}$ *up to time* T to be $\{\lambda \in C_0([0,\infty)), \lambda|_{[0,T(\lambda)]} \in D^{T(\lambda)}(\overline{z})\}$. We keep the notations at the beginning of Section 2.3.2 and those before Lemma 2.12.

Proof of Proposition 2.11. Let $M < \infty$ be the energy of the curve γ passing through all constraint points constructed in Lemma 2.12, $T(\gamma)$ be the hitting time of the last point of γ. We choose θ' s.t. $-8\ln(\sin(\theta')) \ge M + 1$. There exists $R \in \mathbb{R}_+$, such that for every $i \le n$, if $\tau_i > \tau_R$, $f_{\tau_R}(z_i)$ has argument in $(0,\theta') \cup (\pi - \theta', \pi)$ and let $\tau = R^2/2$. By construction, $T(\gamma) < \tau_R(\gamma) \le \tau$. So $D^\tau(\overline{z})$ is non empty, and the infimum of the energy of curves in $D^\tau(\overline{z})$ is less than M.

We know that SLE_κ is almost surely transient and does not hit $\overline{z}$. Thus apart from a set N, with zero W_κ-measure for all $\kappa \le 4$, the symmetric difference between $D(\overline{z})$ and $\mathscr{D}^{\tau_R}(\overline{z})$ consists of driving functions which give different signs to $w_i(\tau_R)$ and the limit of $w_i(t)$

when $t \to \infty$ for some $i \leq n$,

$$\begin{aligned}&D(\overline{z})\Delta\mathscr{D}^{\tau_R}(\overline{z})\\=&\{\lambda \in C_0([0,\infty))|\exists i \leq n, \tau_i = \infty \text{ and } w_i(\tau_R)w_i(t) \to -\infty\} \cup N\end{aligned}$$

where N is W_k-null set for all $\kappa \leq 4$.

By the domain Markov property, $w_i(\tau_R)w_i(t) \to -\infty$ means that the Loewner curve reaches a point with argument θ' or $\pi - \theta'$ starting from time τ_R. Hence the probability of $\sqrt{\kappa}B$ stays in the symmetric difference $D(\overline{z})\Delta\mathscr{D}^{\tau_R}(\overline{z})$ is bounded by $2nF_\kappa(\theta')$, where $F_\kappa(\theta)$ is the probability that SLE_κ is to the right of z with argument θ. The inequality $\tau_R \leq \tau$ holds for all driving functions, gives the upper-bound of the probability on the symmetric difference between $D(\overline{z})$ and $\mathscr{D}^\tau(\overline{z})$:

$$\begin{aligned}&W_\kappa(D(\overline{z})\Delta\mathscr{D}^\tau(\overline{z}))\\\leq&W_\kappa(D(\overline{z})\Delta\mathscr{D}^{\tau_R}(\overline{z}))\\&+W_\kappa(\exists i \leq n \mid w_i(\tau_R)w_i(t) \to +\infty, w_i(\tau)w_i(t) \to -\infty)\\\leq&W_\kappa(D(\overline{z})\Delta\mathscr{D}^{\tau_R}(\overline{z})) + W_\kappa(\exists i \leq n \mid w_i(\tau_R)w_i(t) \to -\infty)\\\leq&2W_\kappa(D(\overline{z})\Delta\mathscr{D}^{\tau_R}(\overline{z})) \leq 4nF_\kappa(\theta').\end{aligned}$$

The inequality also holds for $t \geq \tau$. Using Schilder's theorem (Theorem C) on $[0,t]$ and the fact that $D^t(\overline{z})$ is non empty and closed and $O^t(\overline{z}) \subset D^t(\overline{z})$ is open, we have for any $\varepsilon > 0$,

$$\begin{aligned}\kappa \ln W_\kappa(D(\overline{z})) &\leq \kappa \ln \left[W_\kappa(\mathscr{D}^t(\overline{z})) + 4nF_\kappa(\theta')\right]\\&= \kappa \ln \left[W_\kappa(D^t(\overline{z})) + 4nF_\kappa(\theta')\right]\\&\leq -\inf_{\lambda \in D^t(\overline{z})} I_t(\lambda) + \varepsilon;\\\kappa \ln W_\kappa(D(\overline{z})) &\geq \kappa \ln \left[W_\kappa(O^t(\overline{z})) - 4nF_\kappa(\theta')\right]\\&\geq -\inf_{\lambda \in O^t(\overline{z})} I_t(\lambda) - \varepsilon,\end{aligned}$$

for κ is sufficiently small, using the fact that $\lim_{\kappa \to 0} \ln(F_\kappa(\theta')) = M+1$ and $\inf_{\lambda \in D^t(\overline{z})} I_t(\lambda) \leq M$. By Lemma 2.13, we can therefore conclude that for all $t \geq \tau$,

$$\lim_{\kappa \to 0} -\kappa \ln W_\kappa(D(\overline{z})) = \inf_{\lambda \in D^t(\overline{z})} I_t(\lambda).$$

The set $D^\tau(\overline{z})$ is closed and $I_\tau^{-1}([0, M])$ is compact and non empty, so that there exists $\lambda_0 \in D^\tau(\overline{z})$ such that

$$I(\lambda_0) = \inf_{t \geq \tau} \inf_{\lambda \in D^t(\overline{z})} I_t(\lambda) = \inf_{\lambda \in D(\overline{z})} I(\lambda).$$

□

2.3.3 Proof of reversibility of the energy

We can now conclude the proof of Theorem 2.1. For a curve γ from 0 to ∞ driven by λ, the reversed curve $-1/\gamma$ is driven by a continuous function by Theorem 2.2, we define it as $\mathrm{Rev}(\lambda)$. In particular, by Proposition 2.3, the functional Rev is well-defined for finite energy functions. Let $-1/\overline{z}$ be the constraint set obtained by taking the image of $\overline{z}$ by $z \to -1/z$ equipped with the opposite assignment. Write $Z(\gamma)$ for the collection of all finite constraint sets compatible with γ. We have in particular $\overline{z} \in Z(\gamma)$ if and only if $-1/\overline{z} \in Z(-1/\gamma)$.

Let $\mathscr{I}(\overline{z})$ denote the infimum of $I(\lambda)$ over $\lambda \in D(\overline{z})$ ie. it is the minimal energy needed to fulfill the constraint $\overline{z}$.

Proposition 2.14. *Let γ_0 be a simple curve from 0 to ∞ in $\mathbb{H}$. The energy of γ_0 equals to the supremum of $\mathscr{I}(\overline{z})$ over $\overline{z} \in Z(\gamma_0)$.*

This proposition implies Theorem 2.1. Indeed, by reversibility of SLE_κ (Theorem B), we know that

$$W_\kappa(D(\overline{z})) = W_\kappa(D(-1/\overline{z})).$$

Thus by Proposition 2.11, $\mathscr{I}(\overline{z}) = \mathscr{I}(-1/\overline{z})$. Hence (writing $\overline{z'} =$

$-1/\overline{z})$,

$$I(-1/\gamma_0) = \sup_{\overline{z}' \in Z(-1/\gamma_0)} \mathscr{I}(\overline{z}') = \sup_{\overline{z} \in Z(\gamma_0)} \mathscr{I}(-1/\overline{z})$$
$$= \sup_{\overline{z} \in Z(\gamma_0)} \mathscr{I}(\overline{z}) = I(\gamma_0).$$

Proof of Proposition 2.14. Let $L(\gamma_0) = \sup_{\overline{z} \in Z(\gamma_0)} \mathscr{I}(\overline{z})$. Since we know already that $L(\gamma_0) \leq I(\gamma_0)$, it remains to prove that $L(\gamma_0) \geq I(\gamma_0)$ when $L(\gamma_0)$ is finite.

Let $(\overline{z_n})_{n\geq 1}$ be a increasing sequence of finite point constraints, with assignments compatible with γ_0 and such that $\bigcup_{n\geq 1}\{z_n\} = \mathbb{Q} \times \mathbb{Q}_+$. We pick a minimizer γ_n of the energy in $D(\overline{z_n})$ arbitrarily (the existence of such minimizers follows from Proposition 2.11). In addition, we know that $I(\gamma_n)$ is non-decreasing in n, and bounded from above by $L(\gamma_0)$. In view of Corollary 2.4 (with the same notation as in that corollary), there exists a subsequence n_k such that φ_{n_k} converges uniformly locally to some φ. Note that the energy of $\tilde{\gamma} := \varphi(i\mathbb{R}_+)$ can not be larger than $L(\gamma_0)$.

We now show that $\tilde{\gamma}$ is compatible with $\overline{z_n}$ for all n. Assume for instance that the point constraint on z_n is $+1$. Then, for $n_k \geq n$, $\varphi_{n_k}^{-1}(z_n)$ has non-negative real part. Again by quasiconformality of $\varphi_{n_k}^{-1}$, we know that a subsequence of $(\varphi_{n_k}^{-1})_{k\geq 1}$ has local uniform limit φ^{-1} so that the real part of $\varphi^{-1}(z_n)$ is also non-negative. Thus it follows that $\tilde{\gamma}$ is compatible with all the constraints $\overline{z_n}$. Given that both γ_0 and $\tilde{\gamma}$ are simple curves, it follows readily that they are equal, which in turns implies that $I(\gamma_0) = I(\tilde{\gamma}) \leq L(\gamma_0)$. □

2.4 Comments

In the present final section, we will provide two more direct derivations of results that will provide some insight and information about the energy of a Loewner chain. In fact, it might look at first sight that each of these results could be used in order to get an alternative derivation of Theorem 2.1, but it does not seem to be the case for reasons that we will comment on.

2.4.1 Conformal restriction

In this subsection we study the variation of the energy of a given curve when the domain varies (this is similar to the conformal restriction-type ideas initiated in [LSW03] in the SLE-framework, similar computations can be also found in the Section 9.3 of [Dub07]). Let K be a compact hull at positive distance to 0. The simply connected domain $H_K := \mathbb{H}\backslash K$ coincides with $\mathbb{H}$ in the neighborhoods of 0 and ∞. Let $\gamma \subset \mathbb{H}$ be a finite energy curve connecting 0 and ∞, driven by W with centered flow (f_t), and assumed to be at positive distance to K.

Proposition 2.15. *We have*

$$I_{H_K,0,\infty}(\gamma) - I(\gamma) = 3\ln|\psi_0'(0)| + 12\mathscr{B}(\gamma, K; \mathbb{H}),$$

where ψ_0 is the conformal equivalence $H_K \to \mathbb{H}$ fixing $0, \infty$ and $\psi_0(z)$ being normalized as $z + O(1)$ near ∞; $\mathscr{B}(A, B; \mathbb{H})$ is the measure of Brownian loops in $\mathbb{H}$ intersecting both A and B for $A, B \subset \mathbb{H}$.

The Brownian loop measure is a natural conformally invariant measure on the unrooted loops defined in [LW04]. In particular we have $\mathscr{B}(\gamma, K; \mathbb{H}) = \mathscr{B}(-1/\gamma, -1/K; \mathbb{H})$.

We also remark that $|\psi_0'(0)|$ is the probability of a Brownian excursion in $\mathbb{H}$ starting from 0 not hitting K (this was first observed in [Vir03], see also [Wer04] Lemma 5.4; by a Brownian excursion in $\mathbb{H}$ we mean a process in $\mathbb{H}$ which is Brownian motion in x-direction and an independent 3-dimensional Bessel process in y-direction, see [Wer05] for more information). Note that if γ is a analytic curve such that there exists some K where γ is the hyperbolic geodesic in $(H_K, 0, \infty)$, then $I_{H_K,0,\infty}(\gamma) = 0$. Hence,

$$I(\gamma) = -3\ln|\psi_0'(0)| - 12\mathscr{B}(\gamma, K; \mathbb{H}),$$

which does provide an interpretation of the energy $I(\gamma)$ as a function of "conformal distance" between K and γ, and shows directly reversibility of the energy for such curves γ. However, such analytic curves form a rather small class of curves (for instance, the beginning of the curve fully determines the rest of it) so that it does not seem to allow to deduce the general reversibility result easily from this fact.

Proposition 2.15 is a consequence of the following proposition as $t \to \infty$.

Proposition 2.16. *Let $K_t := f_t(K)$, and $\psi_t : \mathbb{H}\backslash K_t \to \mathbb{H}$ the conformal mapping fixing 0 and ∞, such that $\psi_t(z) - z$ bounded.*

$$I(\psi_0(\gamma[0,t])) = \frac{1}{2}\int_0^t \left[\dot{W}_s - \frac{3\psi_t''(0)}{\psi_t'(0)}\right]^2 \mathrm{d}t.$$

Moreover,

$$\begin{aligned} I(\psi_0(\gamma[0,t])) &- I(\gamma[0,t]) \\ &= 3\ln|\psi_0'(0)| + 12\mathscr{B}(\gamma[0,t], K; \mathbb{H}) - 3\ln|\psi_t'(0)|, \end{aligned}$$

with $3\ln|\psi_t'(0)| \to 0$ when $t \to \infty$.

The first equality is obtained by computing the driving function of $\psi_0(\gamma)$ as in [LSW03] Section 5 and then the difference is identified using the identity:

$$\mathscr{B}(\gamma[0,t], K; \mathbb{H}) = -\frac{1}{3}\int_0^t S\psi_s(0)\,\mathrm{d}s,$$

where S is the Schwarzian derivative. The identification is explained in [LSW03] Section 7.1. Moreover, $-S\psi_s(0)/6$ can be interpreted as the half-plane capacity of K_s seen from 0 as we explained in Section 2.2.2.

2.4.2 Two curves growing towards each other

The commutation relations for SLE [Dub07] are closely related to the coupling of both ends of the SLE curves used to show SLE reversibility. In the present subsection, we will derive directly an analogous commutation relation for the energy of two curves growing towards each other. One may again wonder whether it is possible to deduce the reversibility of the Loewner energy from this commutation, but it seems again to be strictly weaker than Theorem 2.1.

Let us compute the energy of a curve γ^T in $\mathbb{H}\backslash\tilde{\gamma}^S$, where $\gamma^T := \gamma[0,T]$ starting at 0 is driven by W and $\tilde{\gamma}^S := \tilde{\gamma}[0,S]$ is driven by U in $(\mathbb{H},\infty,0)$ at positive distance of γ^T, i.e. the image of $(\tilde{\gamma}_s)$ by $z \mapsto -1/z$ is driven by U. The centered flow $(\phi_{s,0})_s$ associated with $\tilde{\gamma}^S$ is, for every $s \leq S$, the mapping-out function of $\tilde{\gamma}^s$ sending the tip to ∞, fixing 0 and being normalized as $\phi'_{s,0}(0) = 1$. In particular $(\phi_{s,0})_s$ satisfies the Loewner equation:

$$\partial_s(\phi_{s,0})(z) = -\phi_{s,0}(z)^2(2\phi_{s,0}(z) + \dot{U}_s), \quad \phi_{0,0}(z) = z.$$

Thus

$$\partial_s(\phi'_{s,0}(z)) = -6\phi_{s,0}(z)^2\phi'_{s,0}(z) - 2\phi_{s,0}(z)\phi'_{s,0}(z)\dot{U}_s$$

$$\begin{aligned}\partial_s(\phi''_{s,0}(z)) = &-12\phi_{s,0}(z)(\phi'_{s,0}(z))^2 - 6\phi_{s,0}(z)^2\phi''_{s,0}(z)\\ &-2(\phi'_{s,0}(z))^2\dot{U}_s - 2\phi_{s,0}(z)\phi''_{s,0}(z)\dot{U}_s\end{aligned}$$

Since $\phi_{s,0}(0) = 0$ and $\phi'_{s,0}(0) = 1$,

$$\partial_s\left(\frac{\phi''_{s,0}(0)}{\phi'_{s,0}(0)}\right) = \frac{\partial_s\phi''_{s,0}(0)}{\phi'_{s,0}(0)} - \frac{\phi''_{s,0}(0)\partial_s\phi'_{s,0}(0)}{(\phi'_{s,0}(0))^2} = -2\dot{U}_s$$

which implies $\phi''_{s,0}(0)/\phi'_{s,0}(0) = -2U_s$. The map $\phi_{s,t}$ is defined as illustrated in the Figure 2.4, where f_t and f_t^s are the (time reparametrized) centered flow of γ^T and $\phi_{S,0}(\gamma^T)$. We deduce that

$$\phi''_{s,t}(0)/\phi'_{s,t}(0) = -2U_s^t$$

where $(U_s^t)_{s\geq 0}$ is the driving function for $f_t(\tilde{\gamma})$ parameterized by capacity of $\tilde{\gamma}$.

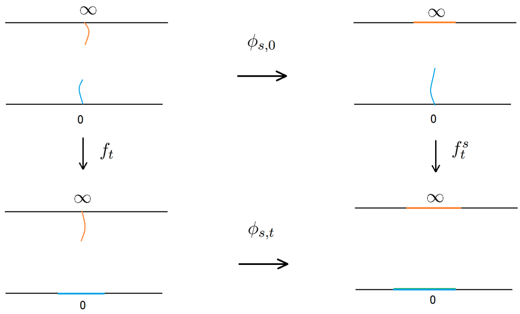

Figure 2.4: f_t and f_t^s are centered mapping-out functions normalized at ∞, and $\phi_{s,0}$ is the centered mapping-out function normalized at 0. The mapping $\phi_{s,t}$ makes the diagram commute.

Lemma 2.17. *With above notations,*

$$\begin{aligned} I_{\mathbb{H}\backslash\tilde\gamma^S,0,\tilde\gamma_S}(\gamma[0,T]) &= \frac{1}{2}\int_0^T \left[\dot W_t - \frac{3\phi''_{S,t}(0)}{\phi'_{S,t}(0)}\right]^2 \mathrm{d}t \\ &= \frac{1}{2}\int_0^T \left[\dot W_t + 6U_S^t\right]^2 \mathrm{d}t. \end{aligned}$$

Furthermore,

$$\begin{aligned} &I_{\mathbb{H}\backslash\tilde\gamma^S,0,\tilde\gamma_S}(\gamma^T) - I(\gamma^T) \\ &= 12\mathscr{B}(\gamma^T,\tilde\gamma^S;\mathbb{H}) - 3\ln(\phi'_{S,T}(0)) \\ &= 12\mathscr{B}(\gamma^T,\tilde\gamma^S;\mathbb{H}) - 3\ln\left(\frac{H(\mathbb{H}\backslash\gamma^T\cup\tilde\gamma^S,\gamma_T,\tilde\gamma_S)H(\mathbb{H},0,\infty)}{H(\mathbb{H}\backslash\gamma^T,\gamma_T,\infty)H(\mathbb{H}\backslash\tilde\gamma^S,0,\tilde\gamma_S)}\right), \end{aligned}$$

where $H(D,x,y)$ is the Poisson excursion kernel in a domain D at boundary points x and y with respect to analytic local coordinates in the neighborhood of each of them. We choose local coordinates to coincide if it is the same neighborhood involved.

The quotient term of Poisson excursion kernel does not depend on the local coordinates (for more discussions on the Poisson excursion kernel, readers may refer to [Dub09] Section 3). The first equality is obtained as in the previous section, and the second one is due to the computation above. In particular, we see how the energy of $\tilde{\gamma}^S$ redistributes on γ^T in $(\mathbb{H} \setminus \tilde{\gamma}^S, 0, \tilde{\gamma}_S)$ in a rather intricate way.

Also notice that the last expression of the difference shows the symmetry in γ^T and $\tilde{\gamma}^S$. Hence we get the following commutation relation:

Corollary 2.18. *The sum of the energy of one slit and the energy of the second slit in the domain left vacant by the first one, targeting at the tip of the first slit, does not depend on the ordering of the slits:*

$$I(\tilde{\gamma}^S) + I_{\mathbb{H}\setminus\tilde{\gamma}^S,0,\tilde{\gamma}_S}(\gamma^T) = I(\gamma^T) + I_{\mathbb{H}\setminus\gamma^T,\infty,\gamma_T}(\tilde{\gamma}^S).$$

To conclude, let us remark that the above corollary is also a simple corollary of Theorem 2.1: Let us define the hyperbolic geodesic γ in the two-slit domain between the two tips. The concatentation of this geodesic with the two slits defines a curve from the origin to infinity in the upper half-plane. Then

$$\begin{aligned} I(\gamma^T \cup \gamma \cup \tilde{\gamma}^S) =& I(\gamma^T) + I_{\mathbb{H}\setminus\gamma^T,\infty,\gamma_T}(\gamma \cup \tilde{\gamma}^S) \\ =& I(\gamma^T) + I_{\mathbb{H}\setminus\gamma^T,\infty,\gamma_T}(\tilde{\gamma}^S), \end{aligned}$$

(the first equality is due to the additivity of I and Theorem 2.1 applied in $(\mathbb{H} \setminus \gamma^T, \infty, \gamma_T)$). Considering the completed curve $\gamma^T \cup \gamma \cup \tilde{\gamma}^S$ in $(\mathbb{H}, \infty, 0)$, we get

$$I(\gamma^T \cup \gamma \cup \tilde{\gamma}^S) = I(\tilde{\gamma}^S) + I_{\mathbb{H}\setminus\tilde{\gamma}^S,0,\tilde{\gamma}_S}(\gamma^T),$$

which provides an alternative proof of the corollary.

2.A One point estimate

Lemma 2.19. *Let* $F(w) = \frac{8w}{w^2+1} \wedge 0$*, and* $\varepsilon_\kappa(w) = -\kappa \frac{h'_\kappa(w)}{h_\kappa(w)} - F(w)$*, then*

$$\varepsilon_\kappa(w) = -\kappa \frac{h'_\kappa(w)}{h_\kappa(w)} - F(w) = \frac{\kappa(w^2+1)^{-\frac{4}{\kappa}}}{\int_w^\infty (s^2+1)^{-\frac{4}{\kappa}}\, \mathrm{d}s} - F(w)$$

converges uniformly to 0 *as* $\kappa \to 0$.

Proof. For $w > 0$, we can write $\varepsilon_\kappa(w)$ as

$$\frac{\kappa(w^2+1)^{-\frac{4}{\kappa}}}{\int_w^\infty (s^2+1)^{-\frac{4}{\kappa}}\, \mathrm{d}s} - \frac{8w}{w^2+1} = \frac{2}{(w^2+1)M_\kappa(w)} - \frac{8w}{w^2+1}$$

after a change of variable $t^2 = \frac{(w^2+1)s^2}{s^2+1}$ in the integral:

$$M_\kappa(w) = \frac{2}{\kappa} \int_w^{\sqrt{w^2+1}} (w^2+1-t^2)^{\frac{4}{\kappa}-\frac{3}{2}}\, \mathrm{d}t.$$

It can be bounded by

$$\begin{aligned} M_\kappa(w) &\leq \frac{1}{\kappa w} \int_w^{\sqrt{w^2+1}} 2t(w^2+1-t^2)^{\frac{4}{\kappa}-\frac{3}{2}}\, \mathrm{d}t \\ &= \frac{1}{\kappa w} \frac{2\kappa}{8-\kappa} [-(w^2+1-t^2)^{\frac{4}{\kappa}-\frac{1}{2}}]_w^{\sqrt{w^2+1}} = \frac{2}{w(8-\kappa)} \end{aligned}$$

For $1 > \varepsilon > 0$,

$$\begin{aligned} M_\kappa(w) &\geq \frac{1}{\kappa\sqrt{w^2+\varepsilon}} \int_w^{\sqrt{w^2+\varepsilon}} 2t(w^2+1-t^2)^{\frac{4}{\kappa}-\frac{3}{2}}\, \mathrm{d}t \\ &= \frac{1}{\kappa\sqrt{w^2+\varepsilon}} \frac{2\kappa}{8-\kappa} [-(w^2+1-t^2)^{\frac{4}{\kappa}-\frac{1}{2}}]_w^{\sqrt{w^2+\varepsilon}} \\ &= \frac{2}{(8-\kappa)\sqrt{w^2+\varepsilon}} (1-(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}}). \end{aligned}$$

Then we obtain bounds on ε_κ:

$$\varepsilon_\kappa(w) \geq \frac{-\kappa w}{w^2+1} \geq -\frac{\kappa}{2},$$

where we used $\frac{w}{w^2+1} \leq \frac{1}{2}$. And

$$\begin{aligned}
\varepsilon_\kappa(w) &\leq \frac{1}{w^2+1}\left(\frac{(8-\kappa)\sqrt{w^2+\varepsilon}}{1-(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}}} - 8w\right) \\
&\leq \frac{1}{w^2+1}\left|\frac{(8-\kappa)\sqrt{w^2+\varepsilon} - 8w + 8w(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}}}{1-(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}}}\right| \\
&\leq \frac{1}{w^2+1}\frac{(8-\kappa)(\sqrt{w^2+\varepsilon}-w) + \kappa w + 8w(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}}}{1-(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}}}
\end{aligned}$$

for every κ, we choose ε such that $(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}} = \kappa$. One can check that $\varepsilon \sim \frac{\kappa \ln(\kappa)}{4}$, and notice that $\sqrt{w^2+\varepsilon} - w \leq \sqrt{\varepsilon}$. The upper-bound of $\varepsilon_\kappa(w)$ becomes

$$\begin{aligned}
\varepsilon_\kappa(w) &\leq \frac{1}{w^2+1}\frac{(8-\kappa)\sqrt{\varepsilon} + \kappa w + 8w\kappa}{1-\kappa} \\
&\leq \frac{(8-\kappa)\sqrt{k\ln(k)}}{1-\kappa} + \frac{9\kappa}{1-\kappa}\frac{1}{2},
\end{aligned}$$

which does not depend on w and converges to 0 with speed $\sqrt{\kappa \ln(\kappa)}$.

For $w \leq 0$, $\varepsilon_\kappa(w)$ can be bounded brutally:

$$0 \leq \varepsilon_\kappa(w) = \frac{\kappa(w^2+1)^{-\frac{4}{\kappa}}}{\int_w^\infty (s^2+1)^{-\frac{4}{\kappa}}\,\mathrm{d}s} \leq \frac{\kappa}{\int_0^\infty (s^2+1)^{-\frac{4}{\kappa}}\,\mathrm{d}s} = \varepsilon_\kappa(0).$$

Using a change of variable $u = \ln(s^2+1)/\kappa$, one gets

$$\begin{aligned}\varepsilon_\kappa(0) =& \frac{2}{\int_0^\infty e^{-4u} e^{u\kappa}/\sqrt{e^{uk}-1}\,\mathrm{d}u} \leq \frac{2}{\int_0^1 e^{-4u}/\sqrt{e^{uk}-1}\,\mathrm{d}u} \\ \leq& \frac{2\sqrt{\kappa}}{\int_0^1 e^{-4u}/\sqrt{2u}\,\mathrm{d}u} = c\sqrt{\kappa}.\end{aligned}$$

The second inequality holds for $\kappa < c'$, where $c, c' > 0$. We conclude that ε_κ converges uniformly to 0 with speed at least $\sqrt{\kappa}$ on $(-\infty, 0]$. $\square$

Chapter 3

Loop energy and regularity correspondence

This chapter corresponds to the joint paper with Steffen Rohde [RW17].

3.1 Introduction

Our first main result is the following.

Theorem 3.1. *The loop Loewner energy is root-invariant.*

In our proof of the root-invariance, we approximate the curve by well-chosen regular curves and are led to the following question: What can we say about the relation between the regularity of the driving function and the regularity of the curve? Prior to this work, only one direction was well understood. Slightly imprecisely, the following results state that C^α driving functions generate $C^{\alpha+1/2}$ curves for $\alpha > 1/2$, where C^α is understood with the usual convention as $C^{n,\beta}$, where n is the integer part of α and $\beta = \alpha - n$ (see Section 3.3.1). More precisely:

Theorem 3.2 ([Won14])**.** *If $0 < \beta \leq 1/2$, $W \in C^{0,1/2+\beta}$, then the Loewner curve η in $\mathbb{H}$ generated by W is a simple curve of class $C^{1,\beta}$ if reparametrized as $t \mapsto \eta(t^2)$. If $W \in C^{1,\beta}$, the curve is in $C^{1,\beta+1/2}$ (weakly $C^{1,1}$ when $\beta = 1/2$).*

We will comment on the choice of parametrization later on. Similarly,

Theorem 3.3 ([Won14, LT16])**.** *If $\alpha > 3/2$ and $W \in C^\alpha$, then W generates a simple curve of class $C^{\alpha+1/2}$ if $\alpha + 1/2 \notin \mathbb{N}$, and in the Zygmund class $\Lambda_*^{\alpha-1/2}$ otherwise.*

The Zygmund class $\Lambda_*^{\alpha-1/2}$ contains the class $C^{\alpha+1/2}$. In the other direction, one can ask about the regularity of the driving function given the regularity of the curve. Here Earle and Epstein proved the following result using a local quasiconformal variation near the tip of the curve:

Theorem 3.4 ([EE01]). *If $n \in \mathbb{Z}$, $n \geq 2$ and $\eta \in C^n$, then its driving function is C^{n-1} on the half-open interval $(0, T]$.*

They stated the result in the radial setting, but using a change of coordinate it is not hard to see that the regularity of the driving function remains the same in the chordal case. Their result precedes the work of Wong, Lind and Tran, which in turn supported the natural conjecture that C^α curves should have $C^{\alpha-1/2}$ driving functions when $\alpha > 1$.

The second main result of this paper is a proof of this conjecture in the case $1 < \alpha \leq 2$. It is the converse of Theorem 3.2 when neither α nor $\alpha - 1/2$ is an integer. We will discuss the remaining cases of higher regularity in Section 3.4.

Conventions: We say that an (arc-length parametrized) simple arc $\gamma : [0, S] \to \mathbb{C} \setminus \mathbb{R}_{>0}$ of regularity at least C^1 is *tangentially attached* to $\mathbb{R}_+$ if $\gamma(0) = 0$, and the right-derivative $\gamma'(0) = -1$. In this paper, the curve γ is always at least C^1 and arc-length parametrized (we use the variable s). Loewner driving functions are defined with respect to capacity parametrization (we use the variable t). We use η for Loewner curves in $\mathbb{H}$, in particular for $\sqrt{\gamma}$, where $\sqrt{\cdot}$ on $\mathbb{C} \setminus \mathbb{R}_+$ is taking values in $\mathbb{H}$. Let T be the half-plane capacity of $\sqrt{\gamma[0, S]}$.

Theorem 3.5. *Let $0 < \beta \leq 1$, and γ be a $C^{1,\beta}$ simple arc tangentially attached to $\mathbb{R}_+$. The driving function W of $\sqrt{\gamma}$ has the following regularity on the closed interval $[0, T]$:*

- *$C^{0,\beta+1/2}$ if $0 < \beta < 1/2$;*
- *weakly $C^{0,1}$, if $\beta = 1/2$;*
- *$C^{1,\beta-1/2}$ with $\dot{W}_0 = 0$, if $1/2 < \beta < 1$;*
- *weakly $C^{1,1/2}$, if $\beta = 1$.*

Their respective norm is bounded by a function of both the local regularity $\|\gamma\|_{1,\beta}$ and constants associated with the global geometry of γ.

The *weak regularity* stands for a logarithmic correction term in the modulus of continuity (see Section 3.3.1). Examples of curves with bottle-necks easily show that the C^α norm of the driving function cannot be bounded solely in terms of the local behavior of γ. The sharpness of the Theorem is addressed in Section 3.4.1.

It is also not hard to deduce from Theorem 3.5 that $C^{1,\beta}$ simple loops have finite energy if $\beta > 1/2$ (Proposition 3.10).

Let us comment on the choice of the simply connected domain $\mathbb{C} \setminus \mathbb{R}_+$ and subtleties in the parametrization chosen. Note that unlike previous results, we study the regularity of the curve on the closed interval $[0, S]$, which requires some regularity of the curve at 0. This is the reason why we work with curves in the complement of $\mathbb{R}_+$ rather than in $\mathbb{H}$. In fact, it is trivial but worth mentioning that a simple curve γ is $C^{1,\beta}$ on $[0, S]$ and tangentially attached to $\mathbb{R}_+$ if and only if $\gamma[0, S] \cup \mathbb{R}_+$ is a $C^{1,\beta}$ simple curve. On the other hand, the driving function $\tilde{W}$ of $\gamma[0, S] \cup [0, 1]$, considered as a chord in the domain $\mathbb{C} \setminus [1, \infty)$, starts with constant function 0 (corresponding to the part $[0, 1]$) and continues with the driving function of γ. Hence it suffices to study the regularity correspondence between $\gamma \cup [0, 1]$ and $\tilde{W}$ which is non-trivial only away from the starting point. Notice that in Theorem 3.2, the parametrization $t \mapsto \eta(t^2)$ is natural since in the half-plane setting, $\eta(t)$ is of order $\sqrt{t}$ for small capacity t. However, $t \mapsto t^2$ is smooth, therefore it does not affect the regularity away from 0. Therefore, considering regularity correspondence in the domain $\mathbb{C} \setminus \mathbb{R}_+$ is more natural than in $\mathbb{H}$ and Theorem 3.2 can be stated as the implication of the regularity $\tilde{W}$ to the regularity of $\gamma \cup [0, 1]$ under the usual capacity parametrization. Note that, according to the above conventions, in our theorem the smoothness assumption of γ is with respect to the arclength-parametrization, while the stated regularity of W refers to the capacity parametrization. Since the arclength parametrization has the highest degree of regularity among all parametrizations that have speed bounded away from zero (to see this, note that for any parametrization, the arclength function and hence its inverse has the same regularity as the curve), it follows from Theorem 3.5 and Theorem 3.2 that both the arclength and capacity parametrizations of the curve have the same degree of smoothness which is also 1/2 higher than the smoothness of the driving function.

Returning to the strategy of the proof of Theorem 3.1: We use concatenated circular arcs to replace a part of the loop and deduce that the energy rooted at two ends of each circular arc are the same if both of them are finite. We use Theorem 3.5 to show that loops appearing in the surgery are regular enough to have finite energy. The proof of the general case uses an approximation by minimal energy loops that are of independent interest (Proposition 3.13). Our proof uses the reversibility of Loewner energy, sometimes implicitly so that we never specify the orientation of loops/arcs and alter freely the orientation. As the reversibility was proved using an interpretation via SLE_{0+}, our proof of Theorem 3.1 is not purely deterministic.

However, it suggests that loop energy has to be a more fundamental quantity. Indeed, in Chapter 4 of very different flavor, we derive equivalent descriptions of the loop energy connecting to Weil-Petersson class of universal Teichmüller space, which give a deterministic proof of both reversibility and root invariance.

3.2 Loop Loewner energy

3.2.1 Basic properties

In this section, we provide more details on the definition and basic properties the rooted loop/arc Loewner energy. As we explained in the introduction, it is a natural generalization of the Loewner energy for chords. In order to distinguish the different types of energy that we are dealing with, we use the superscript C for chords (i.e. $I = I^C$), L for loops and A for arcs.

We first list/recall some properties of the chordal Loewner energy in addition to the list of Section 2.2.2:

- *Regular curves have finite energy.* If $\beta > 1/2$, $S < \infty$ and $\gamma[0, S]$ is an arclength-parametrized $C^{1,\beta}$ curve tangentially attached to $\mathbb{R}_+$, then Theorem 3.5 below implies that the driving function of $\sqrt{\gamma}$ is in $C^{1,\beta-1/2}$ on $[0, T]$. Since the capacity $T < \infty$, γ has finite energy in $(\mathbb{C} \setminus \mathbb{R}_+, 0, \infty)$.
- *Finite energy curves are quasiconformal.* If γ in $(\mathbb{H}, 0, \infty)$ has finite energy, then it is the image of $i[0, 1]$ if $T < \infty$ (or $i\mathbb{R}_+$ if $T = \infty$) under a quasiconformal homeomorphism $\mathbb{H} \to \mathbb{H}$ fixing 0 and ∞ (Proposition 2.3).
- *Finite energy curves are rectifiable.* See [FS17, Thm. 2.iv].
- *Corners have infinite energy.* The reason is that finite energy curves in $(\mathbb{H}, 0, \infty)$ have a vertical tangent at 0 (Proposition 2.10), while a corner with an opening angle different from π generates a curve with non-vertical tangent at 0 when we map out the portion of the curve up to the corner. More generally, it is not hard to see that finite energy curves are asymptotically conformal (see [Pom92, Chap. 11.2]), using the fact that small energy implies small quasiconformal constant.
- *Reversibility.* See Chapter 2. For any simple curve $\gamma \subset D$ connecting a and b,
$$I_{D,a,b}(\gamma) = I_{D,b,a}(\gamma).$$
  Thus when there is no ambiguity of which boundary points we are dealing with, we simplify the notation to $I_D(\gamma)$, and view γ as an unoriented curve.

By the conformal invariance of chordal energy and Proposition 2.15, we also deduce the change of Loewner energy in two domains which coincide in a neighborhood of the marked boundary points. The corollary below is an important tool in the proofs of this chapter.

Corollary 3.6. *Let (D, a, b) and (D', a, b) be two domains coinciding in a neighborhood of a and b, and γ a simple curve in both (D, a, b)*

and (D', a, b)*. Then*

$$\begin{aligned} & I_{D',a,b}(\gamma) - I_{D,a,b}(\gamma) \\ =& 3\ln(\psi'(a)\psi'(b)) + 12\mathscr{B}(\gamma, D\setminus D'; D) - 12\mathscr{B}(\gamma, D'\setminus D; D'), \end{aligned}$$

where $\psi : D' \to D$ *is a conformal map fixing* a, b *and* $\mathscr{B}(A, B; D)$ *is the total mass of Brownian loops contained in* D *and attached to both* A *and* B*.*

Definition 3.7. A *simple loop* is a continuous 1-periodic function $\gamma : \mathbb{R} \to \hat{\mathbb{C}}$, such that $\gamma(s) \neq \gamma(t)$, for $0 \le s < t < 1$. We consider two loops as the same if they differ by an increasing reparametrization.

Proposition 3.8. *Both limits below exist and are equal:*

$$\lim_{\varepsilon\searrow 0} I^C_{\gamma[0,\varepsilon]}(\gamma[\varepsilon, 1]) = \lim_{\delta\searrow 0} I^C_{\gamma[-\delta,0]}(\gamma[0, 1-\delta]) \in [0,\infty].$$

We define the *rooted loop Loewner energy* of a simple loop γ at $\gamma(0)$ to be this limit, denoted as $I^L(\gamma, \gamma(0))$. It is clear that the definition does not depend on the increasing reparametrization fixing $\gamma(0)$. Similarly, the energy of γ rooted at $\gamma(s)$ is

$$I^L(\gamma, \gamma(s)) := I^L(\tilde{\gamma}, \tilde{\gamma}(0)),$$

where $\tilde{\gamma}$ is γ "re-rooted at $\gamma(s)$", defined as $\tilde{\gamma}(t) = \gamma(t+s)$.

Proof. The existence follows from

$$\begin{aligned} I^C_{\gamma[0,\varepsilon]}(\gamma[\varepsilon, 1]) &= I^C_{\gamma[0,\varepsilon]}(\gamma[\varepsilon, \varepsilon']) + I^C_{\gamma[0,\varepsilon']}(\gamma[\varepsilon', 1]) \\ &\ge I^C_{\gamma[0,\varepsilon']}(\gamma[\varepsilon', 1]), \end{aligned}$$

if $\varepsilon' > \varepsilon$. The limit is then an increasing limit as $\varepsilon \to 0$. The proof is the same for $\delta \to 0$.

For the equality, it suffices to show

$$\lim_{\varepsilon \searrow 0} I^C_{\gamma[0,\varepsilon]}(\gamma[-1/3,0] \cup \gamma[\varepsilon, 1/3]) \\ = \lim_{\delta \searrow 0} I^C_{\gamma[-\delta,0]}(\gamma[-1/3,-\delta] \cup \gamma[0,1/3]).$$

The above expressions are two-slit Loewner energies defined as the sum in Corollary 2.18. In fact, it follows from the reversibility and the additivity of chordal Loewner energy that

$$I^C_{\gamma[0,\varepsilon]}(\gamma[\varepsilon, 1]) = I^C_{\gamma[0,\varepsilon]}(\gamma[-1/3,0] \cup \gamma[\varepsilon, 1/3]) \\ + I^C_{\gamma[-1/3,1/3]}(\gamma[1/3, 2/3]).$$

Now assume $\lim_{\varepsilon \searrow 0} I^C_{\gamma[0,\varepsilon]}(\gamma[-1/3,0] \cup \gamma[\varepsilon, 1/3]) = A < \infty$. Then

$$I^C_{\gamma[-\delta,\varepsilon]}(\gamma[\varepsilon, 1/3]) \le A$$

for all $\varepsilon > 0$, and it follows from the definition of chordal Loewner energy that $I^C_{\gamma[-\delta,0]}(\gamma[0,1/3]) \le A$, so that (again from the definition)

$$\lim_{\varepsilon \searrow 0} I^C_{\gamma[-\delta,0]}(\gamma[0,\varepsilon]) = 0.$$

It follows that

$$\begin{aligned} & I^C_{\gamma[-\delta,0]}(\gamma[-1/3,-\delta] \cup \gamma[0,1/3]) \\ = & I^C_{\gamma[-\delta,0]}(\gamma[0,\varepsilon]) + I^C_{\gamma[-\delta,\varepsilon]}(\gamma[-1/3,-\delta] \cup \gamma[\varepsilon,1/3]) \\ = & \lim_{\varepsilon \searrow 0} I^C_{\gamma[-\delta,\varepsilon]}(\gamma[-1/3,-\delta] \cup \gamma[\varepsilon,1/3]) \\ = & \lim_{\varepsilon \searrow 0} I^C_{\gamma[0,\varepsilon]}(\gamma[-1/3,0] \cup \gamma[\varepsilon,1/3]) - I^C_{\gamma[0,\varepsilon]}(\gamma[-\delta,0]) \le A. \end{aligned}$$

We conclude that $\lim_{\delta \searrow 0} I^C_{\gamma[-\delta,0]}(\gamma[-1/3,-\delta] \cup \gamma[0,1/3]) \le A$, and obtain the equality by symmetry. □

Similarly, we define the *Loewner energy of a simple arc* (continuous injective) $\eta : [0,1] \to \hat{\mathbb{C}}$ rooted at $\eta(s)$ as follows:

$$\begin{aligned} I^A(\eta, \eta(s)) &= \lim_{\varepsilon \searrow 0} I^C_{\eta[s,s+\varepsilon]}(\eta[0,s] \cup \eta[s+\varepsilon, 1]) \\ &= \lim_{\delta \searrow 0} I^C_{\eta[s-\delta,s]}(\eta[0,s-\delta] \cup \eta[s,1]). \end{aligned}$$

As the definitions suggests, the loop- and arc energies a priori depend strongly on the root, but we will prove that they are actually independent of it. We first deal with sufficiently regular loops (for instance in the class $C^{1.5+\varepsilon}$, $\varepsilon > 0$). This does not cover all finite energy loops, since there exist such loops which are not even C^1, see the last section for a construction of an example. We will now show that finite energy loops are quasicircles (images of S^1 by quasiconformal homeomorphisms of $\hat{\mathbb{C}}$). On the other hand, notice that quasicircles do not necessarily have finite energy.

Proposition 3.9. *If γ is a finite energy loop when rooted at $\gamma(0)$, then γ is a K-quasicircle, where K depends on $I^L(\gamma, \gamma(0))$.*

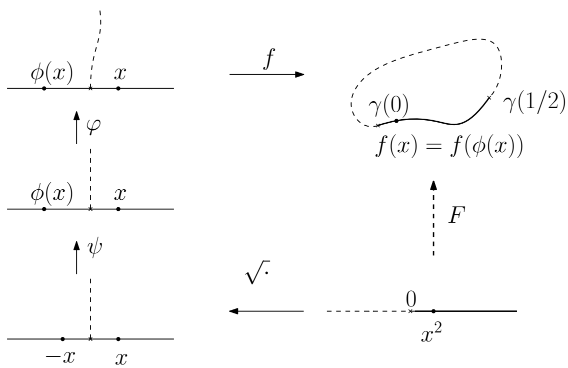

Figure 3.1: Maps in the proof of Proposition 3.9. Solid lines are the boundary of domains.

Proof. It is not hard to see from Carathéodory's theorem that every uniformizing conformal map $f : \mathbb{H} \to \hat{\mathbb{C}} \setminus \gamma[0, 1/2]$ extends continuously to $\mathbb{R}$. Thus we may normalize f such that 0 and ∞ are sent to the two tips of $\gamma[0,1/2]$, say $f(0) = \gamma(1/2)$ and $f(\infty) = \gamma(0)$. Furthermore, f induces a welding function ϕ on $\mathbb{R}$ that is defined by the property that $\phi(x) = y$ if and only if $x = y = 0$ or $f(x) = f(y)$ when $x \neq y$. Thus ϕ is a decreasing involution that encodes which points on the real line are identified by f in order to form $\gamma[0,1/2]$. Since $I^A(\gamma[0,1/2],\gamma(0)) \leq I^L(\gamma,\gamma(0))$, the welding function ϕ is an orientation reversing M-quasisymmetric function, where M depends only on $I^L(\gamma,\gamma(0))$: To see this, fix $x > 0$, set $y = \phi(x)$ and let $t > 0$ be defined by $\gamma(t) = f(x)$. Then the welding function ϕ restricted to $[y, x]$ is the welding function for the slit $\gamma[t, 1/2]$ in the simply connected domain $\hat{\mathbb{C}} \setminus \gamma[0,t]$. Hence Proposition 2.3 implies that both inequalities in Lemma 2.6 hold on the interval $[y, x]$. As we can choose x as large as we want, the inequalities hold on $\mathbb{R}$ and it follows that ϕ is quasisymmetric.

Next, consider the homeomorphism ψ of $\mathbb{R}$ that sends the symmetric pair of points $x, -x$ to the pair $x, \phi(x)$ for all $x \geq 0$. In other words, define $\psi(x) = x$ for $x \geq 0$ and $\psi(x) = \phi(-x)$ for $x < 0$. Then $f(\psi(-x)) = f(\psi(x))$ for all x. It is easy to see, again using both inequalities in Lemma 2.6, that ψ is quasisymmetric (again with constant depending only on $I^L(\gamma,\gamma(0))$). Any quasisymmetric function that fixes 0 can be extended to a quasiconformal map in $\mathbb{H}$ that fixes $i\mathbb{R}_+$ (for instance via the Jerison-Kenig extension, [AIM08] Theorem 5.8.1). Denote such an extension again by ψ.

Now let $\eta = f^{-1}(\gamma[1/2,1])$ and note that $I_{\mathbb{H}}(\eta) \leq I^L(\gamma,\gamma(0))$ so that η is a K-quasislit by Proposition 2.3. In other words, there exists a K-quasiconformal self-map φ of $\mathbb{H}$ fixing 0 and ∞ such that $\varphi(i\mathbb{R}_+) = \eta$, where K depends only on the chordal energy of η. The restriction of φ to $\mathbb{R}$ is a quasisymmetric function. Thus by pre-composing φ with a K-quasiconformal extension of φ^{-1} that fixes $i\mathbb{R}_+$, we can choose φ such that $\varphi(x) = x$ for $x \in \mathbb{R}$.

Finally, define a quasiconformal homeomorphism of the Riemann sphere that maps the real line to the loop γ as follows: Denote $\sqrt{\cdot}$ the branch of the square-root that maps the slit plane $\mathbb{C} \setminus [0,\infty)$ to $\mathbb{H}$ and consider the function

$$F = f \circ \varphi \circ \psi \circ \sqrt{\cdot}\,.$$

As a composition of quasiconformal homeomorphisms, it is quasiconformal in $\mathbb{C} \setminus [0, \infty)$. The negative real line is mapped to $i\mathbb{R}_+$ under $\sqrt{\cdot}$, fixed by ψ, mapped to η under φ and finally mapped to $\gamma[1/2, 1]$ under f. Furthermore, F extends continuously across $\mathbb{R}_+$: Indeed, points $x^2 \in \mathbb{R}_+$ split up into the pair $-x, x$ under $\sqrt{\cdot}$, map to the pair $\psi(-x), \psi(x)$, which is unchanged under φ and mapped to a point $f(\psi(-x)) = f(\psi(x))$ on $\gamma[0, 1/2]$ under f. Thus F is a homeomorphism of the sphere that is quasiconformal in the complement of the real line, and thus quasiconformal on the whole sphere. $\square$

Notice that if $I^L(\gamma, \gamma(0)) = 0$, the above proof can be easily modified to prove that γ is a circle (1-quasicircle).

3.2.2 Root-invariance for regular loops

We first give a sufficient regularity condition for a loop to have finite energy. Essentially, it is a consequence of Theorem 3.5. In this subsection, $\beta > 1/2$ and γ is a $C^{1,\beta}$ simple loop.

Proposition 3.10. *The Loewner energy of γ rooted at $\gamma(0)$ is finite.*

Notice that the regularity does not depend on the choice of root.

Proof. We first prove that $I^A(\eta, \eta(0)) < \infty$ if $\eta : [0, 1] \to \hat{\mathbb{C}}$ is a $C^{1,\beta}$ simple arc. To this end, we extend η by attaching a small piece of straight segment tangentially at $\eta(0)$, denote the new arc $\eta[-1, 1]$, and note that it is again a $C^{1,\beta}$ arc. From the property of Loewner energy on regular chords, we know that

$$I^C_{\eta[-1,0]}(\eta[0, 1]) < \infty.$$

We have also

$$\begin{aligned}
&I^A(\eta[0,1],\eta(0)) \\
&= I^A(\eta[-1,1],\eta(0)) - I^C_{\eta[0,1]}(\eta[-1,0]) \\
&= I^A(\eta[-1,0],\eta(0)) + I^C_{\eta[-1,0]}(\eta[0,1]) - I^C_{\eta[0,1]}(\eta[-1,0]) \\
&\le 0 + I^C_{\eta[-1,0]}(\eta[0,1]) < \infty.
\end{aligned}$$

In particular, $I^A(\gamma[0,1/4],\gamma(0)) < \infty$.

Next, we show that $I^C_{\gamma[0,1/4]}(\gamma[1/4,1]) < \infty$ which then concludes the proof since

$$I^L(\gamma,\gamma(0)) = I^A(\gamma[0,1/4],\gamma(0)) + I^C_{\gamma[0,1/4]}(\gamma[1/4,1]).$$

Since we are now dealing with an infinite capacity chord, the mere regularity of the driving function is not sufficient to guarantee the finiteness of the energy. Instead, we apply Corollary 3.6 with a domain obtained from a carefully chosen modification of γ: From the first part,

$$\begin{aligned}
&I^C_{\gamma[0,1/4]}(\gamma[1/4,3/4]) \\
&= I^A(\gamma[0,3/4],\gamma(0)) - I^A(\gamma[0,1/4],\gamma(0)) < \infty.
\end{aligned}$$

Similarly $I^C_{\gamma[0,1/4]}(\gamma[-1/2,0]) < \infty$. Let $\tilde\gamma$ be the simple loop by completing $\gamma[-1/2,1/4]$ with the hyperbolic geodesic connecting $\gamma(-1/2)$ and $\gamma(1/4)$ in the complement of $\gamma[-1/2,1/4]$, such that $\tilde\gamma(x) = \gamma(x)$ for $x \in [-1/2,1/4]$ (see Figure 3.2). From the reversibility of the chordal Loewner energy,

$$\begin{aligned}
I^C_{\tilde\gamma[0,3/4]}(\tilde\gamma[3/4,1]) &= I^C_{\tilde\gamma[0,1/4]}(\tilde\gamma[1/4,1]) - I^C_{\tilde\gamma[0,1/4]}(\tilde\gamma[1/4,3/4]) \\
&\le I^C_{\tilde\gamma[0,1/4]}(\tilde\gamma[1/4,1]) \\
&= I^C_{\tilde\gamma[0,1/4]}(\tilde\gamma[1/2,1]) < \infty.
\end{aligned}$$

Since $\tilde\gamma$ differs from γ only on the part of the loop parametrized by $[1/4,1/2]$, the domain $\hat{\mathbb{C}} \setminus \tilde\gamma[0,3/4]$ coincides with $\hat{\mathbb{C}} \setminus \gamma[0,3/4]$ in a neighborhood of the two marked boundary points $\gamma(0)$ and $\gamma(3/4)$. We can apply Corollary 3.6 to show

$$I^C_{\tilde\gamma[0,3/4]}(\gamma[3/4,1]) - I^C_{\gamma[0,3/4]}(\gamma[3/4,1]) < \infty.$$

Since $\gamma[3/4,1]$ is at positive distance to $\gamma[1/4,1/2]$ and $\tilde\gamma[1/4,1/2]$, the Brownian loop measure term is finite, and the excursion kernel term is always finite. Hence

$$\begin{aligned} & I^C_{\gamma[0,1/4]}(\gamma[1/4,1]) \\ & = I^C_{\gamma[0,1/4]}(\gamma[1/4,3/4]) + I^C_{\gamma[0,3/4]}(\gamma[3/4,1]) < \infty, \end{aligned}$$

which concludes the proof. □

In particular, any loop formed by concatenating finitely many circular arcs has finite energy if and only if any two adjacent arcs have the same tangent at their common point: Indeed, it is easy to check that such a loop is $C^{1,1}$ and any corner with angle different from π has infinite energy (see Section 3.2.1).

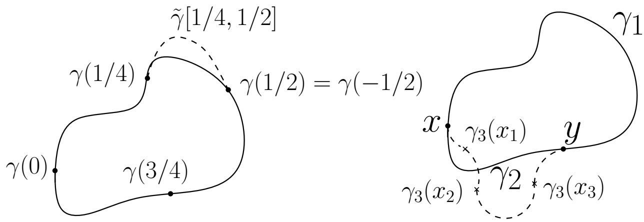

Figure 3.2: Illustrations of the surgeries made in the proof of Proposition 3.10 (left) and Propsition 3.11 (right). Left: $\tilde\gamma$ is the loop obtained from replacing $\gamma[1/4,1/2]$ by the hyperbolic geodesic in the complement of $\gamma[-1/2,1/4]$. Right: x and y separates the solid loop into γ_1 and γ_2, γ_3 is formed by concatenation of circular arcs and replaces γ_2 in the proof.

Proposition 3.11. *If $\beta > 1/2$, the Loewner energy of a $C^{1,\beta}$ loop γ is independent of the root.*

Proof. Two distinct points $x, y \in \gamma$ separate γ into two arcs which we denote by γ_1 and γ_2. The additivity gives

$$I^L(\gamma, x) = I^A(\gamma_1, x) + I^C_{\gamma_1}(\gamma_2)$$

and similarly

$$I^L(\gamma, y) = I^A(\gamma_1, y) + I^C_{\gamma_1}(\gamma_2).$$

Since $I^L(\gamma, x)$ and $I^L(\gamma, y)$ are finite, it suffices to prove the equality of the arc Loewner energy on the right hand side.

We complete γ_1 by another arc γ_3 to form a loop with continuous tangent (see Figure 3.2), where $\gamma_3[0,1]$ is a finite concatenation of circular arcs: there exists a sequence $0 = x_0 < x_1 < \cdots < x_n = 1$, such that $\gamma_3[x_i, x_{i+1}]$ is an circular arc for every i (we consider segments as circular arcs).

We give an explicit construction of γ_3: since γ_1 is a $C^{1,\beta}$ arc, we can first construct a simple, piecewise linear arc $\tilde{\gamma}_3$ that connects two tips of γ_1, being tangent to γ_1 at tips and contained in $\hat{\mathbb{C}} \setminus \gamma_1$. Then replace each corner of $\tilde{\gamma}_3$ by a (very) small circular arc smoothing out the corner without intersecting other parts of the loop.

Tangentially concatenated circular arcs form a $C^{1,1}$ arc therefore the new loop has finite energy by Proposition 3.10. The above energy decomposition tells us

$$\begin{aligned} I^A(\gamma_1, x) = I^A(\gamma_1, y) &\iff I^L(\gamma_1 \cup \gamma_3, x) = I^L(\gamma_1 \cup \gamma_3, y) \\ &\iff I^A(\gamma_3, x) = I^A(\gamma_3, y). \end{aligned}$$

We know that for every circular arc $\eta[0,1]$, the arc energy

$$I^A(\eta, \eta(s)) = 0, \quad \forall s \in [0,1].$$

It is in particular root-invariant. Hence, for $0 \le i \le n-1$,

$$\begin{aligned}
&I^A(\gamma_3[0,1], \gamma_3(x_i)) \\
&= I^A(\gamma_3[x_i, x_{i+1}], \gamma_3(x_i)) + I^C_{\gamma_3[x_i,x_{i+1}]}(\gamma_3[0,x_i] \cup \gamma_3[x_{i+1},1]) \\
&= I^A(\gamma_3[x_i, x_{i+1}], \gamma_3(x_{i+1})) + I^C_{\gamma_3[x_i,x_{i+1}]}(\gamma_3[0,x_i] \cup \gamma_3[x_{i+1},1]) \\
&= I^A(\gamma_3[0,1], \gamma_3(x_{i+1})).
\end{aligned}$$

Hence

$$I^A(\gamma_3, x) = I^A(\gamma_3, \gamma_3(0)) = I^A(\gamma_3, \gamma_3(1)) = I^A(\gamma_3, y),$$

which concludes the proof. □

3.2.3 Root-invariance for finite energy loops

We are now ready to prove the general root-invariance of the loop Loewner energy. We start with the lower-semicontinuity of the loop Loewner energy.

Lemma 3.12. *Let $(\gamma_n : [0,1] \to \hat{\mathbb{C}})_{n\ge 0}$ be a family of simple loops such that $\gamma_n(k/2) = \gamma_0(k/2)$ for $k = 0, 1$. If there exists a simple loop γ such that γ_n converges uniformly to γ, then*

$$\liminf_{n\to\infty} I^L(\gamma_n, \gamma_n(0)) \ge I^L(\gamma, \gamma(0)).$$

Proof. Without loss of generality, we assume that

$$\liminf_{n\to\infty} I^L(\gamma_n, \gamma_n(0)) = A < \infty,$$

and $\sup_{n\ge 0} I^L(\gamma_n, \gamma_n(0)) = B < \infty$.

For every $0 < \varepsilon < 1/4$, consider the family of uniformizing conformal maps $(\psi_n)_{n\ge 0}$, where ψ_n maps $\hat{\mathbb{C}} \setminus \gamma_n[0,\varepsilon]$ to $\mathbb{H}$, sending the two boundary points $\gamma_n(\varepsilon)$ and $\gamma_n(0)$ to 0 and ∞, respectively, and the interior point $\gamma_n(1/2) = \gamma(1/2)$ to a point of modulus 1. Let $\eta_n(s)$ denote the image in $\mathbb{H}$ of $\gamma_n(s)$ under ψ_n. The curve η_n is a chord in

$\mathbb{H}$ connecting 0 and ∞, parametrized by $[\varepsilon, 1]$. Similarly, we define ψ and η corresponding to γ.

By the definition of loop Loewner energy,

$$I^C_{\gamma_n[0,\varepsilon]}(\gamma_n[\varepsilon, 1]) = I^C_{\mathbb{H}}(\eta_n) \leq B,$$

so that all η_n are quasiconformal curves with a fixed constant K depending only on B.

By Carathéodory kernel theorem, ψ_n^{-1} converges uniformly on compacts of $\mathbb{H}$ to ψ^{-1}. In fact, since the γ_n are uniformly locally connected, the convergence of ψ_n^{-1} is uniform (with respect to the spherical metric), by [Pom92], Cor. II.2.4. It follows that η_n, viewed as $[\varepsilon, 1]$ parametrized curves, converge uniformly to η on every interval $[\varepsilon, r]$ with $r < 1$. Let λ_n be the capacity-parametrized driving function of η_n. We claim that λ_n converges uniformly on compacts to the driving function of η. To see this, notice that by [MR05] the λ_n are uniformly Hölder-1/2, with constant only depending on B. By Theorem 4.1 and Lemma 4.2 of [LMR10], every subsequential limit of λ_n is the driving function of a limit of η_n, and the only such limit is the capacity parametrization of η.

From the lower semicontinuity of the Dirichlet energy on driving functions we get

$$\liminf_{n\to\infty} I^C_{\mathbb{H}}(\eta_n) \geq I^C_{\mathbb{H}}(\eta) = I^C_{\gamma[0,\varepsilon]}(\gamma[\varepsilon, 1]),$$

which implies the claim

$$A \geq I^L(\gamma, \gamma(0))$$

by letting ε to 0, since

$$A = \liminf I^L(\gamma_n, \gamma_n(0)) \geq \liminf I^C_{\mathbb{H}}(\eta_n).$$

$\square$

Next, we will introduce the curves that we will use to approximate a given finite energy loop. They are minimizers of loop energy among all curves that pass through a given collection of points. In Section 3.4.3 below, we will discuss a generalization to the setting of isotopy classes of curves. Let $\overline{z} = (z_0, z_1, z_2, \cdots, z_n)$ be a finite collection of points in $\hat{\mathbb{C}}$, $\mathscr{L}(\overline{z})$ be the set of Jordan curves passing through $z_0, z_1, \cdots, z_n, z_0$ in order. We say that curves in $\mathscr{L}(\overline{z})$ are *compatible with* $\overline{z}$. Define

$$I^L(z_0, \{\overline{z}\}) := \inf_{\gamma \in \mathscr{L}(\overline{z})} I^L(\gamma, z_0).$$

From Lemma 2.12 we know that $I^L(z_0, \{\overline{z}\})$ is finite. In fact, one can easily construct a loop which is a small circular arc in a neighborhood of z_0, has finite chordal energy, and passes through the other points in order. We will now show that minimizers exist and are weakly $C^{1,1}$ from the regularity of its driving function. (Explicit computation of energy minimizers are also derived in [MRW+], one may obtain the precise regularity directly from explicit formula.)

Proposition 3.13. *There exists $\gamma \in \mathscr{L}(\overline{z})$ such that*

$$I^L(\gamma, z_0) = I^L(z_0, \{\overline{z}\}).$$

Moreover, every such energy minimizer γ is at least weakly $C^{1,1}$.

Proof. We first prove the existence. When $\overline{z}$ has no more than 3 points, a circle passing through all points is a minimizer of the energy. Now assume that $\overline{z}$ has more than 3 points. Let (γ_n) be a sequence of finite energy loops compatible with $\overline{z}$ whose energy rooted at z_0 converges to $I^L(z_0, \{\overline{z}\})$. Let A be the supremum of their energies. Then all γ_n are $K(A)$-quasicircles for some constant $K \geq 1$ due to Proposition 3.9. Let φ_n be a $K(A)$-quasiconformal map such that $\varphi_n(S^1) = \gamma_n$ and $\varphi_n(e^{2i\pi k/3}) = z_k$ for $k = 0, 1, 2$. We obtain a normal family of quasiconformal maps which converges uniformly on a subsequence to some φ. In particular, along this subsequence, γ_n converges uniformly to $\gamma = \varphi(S^1)$ viewed as a curve parametrized by

S^1. From Lemma 3.12, we have

$$I^L(z_0, \{\overline{z}\}) = \liminf_{n\to\infty} I^L(\gamma_n, z_0) \geq I^L(\gamma, z_0).$$

Since γ is compatible with $\overline{z}$, it is a minimizer in $\mathscr{L}(\overline{z})$.

To obtain the regularity of γ, notice that γ has the following remarkable property:

For $i \in \{0, 1, \cdots, n\}$, z_i and z_{i+1} split γ into two arcs $a_{i,1}$ and $a_{i,2}$, where $a_{i,1}$ does not contain other points than z_i and z_{i+1} (we set $z_{n+1} = z_0$). It is not hard to see that $a_{i,1}$ is the hyperbolic geodesic in the complement of $a_{i,2}$: Otherwise we could replace $a_{i,1}$ by the hyperbolic geodesic, since

$$I^L(\gamma, z_0) = I^A(a_{i,2}, z_0) + I^C_{a_{i,2}}(a_{i,1})$$

by Corollary 2.18. Thus $a_{i,1} \cup a_{i+1,1}$ is a *geodesic pair* in the simply connected domain $D = \hat{\mathbb{C}} \setminus (a_{i,2} \cap a_{i+1,2})$ between the two marked boundary points z_i and z_{i+2} and passing through z_{i+1}, namely $a_{i,1}$ is the hyperbolic geodesic in $D \setminus a_{i+1,1}$ between z_i and z_{i+1}, and $a_{i+1,1}$ is the hyperbolic geodesic in $D \setminus a_{i,1}$ between z_{i+1} and z_{i+2}. Such geodesic pairs have been characterized in [MRW+], and we know that either $a_{i,1} \cup a_{i+1,1}$ form a logarithmic spiral at z_{i+1}, or it is the energy minimizing chord in (D, z_i, z_{i+2}) passing through z_{i+1}. In Chapter 2, minimizers are identified and by explicit computation, it is not hard to see that their driving function is $C^{1,1/2}$ which implies weak $C^{1,1}$ trace. Only the latter case is possible for a minimizing loop γ with constraint $\overline{z}$, as the logarithmic spirals have infinite energy as can be seen by using their self-similarity. $\square$

To keep this paper self-contained, we outline a proof of the classification of geodesic pairs, and refer to [MRW+] for details: Assume that η_1 and η_2 are two curves in a simply connected domain D, forming a geodesic pair through a point $A \in D$. Let B be the boundary point of D on η_2. The pair separates D into two domains H_+ and H_-. Let R_i be the conformal reflection in η_i, which is well-defined in $D \setminus \eta_{i+1}$

$(i \in \mathbb{Z}_2)$. Define $F(z) = R_2 \circ R_1(z)$ in H_+, and note that F is a conformal automorphism of H_+ fixing the boundary point A. From the map F one can recover the welding functions of η_1 and of η_2 as follows: Let φ be a conformal map from $D \setminus \eta_1$ to $\hat{\mathbb{C}} \setminus \mathbb{R}_-$ such that $\varphi(A) = \infty$, $\varphi(B) = 0$. Assume without loss of generality that $\varphi(H_+) = \mathbb{H}$. From the geodesic property, $\varphi(\eta_2) = \mathbb{R}_+$. The map $g := \varphi \circ F \circ \varphi^{-1}|_{\mathbb{H}}$ defined on the upper half-plane is a Möbius map fixing ∞, hence

$$g(x) = ax + b, \quad \text{where } a, b \in \mathbb{R} \quad \text{and } a > 0.$$

Moreover, if $[-\infty, -t]$ is the image of $\eta_1 \subset \partial H$ under φ, it is not hard to see that $g|_{[-\infty,-t]}$ is the welding map of η_1. Indeed, denoting by φ_+ resp. φ_- the restrictions of φ to H_+ resp. H_-, we have

$$\varphi_+^{-1}(x) = \varphi_-^{-1} \circ g(x), \quad \forall x \in (-\infty, -t].$$

Since the welding determines the curve (up to conformal change of coordinates), it is then not hard to see that we have the following dichotomy:

1. $a = 1$ corresponds to the minimal energy curve in D passing through A. See Section 2.3.1 and the simulation by Brent Werness in Figure 3.3.
2. $a \neq 1$ corresponds to a geodesic pair with a logarithmic spiral at A.

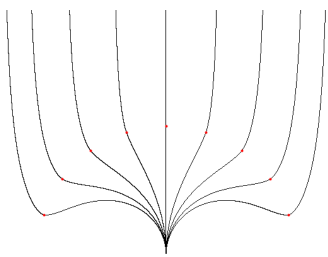

Figure 3.3: Finite energy geodesic pairs in $\mathbb{H}$ between 0 and ∞ passing through different points on the unit circle. Simulation by Brent Werness.

The following corollary is an immediate consequence of Propositions 3.11 and 3.13:

Corollary 3.14. *If γ minimizes the energy rooted at z_0 among all loops in $\mathscr{L}(\overline{z})$, then its energy is root-invariant. Therefore it also minimizes the energy rooted at z_k for $k \in \{1, \cdots, n\}$, and $I^L(z_k, \{\overline{z}\}) = I^L(z_0, \{\overline{z}\})$.*

Theorem 3.1 is then an immediate consequence of Corollary 3.14 and the following

Proposition 3.15. *Let γ be a Jordan curve. The energy of γ rooted at $\gamma(0)$ is the supremum of $I^L(z_0, \{\overline{z}\})$, where $\overline{z}$ is taken over all finite collections of points on γ which are compatible with γ and have $z_0 = \gamma(0)$.*

Proof. Let A denote the supremum of such $I^L(z_0, \{\overline{z}\})$. It is obvious that $A \leq I^L(\gamma, \gamma(0))$. Now we assume that $A < \infty$.

Let $(\overline{z^n})_{n\in\mathbb{N}}$ be a sequence of increasing $(n+3)$-tuples of points (i.e. a point in $\overline{z^n}$ is also in $\overline{z^{n+1}}$), such that the union of points in the sequence is a dense subset of γ, $\overline{z^0} = (\gamma(0), \gamma(1/3), \gamma(2/3))$, and the increasing sequence $I^L(z_0, \{\overline{z^n}\})$ converges to A.

Let γ_n be a minimizer of the energy (independent of the root due to Corollary 3.14) in $\mathscr{L}(\overline{z^n})$, all of them pass through $\gamma(0), \gamma(1/3)$ and $\gamma(2/3)$. Proposition 3.9 tells us that γ_n are all K-quasicircle, where K is independent of n. Let φ_n be a K-quasiconformal map of $\hat{\mathbb{C}}$ such that $\gamma_n = \varphi_n(S^1)$ as subsets of $\hat{\mathbb{C}}$. By pre-composing with a Möbius map, we assume that $\varphi_n(\exp(2i\pi k/3)) = \gamma(k/3)$ for all $n \geq 0$ and $k = 0, 1, 2$. Hence $(\varphi_n)_{n\geq 0}$ is a normal family (see e.g. [Leh12] Thm. 2.1), and a subsequence of φ_n converges uniformly to a K-quasiconformal map φ with respect to the spherical metric. The limiting curve γ passes through all points in $\overline{z^n}$ for every n. From the density of points in the union of $\overline{z^n}$, $\varphi(S^1) = \gamma$.

From Lemma 3.12, $I^L(\gamma, \gamma(0)) \leq \liminf_{n\to\infty} I^L(\gamma_n, \gamma(0)) = A$ which concludes the proof. □

3.3 Regularity correspondence

In this section we prove Theorem 3.5, which was an important tool in our proof of the root-invariance of the Loewner energy. It also is of independent interest, since it gives the optimal regularity of the driving function of an $C^{1,\beta}$ curve in most of the cases, see Section 3.4.1.

In Section 3.3.2 we study the regularity of the mapped-out curve, the main results are Corollary 3.20 (for $\beta \in (0, 1/2]$) and Corollary 3.21 (for $\beta \in (1/2, 1]$), which state that up to a Mobius transform in the latter case, the mapped-out curve has the same regularity as the initial curve. Therefore it suffices to study the displacement of the Loewner driving function for small times and we see the 1/2-shift in the regularity (Section 3.3.3). We complete the proof of Theorem 3.5 in Section 3.3.4.

3.3.1 Notations

Fix $n \in \mathbb{N}$ and $0 < \beta \leq 1$. A function $f : I \to \mathbb{R}$ is $C^{n,\beta}$ if there is $C > 0$ such that the modulus of continuity $\omega(\delta; f^{(n)})$ of $f^{(n)}$ on the interval I is bounded by $C\delta^\beta$ for $\delta \leq 1/2$, where

$$\omega(\delta; g) = \sup_{|s-s'|\leq\delta} |g(s) - g(s')|.$$

We denote $\|f\|_{n,\beta}$ the smallest such C. When $\beta = 0$, the class $C^{n,0}$ corresponds to continuous $f^{(n)}$.

A function f is said to be *weakly* $C^{n,\beta}$ if there is $C > 0$ such that for all $\delta \leq 1/2$,

$$\omega(\delta; f^{(n)}) \leq C\delta^\beta \log(1/\delta).$$

Sometimes we also write C^α when $\alpha > 1$, as in Theorem 3.3 above. This stands for $C^{n,\beta}$, where n is the largest integer less than or equal to α, and $\beta = \alpha - n$.

Throughout Section 3.3, γ is a $C^{1,\beta}$ *arclength-parametrized simple curve tangentially attached to* $\mathbb{R}_+$ for some $\beta \in (0, 1]$, that is an injective $C^{1,\beta}$ function $\gamma : [0, S] \to \mathbb{C} \setminus \mathbb{R}_+^*$, such that $\gamma(0) = 0$, $\gamma'(0) = -1$ and

$|\gamma'(s)| = 1$ for all $s \in [0, S]$. We abbreviate $\omega(\delta, \gamma')$ to $\omega(\delta)$.

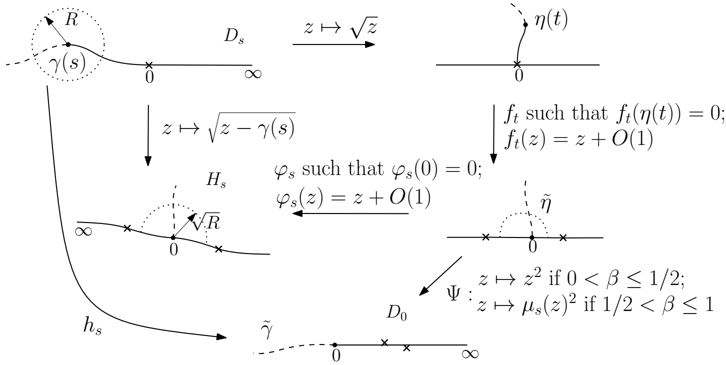

Figure 3.4: Illustration of different maps considered in Section 3.3. We define the map Ψ according to the value of β, and μ_s is the Möbius function defined in Corollary 3.21.

Maps and domains that we use frequently are illustrated in Figure 3.4, where

- D_s denotes the slitted sphere $\mathbb{C} \setminus (\gamma[0, s] \cup \mathbb{R}_+)$;
- H_s is the image of D_s under $z \to \sqrt{z - \gamma(s)}$;
- $z \mapsto \sqrt{z}$ maps $\gamma[0, S]$ to a slit η in the upper half plane $\mathbb{H}$;
- $t = t(s)$ is the half-plane capacity parametrization of η, that is

$$cap(\sqrt{\gamma[0,s]}) = cap(\eta[0, t(s)]) = 2t(s),$$

  where the mapping-out function g_t of $\eta[0, t(s)]$ satisfies

$$g_t(z) = z + 2t(s)/z + o(1/z);$$

- $(W_t)_{0 \leq t \leq T}$ is the Loewner driving function of η and $T = t(S)$;
- we also write $\gamma(-s) = s$ and $W_{-s} = 0$ for $s \geq 0$;
- $\Psi(z)$ is defined as z^2, if $\beta \leq 1/2$ and $\mu_s(z)^2$ if $\beta > 1/2$, where μ_s is a well-chosen Möbius map (Corollary 3.21);
- the sphere mapping-out function $h_s(z)$ is given by $\Psi \circ f_t(\sqrt{z})$;

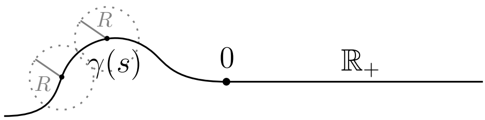

Figure 3.5: C^1 curve γ without bottle-necks $\leq R$.

- $\tilde{\gamma}$ is the image of $\gamma[s, S]$ by h_s;
- let $\varphi_s : \mathbb{H} \to H_s$ be the conformal map such that $\varphi_s(0) = 0$ and $\varphi_s(z) = z + O(1)$ as $z \to \infty$.

The existence and uniqueness of φ_s are discussed in Lemma 3.18. This map is closely related to the centered mapping-out function $f_t : \mathbb{H} \setminus \eta[0, t] \to \mathbb{H}$, that is

$$f_t(z) = g_t(z) - W_t = \varphi_s^{-1}\left(\sqrt{z^2 - \gamma(s)}\right), \tag{3.1}$$

where $t = t(s)$. Indeed, it suffices to check that $f_t(\eta(t)) = 0$, and $f_t(z) = z + O(1)$ as $z \to \infty$ which is straightforward.

Regarding the global geometry of γ, we assume that there exists $R > 0$ such that for all $s \in [0, S]$ and for all $r \leq R$, the intersection of the disc of radius r centered at $\gamma(s)$ with $\gamma(-\infty, S]$ is connected (Figure 3.5). Intuitively, this rules out bottle-necks of scale less than R. By taking perhaps a smaller R, we assume that $\omega(R) \leq 1/5$ and $R \leq 1/2$ (so that our bound for $\omega(\delta)$ applies for all $\delta \leq R$). Using the compactness of $\gamma[0, S]$, such R can always be found if γ is C^1, and we say that γ is *R-regular*.

3.3.2 Regularity of mapped-out curves.

The main goal of this section is to study the regularity of the image of γ under the function h_s. It is proven in Corollary 3.20 and Corollary 3.21 that, apart from a minor difference when the regularity is an integer, $\gamma \cup \mathbb{R}_+$ and $h_s(\gamma[s, S]) \cup \mathbb{R}_+$ are in the same class of regularity modulo a Möbius transform μ_s when $\beta > 1/2$. Notice that the only non-trivial part of the proof is the regularity of the new curve near the image 0 of the tip $\gamma(s)$.

One of our main tools is the Kellog-Warschawski theorem. Roughly speaking, it states that the conformal parametrization of a smooth Jordan curve (that is, the boundary extension of a conformal map of the disc onto the interior of the curve) has the same regularity as the arc-length parametrization of the curve, see for instance [Pom92] or [GM05]. We also need to keep track of the $C^{1,\beta}$-norm of the extension, and this norm depends not only on the local regularity of the curve, but also on a global property (roughly speaking, the absence of bottlenecks, which can be quantified for instance by the quasidisc-constant). To give a precise statement, define the chord-arc constant of a Jordan curve γ as

$$c_1(\gamma) = \sup_{z,w\in\gamma} \frac{\ell(\gamma(z,w))}{|z-w|},$$

where ℓ denotes length and $\gamma(z,w)$ is the subarc of γ from z to w (in case of a closed Jordan curve, $\gamma(z,w)$ is the shorter of the two arcs). Note that the chord-arc constant $c_1(\gamma(-\infty,S])$ is bounded in terms of R, S and $\|\gamma\|_{1,\beta}$: If $|z-w|$ is small and $\ell(\gamma(z,w))/|z-w|$ large, then $\gamma(z,w)\cap D_r(z)$ cannot be connected for suitable r.

The following quantitative version is a combination of results from [War32](“Zusatz 1 zum Satze 10”, inequality (10,16), p. 440, and “Zusatz zu Satz 11”, p. 451).

Theorem 3.16. *If f is a conformal map of the unit disc $\mathbb{D}$ onto the interior domain of a Jordan curve γ, if D, ℓ, c_1, K, ρ and $0 < \alpha < 1$ are such that* $\operatorname{diam}\gamma \le D, \ell(\gamma) \ge \ell$*, the chord-arc constant* $c_1(\gamma) \le c_1$, $\operatorname{dist}(f(0),\gamma) \ge \rho$*, and $\omega(\delta, \arg\gamma') \le K\delta^\alpha$ for its arc-length parametrization, then there are constants μ_1, μ_2 and L depending only on D, ℓ, c_1, K, ρ and α such that*

$$\mu_1 \le |f'(z)| \le \mu_2 \quad \textit{for all} \quad z \in \overline{\mathbb{D}}$$

and

$$|f'(z) - f'(w)| \le L|z-w|^\alpha \quad \textit{for all} \quad z,w \in \overline{\mathbb{D}}.$$

Let us explain the argument in this subsection. The sphere mapping-out function h_s is closely related to the conformal map φ_s, as $h_s(z) = \Psi \circ \varphi_s^{-1}(\sqrt{z - \gamma(s)})$. Lemma 3.17 studies the boundary regularity of H_s, then Lemma 3.18 applies Theorem 3.16 to H_s which allows us to compute the angular derivatives of φ_s at 0 in Proposition 3.19. Since the curve γ is contained in a cone at 0, knowing the angular derivatives is enough to compute the regularity of η which in turn gives us the regularity of $\tilde{\gamma}$ (Corollary 3.20 and Corollary 3.21).

We start with some trivial but useful estimates on γ. For every $s \in [0, S]$, $h > 0$,

$$\gamma(s+h) - \gamma(s) - h\gamma'(s) = \int_0^h \gamma'(s+r) - \gamma'(s)\, dr.$$

Since $|\gamma'(s+r) - \gamma'(s)| \leq \omega(r) \leq \omega(h)$ for $r \leq h$, we have

$$|\gamma(s+h) - \gamma(s) - h\gamma'(s)| \leq |h|\, \omega(|h|). \tag{3.2}$$

In particular, if $0 \leq h \leq R$, then

$$|\gamma(s+h) - \gamma(s)| \geq h - h\omega(h) \geq 4h/5. \tag{3.3}$$

Lemma 3.17. *Let γ be a $C^{1,\beta}$ curve tangentially attached to $\mathbb{R}_+$ of total length S, R-regular. For $s \in (0, S]$, the boundary Γ of H_s, parametrized by arclength, is a $C^{1,\beta}$ curve whose norm is bounded independently of s. Furthermore, there exists a constant $C > 0$, depending only on R, S and $\|\gamma\|_{1,\beta}$, such that*

$$|\arg(\Gamma'(l)) - \arg(\Gamma'(0))| \leq C(l^{2\beta} \wedge 1) \quad \text{for all} \;\; l \in \mathbb{R},$$

where $\Gamma(0) = 0$.

Proof. Define

$$\tilde{\Gamma}(r) = \sqrt{\gamma(s - r^2) - \gamma(s)} \quad \text{for } r \geq 0$$

and set $\tilde{\Gamma}(r) = -\tilde{\Gamma}(-r)$ for $r < 0$. Since γ has finite chord-arc constant,

$|\gamma(s-r^2)-\gamma(s)|$ is comparable to r^2, and consequently

$$\tilde{\Gamma}'(r) = \frac{-r\gamma'(s-r^2)}{\sqrt{\gamma(s-r^2)-\gamma(s)}}$$

is bounded above and away from zero. Since Γ is the arc-length parametrization of $\tilde{\Gamma}$, it easily follows that the modulus of continuity of Γ is bounded in terms of the modulus of continuity of $\tilde{\Gamma}$, $\omega_\Gamma(r) \le C\omega_{\tilde{\Gamma}}(Cr)$. Hence it suffices to prove the claims of the proposition for $\tilde{\Gamma}$ instead of Γ.

If $\varepsilon > 0$ and $r > 0$,

$$\begin{aligned}
&\left|\tilde{\Gamma}'(r+\varepsilon) - \tilde{\Gamma}'(r)\right| \\
&= \left| \frac{-(r+\varepsilon)\gamma'(s-(r+\varepsilon)^2)}{\sqrt{\gamma(s-(r+\varepsilon)^2)-\gamma(s)}} - \frac{-r\gamma'(s-r^2)}{\sqrt{\gamma(s-r^2)-\gamma(s)}} \right| \\
&\le \left| \frac{-(r+\varepsilon)[\gamma'(s-(r+\varepsilon)^2)-\gamma'(s-r^2)]}{\sqrt{\gamma(s-(r+\varepsilon)^2)-\gamma(s)}} \right| \\
&\quad + \left| \frac{-(r+\varepsilon)}{\sqrt{\gamma(s-(r+\varepsilon)^2)-\gamma(s)}} + \frac{r}{\sqrt{\gamma(s-r^2)-\gamma(s)}} \right| \\
&\le C\omega(2r\varepsilon+\varepsilon^2) + |f(r+\varepsilon)-f(r)|,
\end{aligned}$$

where $f(r) = r/\sqrt{\gamma(s-r^2)-\gamma(s)}$. By (3.2) and the aforementioned comparability of $|\gamma(s-r^2)-\gamma(s)|$ and r^2, we obtain

$$\begin{aligned}
|f'(r)| &= \left| \frac{\gamma(s-r^2)-\gamma(s)+r^2\gamma'(s-r^2)}{(\gamma(s-r^2)-\gamma(s))^{3/2}} \right| \\
&\le \frac{r^2\omega(r^2)}{(Cr^2)^{3/2}} = C_1\omega(r^2)/r.
\end{aligned}$$

Since γ is a $C^{1,\beta}$ curve and for all $\delta \le 1/2$, $\omega(\delta) \le \|\gamma\|_{1,\beta}\delta^\beta$ so that

$$|f'(r)| \le C_1\|\gamma\|_{1,\beta} r^{2\beta-1} \quad \text{for } r \le 1/2.$$

It follows that

$$|f(r+\varepsilon)-f(r)| \le C_2 \left|(r+\varepsilon)^{2\beta}-r^{2\beta}\right| \le C_3\varepsilon^{2\beta\wedge 1}.$$

Letting $r \to 0$ we obtain

$$\left|\tilde{\Gamma}'(\varepsilon)-\tilde{\Gamma}'(0)\right| \le C\omega(\varepsilon^2)+|f(\varepsilon)-f(0)| \le (C\|\gamma\|_{1,\beta}+C_2)\varepsilon^{2\beta},$$

while for $r < 2S$ and $\varepsilon < 1/2$ we get

$$\left|\tilde{\Gamma}'(r+\varepsilon)-\tilde{\Gamma}'(r)\right| \le C_4\varepsilon^{\beta}.$$

Direct computation shows that for $r > 2S$ we have

$$\left|\tilde{\Gamma}'(r+\varepsilon)-\tilde{\Gamma}'(r)\right| \le C_5\varepsilon,$$

and we deduce that Γ is a $C^{1,\beta}$ curve. □

Lemma 3.18. *There exists a unique conformal map $\varphi_s : \mathbb{H} \to H_s$ such that $\varphi_s(0) = 0$ and $\varphi_s(z) = z(1+o(1))$ as $z \to \infty$. Moreover, φ_s extends by continuity to a $C^{1,\beta}$ map $\overline{\mathbb{H}} \to \overline{H_s}$, and*

$$\frac{1}{C} \le |\varphi_s'(r)| \le C$$

for all $r \in \mathbb{R}$ and some constant C depending only on R, S and $\|\gamma\|_{1,\beta}$.

Proof. The points $z_0 := 3i\sqrt{S} \in H_s$ and $-z_0$ have distance at least $\sqrt{S}$ from the boundary Γ of H_s. The Möbius transformation $T_1(z) = (z-z_0)/(z+z_0)$ maps Γ to a (closed) Jordan curve $\sigma = T_1(\Gamma)$. We will first show that σ satisfies the assumptions of Theorem 3.16, with constants depending only on R, S and $\|\gamma\|_{1,\beta}$. Since σ is contained in the image under T_1 of the circle of radius $\sqrt{S}$ centered at $-z_0$, a simple calculation shows that the diameter of σ is bounded above by 12. Similarly, the distance $\mathrm{dist}(0,\sigma)$ is bounded below by the inradius $1/5$ of the image of the circle of radius $\sqrt{S}$ centered at z_0. The length of σ is bounded below since $T_1(\infty) = 1$ and $T_1(0) = -1$ are in σ. We already noted that the chord-arc constant $c_1(\gamma)$ is bounded in terms of R, S and $\|\gamma\|_{1,\beta}$. It is an exercise to show that the image under the square-root map of a chord-arc curve from 0 to ∞ is chord-arc with comparable constant, so that $c_1(\Gamma)$ is uniformly bounded. It easily follows that $c_1(\sigma)$ is bounded as well. Finally, from Lemma 3.17 we know that the regularity of σ is $C^{1,\beta}$ away from $T_1(\infty) = 1$. But from a straightforward computation, we see that σ is also at least $C^{1,\beta}$ near 1. Thus $T_1(H_s)$ is bounded by a $C^{1,\beta}$ Jordan curve.

Consider the conformal map $f : \mathbb{D} \to T_1(H_s)$ that is normalized by $f(0) = 0$ and $f(1) = 1$, and denote $p = f^{-1}(-1)$. By Theorem 3.16, the derivative of f is bounded above, so that $|p-1|$ is bounded away from zero. Denote $T_2 : \mathbb{H} \to \mathbb{D}$ the Möbius transformation that sends ∞ to 1, 0 to p, and is furthermore normalized by $T_2(z) = 1 + c/z + O(1/z^2)$ where $|c| = 2|z_0/f'(1)|$. Then either $\varphi_s = T_1^{-1} \circ f \circ T_2$ or $-\varphi_s$ is the conformal map from $\mathbb{H}$ to H_s with the desired normalization, and the regularity claims about φ_s follow from Theorem 3.16. $\square$

Now we are ready to compute the angular derivatives of φ_s at 0. It is not surprising that the highest order that we need to consider is related to the value of β. Heuristically, since the boundary Γ of the domain behaves like a $C^{1+2\beta}$ curve at 0 thanks to Lemma 3.17, one expects that φ_s has angular derivatives up to the order $1 + 2\beta$. The precise statement is the following:

Proposition 3.19. *There exist $L_s > 0$ and $C_1(\beta, R, S, \|\gamma\|_{1,\beta})$, such that for all $0 \le |x| \le y \le 1/2$,*

$$|\varphi_s'(x+iy) - \varphi_s'(0)| \le C_1 y^{2\beta}, \qquad \text{if } 0 < \beta < 1/2, \tag{3.4}$$

$$|\varphi_s'(x+iy) - \varphi_s'(0)| \le C_1 y \log(1/y), \qquad \text{if } \beta = 1/2, \tag{3.5}$$

$$\left| \frac{\varphi_s''(x+iy)}{\varphi_s'(x+iy)} - L_s \right| \le C_1 y^{2\beta-1}, \qquad \text{if } 1/2 < \beta < 1, \tag{3.6}$$

$$\left| \frac{\varphi_s''(x+iy)}{\varphi_s'(x+iy)} - L_s \right| \le C_1 y \log(1/y), \qquad \text{if } \beta = 1, \tag{3.7}$$

where φ_s is defined in Lemma 3.18. Moreover, let

$$v(r) := \operatorname{Im} \log(\varphi_s'(r)), \quad \forall r \in \mathbb{R},$$

then we have the explicit expression

$$L_s = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(r) - v(0)}{r^2} dr. \tag{3.8}$$

Proof. We denote the harmonic extension of v to $\overline{\mathbb{H}}$ also by v. More precisely, for $x \in \mathbb{R}$ and $y > 0$,

$$v(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(r-x)^2 + y^2} v(r) dr.$$

We have $v = \operatorname{Im} \log \varphi_s'$: Indeed, if u is a harmonic conjugate of v on $\mathbb{H}$, then $\phi(z) := \log \varphi_s'(z) - (u(z) + iv(z))$ is holomorphic in $\mathbb{H}$, with $\operatorname{Im}(\phi(r)) = 0$ if $r \in \mathbb{R}$. By Schwarz reflection, ϕ extends to an entire function with real coefficients. Since both $\operatorname{Im} \log \varphi_s'$ and v are bounded in $\mathbb{H}$ (the boundedness of $\operatorname{Im} \log \varphi_s'$ near ∞ easily follows from the smoothness of $\sigma = T_1(\Gamma)$ established in the proof of Lemma 3.18), the imaginary part of ϕ is bounded so that ϕ is a real constant which we may assume to be zero by adjusting u. Consequently, u and v are the real and imaginary part of $\log \varphi'$.

Since $\varphi_s'(r)$ is bounded away from 0 and ∞, by Lemma 3.17 and Lemma 3.18, there exists C depending on S, R and $\|\gamma\|_{1,\beta}$, such that

$$|w(r)| \le C(r^{2\beta} \wedge 1),$$

where $w(r) := v(r) - v(0)$. We also have

$$\begin{aligned}
\partial_x u(x+iy) &= \partial_y v(x+iy) \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(r-x)^2 - y^2}{[(r-x)^2 + y^2]^2} v(r) dr \\
&= \frac{1}{\pi y} \int_{-\infty}^{\infty} \frac{t^2 - 1}{(t^2+1)^2} w(ty + x) dt,
\end{aligned}$$

and

$$
\begin{aligned}
-\partial_y u(x+iy) &= \partial_x v(x+iy) \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2y(r-x)}{[(r-x)^2+y^2]^2} v(r)dr \\
&= \frac{1}{\pi y}\int_{-\infty}^{\infty} \frac{2t}{(t^2+1)^2} w(ty+x)dt.
\end{aligned}
$$

For $\beta < 1/2$, we use the bound of $w(r)$ in the above expressions and obtain for (x,y) with $0 \le x \le y \le 1/2$,

$$
\begin{aligned}
&|\partial_x u(x+iy)| \\
&\le \frac{2C}{\pi y}\left|\int_0^{\infty} \frac{|t^2-1|}{(t^2+1)^2}\left(t+\frac{x}{y}\right)^{2\beta} y^{2\beta}dt + \int_{(1-x)/y}^{\infty} \frac{|t^2-1|}{(t^2+1)^2}dt\right| \\
&\le \frac{2C}{\pi y}\left|\int_0^{\infty} \frac{1}{t^2+1}(t+x/y)^{2\beta} y^{2\beta}dt + \int_{(1-x)/y}^{\infty} \frac{1}{t^2+1}dt\right| \\
&\le y^{2\beta-1}\frac{2C}{\pi}\left|\int_0^{\infty} \frac{1}{t^2+1}(t+1)^{2\beta}dt\right| + \frac{2C}{\pi y}\left|\arctan\left(\frac{y}{1-x}\right)\right| \\
&\le C_2 y^{2\beta-1} + \frac{2C}{\pi(1-x)} \\
&\le C_2 y^{2\beta-1} + C',
\end{aligned}
$$

where $C_2 = 2C\int_0^{\infty}(t+1)^{2\beta}/(t^2+1)dt/\pi$ and $C' = 8C/\pi$. Similarly,

$$
\begin{aligned}
&|\partial_y u(x+iy)| \\
&\le \frac{2C}{\pi y}\left|y^{2\beta}\int_0^{\infty} \frac{2t}{(t^2+1)^2}(t+x/y)^{2\beta}dt + \int_{(1-x)/y}^{\infty} \frac{2t}{(t^2+1)^2}dt\right| \\
&\le C_3 y^{2\beta-1} + \frac{2C}{\pi y}\frac{y^2}{(1-x)^2+y^2} \\
&\le C_3 y^{2\beta-1} + C'y,
\end{aligned}
$$

where $C_3 = 2C \int_0^\infty t(t+1)^{2\beta}/(t^2+1)^2 dt/\pi$. Consequently,

$$
\begin{aligned}
&|u(x+iy) - u(0)| \\
&\leq \left| \int_0^y \partial_r u(ir) dr \right| + \left| \int_0^x \partial_r u(r+iy) dr \right| \\
&\leq C_3 \int_0^y r^{2\beta-1} dr + C' y^2 + x \left[C_2 y^{2\beta-1} + C' \right] \leq C_1 y^{2\beta},
\end{aligned}
$$

where C_1 does not depend on s. Similarly, for the imaginary part,

$$
\begin{aligned}
&|v(x+iy) - v(0)| \\
&= \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(r-x)^2 + y^2} (v(r) - v(0)) dr \right| \\
&= \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2+1} w(ty+x) dt \right| \\
&\leq y^{2\beta} \left| \frac{2C}{\pi} \int_0^\infty \frac{1}{t^2+1} (t+1)^{2\beta} dt \right| + \frac{2C}{\pi} \left| \arctan\left(\frac{y}{1-x} \right) \right| \\
&\leq C_2 y^{2\beta} + C' y \leq C_1 y^{2\beta}.
\end{aligned}
$$

In the case $\beta = 1/2$, we need to estimate more carefully, since some of the above integrals diverge. Again, for $0 \le x \le y \le 1/2$,

$$
\begin{aligned}
&|\partial_x u(x+iy)| \\
&\le \frac{1}{\pi y}\int_{-\infty}^{\infty} \frac{|t^2-1|}{(t^2+1)^2}\,|w(ty+x)|\,dt \\
&\le \frac{C}{\pi y}\left(\int_{I(y)} \frac{|t^2-1|}{(t^2+1)^2}|ty+x|dt + \int_{\mathbb{R}\setminus I(y)} \frac{|t^2-1|}{(t^2+1)^2}dt\right) \\
&\le \frac{2C}{\pi y}\left(\int_0^{(x+1)/y} y\frac{t+1}{t^2+1}dt + \int_{(1-x)/y}^{\infty} \frac{1}{t^2+1}dt\right) \\
&= \frac{C}{\pi}\left[\log(t^2+1)+2\arctan(t)\right]_0^{(x+1)/y} + \frac{C}{\pi y}\arctan\left(\frac{y}{1-x}\right) \\
&\le \frac{2C}{\pi}\log\left(\frac{1}{y}\right) + \frac{C}{\pi}\log\left(\frac{5}{2}\right) + C + C' \le C''\log(1/y),
\end{aligned}
$$

where $I(x,y) = [-(x+1)/y, (1-x)/y]$. For $\partial_y u(x+iy)$, the same bound obtained for $\beta < 1/2$ also holds for $\beta < 1$, namely

$$|\partial_y u(x+iy)| \le C_2 y^{2\beta-1} + C'.$$

Hence there exists C_1 such that for $0 \le x \le y \le 1/2$,

$$|u(x+iy) - u(0)| \le C_1 y \log(1/y).$$

A similar calculation also holds for v, i.e.

$$|v(x+iy) - v(0)| \le C_1 y \log(1/y).$$

For $1/2 < \beta < 1$, $0 \le x \le y \le 1/2$, we have already seen in the above computation that

$$|\partial_x v(x+iy)| = |\partial_y u(x+iy)| \le C_3 y^{2\beta-1} + C' y \le C_4 y^{2\beta-1}.$$

We define

$$L_s = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{w(r)}{r^2}dr \tag{3.9}$$

and obtain

$$\begin{aligned}\partial_x u(x+iy) - L_s &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{(r-x)^2-y^2}{[(r-x)^2+y^2]^2} - \frac{1}{r^2}\right] w(r)dr \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{P(r)}{[(r-x)^2+y^2]^2 r^2}\right] w(r)dr,\end{aligned}$$

where P is a polynomial of degree 3 with coefficients in $\mathbb{R}[x,y]$. After the change of variable $r = ty + x$, and set $\xi = x/y$, we get

$$\begin{aligned}&|\partial_x u(x+iy) - L_s| \\ \leq & y^{2\beta-1}\frac{C}{\pi}\int_{-\infty}^{\infty}\left|\frac{\tilde{P}(t,\xi)}{\tilde{Q}(t,\xi)}\right|(|t|+1)^{2\beta}dt + \frac{C}{\pi y}\int_{\mathbb{R}\setminus I(x,y)}\left|\frac{\tilde{P}(t,\xi)}{\tilde{Q}(t,\xi)}\right|dt \\ \leq & y^{2\beta-1}\frac{C}{\pi}\int_{-\infty}^{\infty}\left|\frac{\tilde{P}(t,\xi)}{\tilde{Q}(t,\xi)}\right|(|t|+1)^{2\beta}dt + \frac{C_5}{y}\int_{(1-x)/y}^{\infty}\frac{dt}{t^3} \\ = & y^{2\beta-1}\frac{C}{\pi}\int_{-\infty}^{\infty}\left|\frac{\tilde{P}(t,\xi)}{\tilde{Q}(t,\xi)}\right|(|t|+1)^{2\beta}dt + C_6 y,\end{aligned}$$

where C_5 and C_6 are universal constants. Both $\tilde{P}$ and $\tilde{Q}$ have degree 6 in the second variable, and degree 3 and 6 respectively in the first variable. Since $\xi \in [-1,1]$, and $\tilde{P}(t,\xi)/\tilde{Q}(t,\xi)(|t|+1)^{2\beta}$ can be uniformly bounded by an integrable function ($\propto (1+t)^{2\beta-3}$), we know that there exists $C_1 = C_1(\beta, S, R, \|\gamma\|_{1,\beta}) > 0$ such that

$$|\partial_x u(x+iy) - L_s| \leq C_1 y^{2\beta-1},$$

and similarly

$$|\partial_x v(x+iy)| \leq C_1 y^{2\beta-1}.$$

In terms of φ_s,

$$\frac{\varphi_s''}{\varphi_s'}(x+iy) = \log(\varphi_s')'(x+iy) = \partial_x u + i\partial_x v.$$

We have thus obtained the bound (3.6).

The case where $\beta = 1$ is similar to the case $\beta = 1/2$. Integration of

dt/t on the interval $I(x,y) = [-(x+1)/y, (1-x)/y]$ gives the $\log(1/y)$ term. □

We define $\nabla := \{z = x + iy \in \mathbb{H},\, y \le 1/2 \text{ and } |x| \le y\}$. Let γ be a $C^{1,\beta}$ curve. From the above proposition, it is easy to see that there exists $R_0 > 0$ such that for all $s \in [0,S]$, $\sqrt{\gamma(s+r) - \gamma(s)} \in \varphi_s(\nabla)$ for all $r \in [0, R_0]$ where the map φ_s is as defined in Lemma 3.18.

Corollary 3.20. *If $0 < \beta \le 1/2$, the image $\tilde{\gamma}$ of $\gamma[s,S]$ under the conformal map*

$$h_s(z) = \left[\varphi_s^{-1}\left(\sqrt{z - \gamma(s)}\right)\right]^2, \quad D_s \to D_0$$

is also a $C^{1,\beta}$ curve (weak $C^{1,\beta}$ curve if $\beta = 1/2$). More precisely, its behavior near 0 under arclength parametrization is

$$\begin{aligned} |\tilde{\gamma}'(r) + 1| &\le C_2 r^{\beta} && \text{if } 0 < \beta < 1/2, \\ |\tilde{\gamma}'(r) + 1| &\le C_2 r^{\beta}\log(1/r) && \text{if } \beta = 1/2, \end{aligned}$$

or all $r \le R_0$, where C_2 is independent of s.

Proof. It is obvious that the image of $\gamma[s+\varepsilon, S]$ under h_s is a $C^{1,\beta}$ curve. We only need to check that the limit of $\partial_r h_s(\gamma(s+r))$ as $r \to 0$ is in $\mathbb{R}_-$, with convergence rate r^β if $\beta < 1/2$ and $r^{1/2}\log(1/r)$ if $\beta = 1/2$.

We use the same notation

$$\log \varphi_s'(z) = u(z) + iv(z)$$

as before. Set $\psi := \varphi_s^{-1}$, we have $\psi'(z) = 1/\varphi_s'(\psi(z))$. Thus

$$\psi'(0)^2\gamma'(s) = \varphi_s'(0)^{-2}\gamma'(s) = -\exp(-2u(0)) < 0.$$

For $0 < \beta < 1/2$ and $z \in \varphi_s(\nabla)$, from (3.4) and the boundedness of

$|\varphi_s'|$ we have

$$\left|\psi'(z)-\psi'(0)\right| = \left|\frac{\varphi_s'(0)-\varphi_s'(\psi(z))}{\varphi_s'(\psi(z))\varphi_s'(0)}\right| \le C_1 \left|\psi(z)\right|^{2\beta} \le \tilde{C}_2 \left|z\right|^{2\beta}$$

hence

$$\left|\psi(z)-z\psi'(0)\right| \le \tilde{C}_2 \left|z\right|^{1+2\beta}.$$

We know that

$$\left|\gamma'(s+r)-\gamma'(s)\right| \le \|\gamma\|_{1,\beta} \left|r\right|^{\beta}$$

and

$$\left|\gamma(s+r)-\gamma(s)-r\gamma'(s)\right| \le \|\gamma\|_{1,\beta} \left|r\right|^{1+\beta}.$$

By the definition of R_0, we have $\Gamma_r := \sqrt{\gamma(s+r)-\gamma(s)} \in \varphi_s(\nabla)$ for all $r \le R_0$ with $s+r \le S$. For such r, the estimate (3.4) yields

$$\begin{aligned}
&\left|\partial_r(h_s(\gamma(s+r))) - \psi'(0)^2\gamma'(s)\right| \\
=&\left|\psi\left(\Gamma_r\right)\psi'\left(\Gamma_r\right)\gamma'(s+r)/\Gamma_r - \psi'(0)^2\gamma'(s)\right| \\
\le&\left|\psi(\Gamma_r)-\psi'(0)\Gamma_r\right|\left|\psi'(\Gamma_r)\right|/\Gamma_r + \left|\psi'(\Gamma_r)-\psi'(0)\right|\left|\psi'(0)\right| \\
&+\left|\gamma'(s+r)-\gamma'(s)\right|\left|\psi'(0)\right|^2 \\
\le&\tilde{C}_2\left(\left|\psi'(\Gamma_r)\right|\Gamma_r{}^{2\beta+1}/\Gamma_r + \left|\psi'(0)\right|\left|\Gamma_r\right|^{2\beta} + \left|\psi'(0)\right|^2 r^{\beta}\right) \\
\le&C_2\left|r\right|^{\beta},
\end{aligned}$$

since ψ' is uniformly bounded. In particular,

$$\partial_r(h_s(\gamma(s+r)))|_{r=0} = -\left|\psi'(0)\right|^2$$

and $r \mapsto h_s(\gamma(s+r))$ is a $C^{1,\beta}$ function. It is easy to see that $\partial_r(h_s(\gamma(s+r)))$ is bounded away from 0 and ∞, the above estimate suffices to conclude that $\tilde{\gamma} = h_s(\gamma[s,S])$ is also $C^{1,\beta}$ when parametrized by arclength.

In the case $\beta = 1/2$, the argument for the behavior at $h_s(\gamma(s))$ is the same by using the bounds

$$\left|\psi'(z)-\psi'(0)\right| \le C_1 \left|z\right| \log(1/\left|z\right|)$$

and

$$|\psi(z) - \psi'(0)z| \leq C_1 |z|^2 \log(1/|z|)$$

in the above computation of $\left|\partial_r(h_s(\gamma(s+r))) - \psi'(0)^2\gamma'(s)\right|$. The latter of the two inequalities is obtained from an integration. □

We now turn to the case $1/2 < \beta \leq 1$. Let μ_s be the Möbius transform $\mathbb{H} \to \mathbb{H}$ with $\mu_s(0) = 0$, $\mu_s'(0) = 1$ and $\mu_s''(0) = L_s$.

Corollary 3.21. *The angular limit as $z \to 0$ of $[\mu_s \circ \varphi_s^{-1}]''/[\mu_s \circ \varphi_s^{-1}]'(z)$ is 0, with the same rate of convergence as in Proposition 3.19. The image $\tilde{\gamma}$ of $\gamma[s, (s+R_0) \wedge S]$ under the conformal map*

$$h_s(z) = \left[\mu_s \circ \varphi_s^{-1}\left(\sqrt{z - \gamma(s)}\right)\right]^2$$

satisfies

$$\omega(\delta; \tilde{\gamma}') \leq \begin{cases} C_2\delta^\beta & \text{if } 1/2 < \beta < 1 \\ C_2\delta^\beta \log(1/\delta) & \text{if } \beta = 1, \end{cases}$$

where R_0 and C_2 depend only on β, R, S and $\|\gamma\|_{1,\beta}$ (in particular do not depend on s).

Proof. We first check that $[\mu_s \circ \varphi_s^{-1}]''/[\mu_s \circ \varphi_s^{-1}]'(z)$ has angular limit 0. Again denoting $\psi = \varphi_s^{-1}$, we have

$$0 = [\varphi_s \circ \psi]''/[\varphi_s \circ \psi]'(z) = [\varphi_s''/\varphi_s'(\psi(z))]\psi'(z) + \psi''/\psi'(z).$$

For $1/2 < \beta < 1$ and $z \in \varphi_s(\nabla)$,

$$\begin{aligned} &\left|[\mu_s \circ \varphi_s^{-1}]''/[\mu_s \circ \varphi_s^{-1}]'(z)\right| \\ =&|\mu_s''/\mu_s'(\psi(z))\psi'(z) + \psi''/\psi'(z)| \\ =&|[L_s + R_1(z)]\,\psi'(z) - \varphi_s''/\varphi_s'(\psi(z))\psi'(z)| \\ \leq& C' |z|^{2\beta-1}, \end{aligned}$$

where $C' > 0$ does not depend on s. It yields the angular limit 0 with convergence rate as in Proposition 3.19.

The analysis of the behavior of $|\tilde{\gamma}'(r) - \tilde{\gamma}'(0)|$ near 0 is the same as in Corollary 3.20. But unlike Corollary 3.20, we need to bound in addition the modulus of continuity of $\tilde{\gamma}'$ on a small neighborhood of 0. To this end, we first estimate the Lipschitz constant of $\phi(z)/z$ where $\phi(z) = \mu_s \circ \varphi_s^{-1}$.

Since $\phi'(z), z \in \varphi_s(\nabla)$ is bounded by a constant independently of s, we have

$$|\phi''(z)| \leq C'' \, |z|^{2\beta-1} \, .$$

Hence for $z, h \in \mathbb{C}$ such that the segment $[z, z+h] \subset \varphi_s(\nabla)$,

$$\begin{aligned} |\phi'(z+h) - \phi'(z)| &\leq C'' \int_0^{|h|} (|z| + u)^{2\beta-1} \, \mathrm{d}u \\ &\leq C''' \, |h| \, (|z| + |h|)^{2\beta-1}. \end{aligned}$$

For $z_1, z_2 \in \varphi_s(\nabla)$ such that $[tz_1, tz_2] \subset \varphi_s(\nabla)$ for all $t \in [0,1]$,

$$\begin{aligned} \left| \frac{\phi(z_1)}{z_1} - \frac{\phi(z_2)}{z_2} \right| \leq & \int_0^1 |\phi'(tz_1) - \phi'(tz_2)| \, \mathrm{d}t \\ \leq & C''' \int_0^1 t^{2\beta} \, |z_1 - z_2| \, (|z| + |z_1 - z_2|)^{2\beta-1} \, \mathrm{d}t \\ \leq & C''' \, |z_1 - z_2| \, (|z| + |z_1 - z_2|)^{2\beta-1}. \end{aligned}$$

Now the analysis of $\tilde{\gamma}'$ is straightforward: write

$$\Gamma_r := \sqrt{\gamma(s+r) - \gamma(s)},$$

then

$$\partial_r h_s(\gamma(s+r)) = \phi(\Gamma_r)\phi'(\Gamma_r)\gamma'(s+r)/\Gamma_r.$$

If $0 < r' < r < R$,

$$|\Gamma_r - \Gamma_{r'}| = |(\gamma(s+r) - \gamma(s+r'))/(\Gamma_r + \Gamma_{r'})| \leq c \, |r - r'| \, / \sqrt{r},$$

since $\Gamma_r \geq \sqrt{4r/5}$ (see (3.3)). Now we choose furthermore $0 < R_0 \leq R$

such that for all s, the convex hull of $\{\Gamma_r; r \leq R_0\}$ is contained in $\varphi_s(\nabla)$. Thus for every $r, r' \leq R_0$, $t \in [0,1]$, the segment $[t\Gamma_r, t\Gamma_{r'}]$ is in $\varphi_s(\nabla)$. Hence

$$\begin{aligned}
&|\partial_r h_s(\gamma(s+r)) - \partial_r h_s(\gamma(s+r'))| \\
\leq& |\phi(\Gamma_r)\phi'(\Gamma_r)\gamma'(s+r)/\Gamma_r - \phi(\Gamma_{r'})\phi'(\Gamma_{r'})\gamma'(s+r')/\Gamma_{r'}| \\
\leq& C\big(|\phi(\Gamma_r)/\Gamma_r - \phi(\Gamma_{r'})/\Gamma_{r'}| + |\phi'(\Gamma_r) - \phi'(\Gamma_{r'})| \\
&+ |\gamma'(s+r) - \gamma'(s+r')| \big) \\
\leq& C_3 \left[|\Gamma_r - \Gamma_{r'}| \left(|\Gamma_r| + |\Gamma_r - \Gamma_{r'}|\right)^{2\beta-1} + |r-r'|^\beta \right] \\
\leq& C_4 \left[\frac{|r-r'|}{\sqrt{r}} (\sqrt{r} + \frac{|r-r'|}{\sqrt{r}})^{2\beta-1} + |r-r'|^\beta \right] \\
\leq& C_4 \left[\frac{|r-r'|}{\sqrt{r}} (2\sqrt{r})^{2\beta-1} + |r-r'|^\beta \right] \leq C_2 |r-r'|^\beta,
\end{aligned}$$

where all constants do not depend on s. We also used the fact that $|r - r'| \leq |r|$, and $r^{\beta-1} \leq |r-r'|^{\beta-1}$ since $1/2 < \beta < 1$.

The case $\beta = 1$ is similar. $\square$

3.3.3 Driving function of the initial bit

In this subsection we study the driving function of η in a neighborhood of 0. By comparing to an affine line (Corollary 3.23, Lemma 3.24), we deduce that W_t is bounded above by constant times $\operatorname{Re}\eta(t)$ that is again comparable to $\operatorname{Im}(\eta(t))\sqrt{t}^{2\beta} \approx t^{\beta+1/2}$ (Lemma 3.26).

Lemma 3.22 ([KNK04] Sec. 4.1)**.** *Let $0 \leq \theta \leq \pi/4$. There exists $k = k(\theta) \leq (16/\sqrt{3}\pi)\theta$ such that the straight line $\eta = \{re^{i(\pi/2-\theta)}, r \geq 0\}$ has the Loewner driving function $t \mapsto k(\theta)\sqrt{t}$, and the capacity parametrized line $(\eta(t))_{t\geq 0}$ satisfies*

$$\eta(t) = B(k)\sqrt{t},$$

where $|B(k)| \geq 2$ and $|B(k)| \to 2$ as $\theta \to 0$.

Proof. From the explicit computations in [KNK04], the Loewner chain η generated by $t \to k\sqrt{t}$ is the ray with argument $\pi/2 - \theta(k)$, where

$$\theta(k) = \frac{\pi}{2} \frac{k}{\sqrt{k^2 + 16}}.$$

The capacity parametrization of η is also explicit:

$$\eta(t) = B(k)\sqrt{t},$$

where

$$\begin{aligned}
B(k) &= 2 \left(\frac{\sqrt{k^2+16}+k}{\sqrt{k^2+16}-k} \right)^{\frac{k}{2\sqrt{k^2+16}}} \exp(i(\pi/2 - \theta(k))) \\
&= 2 \left(\frac{\pi/2 + \theta(k)}{\pi/2 - \theta(k)} \right)^{\theta(k)/\pi} \exp\left(i(\pi/2 - \theta(k))\right) \\
&= (2 + O(k^2)) \exp\left(i(\pi/2 - \theta(k))\right).
\end{aligned}$$

We see that $|B(k)| \geq 2$ and the claimed convergence as $k \to 0$.

For every $0 \leq \theta \leq \pi/4$, we have

$$k^2 + 16 = (\pi/2\theta)^2 k^2$$

which implies

$$k = 8\theta/\sqrt{\pi^2 - 4\theta^2} \leq (16/\sqrt{3}\pi)\theta$$

as claimed. □

Corollary 3.23. *There is a universal constant $C > 0$ such that for all $0 \leq |x| \leq y$, the image of $x + iy$ under the mapping-out function g of the segment $\eta = [0, x + iy]$ satisfies*

$$|g(x + iy)| \leq C\,|x|\,.$$

Proof. Without loss of generality, assume that $x \geq 0$. Let $l = \sqrt{x^2 + y^2}$, $T = cap(\eta)$, $\theta = \arctan(x/y)$ and $k = k(\theta)$. We know

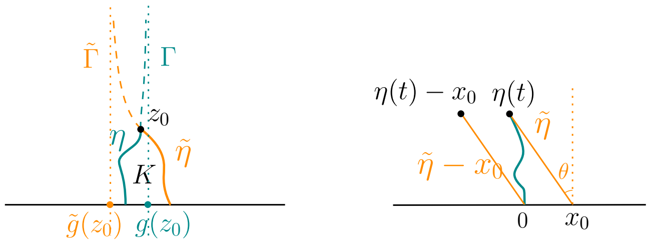

Figure 3.6: Left: The dashed line Γ ($\tilde{\Gamma}$) is the hyperbolic geodesic between z_0 and ∞ in the domain $\mathbb{H} \setminus \eta$ ($\mathbb{H} \setminus \tilde{\eta}$) and dotted lines are their vertical asymptotes as in the proof of Lemma 3.24. Right: Curves in the proof of Lemma 3.26.

that

$$|B(k)| \sqrt{T} = |x + iy| = l$$

and therefore

$$T = l^2 / |B(k)|^2 \leq l^2/4.$$

By definition of the driving function,

$$g(x+iy) = k\sqrt{T} \leq \frac{16\theta}{\sqrt{3}\pi}\frac{l}{2} = \frac{8}{\sqrt{3}\pi}\theta l \leq \frac{8}{\sqrt{3}\pi} 2\sin(\theta) l = \frac{16}{\sqrt{3}\pi} x,$$

where we have used $\theta \leq \pi/4$. □

Lemma 3.24. *Let K be a compact $\mathbb{H}$-hull whose boundary is a Jordan curve, and let $z_0 \in \partial K \cap \mathbb{H}$. Denote η (resp. $\tilde{\eta}$) the left (resp. right) boundary of K connecting $\mathbb{R}$ and z_0, and let g and $\tilde{g}$ be their mapping-out functions. Then we have $g(z_0) \geq \tilde{g}(z_0)$.*

Proof. Recall that the mapping-out function g of η satisfies $g(z) = z + o(1)$. The hyperbolic geodesic Γ in $\mathbb{H} \setminus \eta$ between z_0 and ∞ is the image of $g(z_0) + i\mathbb{R}$ under g^{-1}. Hence Γ has the vertical asymptote $g(z_0) + i\mathbb{R}$. In other words, we can read off $g(z_0)$ from the geodesic. Let $\partial_-(\eta)$ (resp. $\partial_+(\eta)$) be the boundary of $\mathbb{H} \setminus \eta$ between z_0 and $-\infty$ (resp. between z_0 and $+\infty$). The complement of $\Gamma \cup \eta$ in $\mathbb{H}$ has two connected components, $H_-(\eta)$ and $H_+(\eta)$, whose boundaries contain $\partial_-(\eta)$ and $\partial_+(\eta)$ respectively.

For $z \in \mathbb{H}$, let B_z be a Brownian motion starting from z. By the conformal invariance of Brownian motion, $z \in H_-$ if and only if B_z has larger probability of first hitting ∂_- than ∂_+. And $z \in \Gamma$ if and only if these probabilities are equal. It is then not hard to see that for all $z \in \tilde{\Gamma} \setminus K$, we have $z \in H_-(\eta)$, where $\tilde{\Gamma}$ is the geodesic in $\mathbb{H} \setminus \tilde{\eta}$. In fact, the Brownian motion starting from z has equal probability to hit first $\partial_-(\tilde{\eta})$ or to hit $\partial_+(\tilde{\eta})$. Besides, every sample path hitting $\partial_-(\tilde{\eta})$ hits already $\partial_-(\eta)$, but not $\partial_+(\eta)$. Hence, if we stop the Brownian motion when it hits $\eta \cup \mathbb{R}$, it has probability bigger than $1/2$ to hit $\partial_-(\eta)$.

By comparing asymptotes for Γ and $\tilde{\Gamma}$, we have $\tilde{g}(z_0) \le g(z_0)$. $\square$

Lemma 3.25. *If γ is a R-regular curve tangentially attached to $\mathbb{R}_+$, then the arclength parametrization s of γ and the capacity parametrization $t(s)$ of $\eta = \sqrt{\gamma}$ satisfy $s/5 \le t \le s/2$, $\forall s \in [0, R]$.*

Proof. For every $s \in [0, S]$,

$$2t = cap(\sqrt{\gamma[0,s]}) \le cap(\{z \in \mathbb{H}, |z| \le \sqrt{s}\}) = s.$$

To see the other inequality, set

$$(X_r, Y_r) = (\operatorname{Re} g_r(\eta(t)), \operatorname{Im} g_r(\eta(t)))$$

for $r \in [0, t)$. By the Loewner differential equation,

$$\partial_r Y_r = \frac{-2Y_r}{(X_r - W_r)^2 + Y_r^2} \ge \frac{-2}{Y_r}.$$

Hence

$$\partial_r (Y_0^2 - Y_r^2) = -2Y_r \partial_r Y_r \le 4$$

so that

$$4r \ge Y_0^2 - Y_r^2.$$

We also know that $Y_{t(s)} = 0$, hence

$$t(s) \geq Y_0^2/4.$$

Since $\omega(R) \leq 1/5$, we have from (3.2)

$$|\gamma(s) + s| \leq s\omega(s) \leq s/5.$$

We conclude that

$$Y_0 = \operatorname{Im} \sqrt{\gamma(s)} \geq \sqrt{4/5}\sqrt{s},$$

and $t \geq s/5$ follows. $\square$

Lemma 3.26. *Using the same notation and assumption as in Lemma 3.25, there exists a universal constant $c > 0$, such that if η satisfies in addition $|\pi/2 - \arg(\eta'(t))| \leq \theta$ for some $0 \leq \theta < \pi/4$ and all $t \in [0, T]$, then the driving function W is bounded by*

$$|W_{t(s)}| \leq c\theta\sqrt{s}.$$

It implies that for all $t \leq R/5$,

$$|\lambda_t| \leq c\omega(5t)t^{1/2}, \tag{3.10}$$

where we recall ω is the modulus of continuity of γ'.

Proof. Let (x, y) denote $(\operatorname{Re}\eta(t), \operatorname{Im}\eta(t))$. Consider the straight line segment $\tilde{\eta}$ that passes through $\eta(t)$ and makes an angle of θ with the vertical line, as shown in Figure 3.6. Let $x_0 = x + y\tan(\theta)$ be the intersection of $\tilde{\eta}$ and $\mathbb{R}$. Denote $\tilde{g}$ the mapping-out function of the segment $[x_0, \eta(t)]$, g_t of $\eta[0, t]$ and g of $[0, \eta(t) - x_0]$. By assumption on $\arg(\eta')$, the segment $[0, x_0]$, the curve $\eta[0, t]$ and the segment $[x_0, \eta(t)]$ form the boundary of a compact $\mathbb{H}$-hull. In fact, for all $y \in (0, \operatorname{Im}\eta(t))$, there exists a unique point $\eta(t')$ on η and a unique point $\tilde{z}$ on the segment $\tilde{\eta}$ with imaginary part y. It is easy to see that $\operatorname{Re}\eta(t') \leq \operatorname{Re}\tilde{z}$.

It then follows from Corollary 3.23 and Lemma 3.24,

$$W_t = g_t(\eta(t)) \geq \tilde{g}(\eta(t)) \geq g(\eta(t) - x_0) \geq -Cy\tan(\theta).$$

The upper bound is similar, and we have

$$|W_t| \leq Cy\tan(\theta) \leq (4C/\pi)\theta\sqrt{s}$$

with $C = 16/(\sqrt{3}\pi)$, where in the last inequality we have used $t = t(s)$, $y \leq \sqrt{s}$ and $\tan(\theta) \leq 4\theta/\pi$.

In terms of ω, we first compute the difference between $\arg(\eta')$ and $\pi/2$:

$$\begin{aligned}
\arg(\eta'(t)) &= \operatorname{Im}\log(\gamma'(s)/2\sqrt{\gamma(s)}) \\
&= \operatorname{Im}\log(\gamma'(s)) - \operatorname{Im}(\log\gamma(s))/2 \\
&= \arg(\gamma'(s)) - \arg(\gamma(s))/2.
\end{aligned}$$

Hence from (3.2),

$$|\arg(\eta'(t)) - \pi/2| = |\arg(\gamma'(s)) - \pi - (\arg(\gamma(s)) - \pi)/2| \leq 2\omega(s).$$

Since $2\omega(R) \leq 2/5 < \pi/4$, we can apply the above estimate of W to the interval $[0,t]$ with $s \leq R$, $\theta = 2\omega(s)$, and obtain that the driving function λ of η satisfies

$$|\lambda_t| \leq 2c\omega(s)s^{1/2} \leq c'\omega(5t)t^{1/2}.$$

It suffices to replace c by the maximum of c and c'. $\square$

3.3.4 Proof of Theorem 3.5

Now we proceed to the proof of Theorem 3.5. We assume that γ is a $C^{1,\beta}$ curve tangentially attached to the positive real line. Without loss of generality, γ is also assumed to be R-regular.

- For $0 < \beta \leq 1/2$: We would like to compare $|\lambda_{t+r} - \lambda_t|$ to $r^{\beta+1/2}$ for every $t \in [0, T]$ and every r in a small but uniform neighborhood $[0, R_0]$ (as far as it is defined). The constant R_0 is chosen as in Corollary 3.20.

The case $t = 0$ is already given by the inequality (3.10). Fix $s \in (0, S]$, $t := t(s)$. The centered mapping out function f_s, defined as

$$f_s(z) = \varphi_s^{-1}\left(\sqrt{z^2 - \gamma(s)}\right), \quad f_s : \mathbb{H} \setminus \eta[0, t] \to \mathbb{H},$$

maps the curve $\eta[t, T]$ to a curve $\tilde{\eta}$ whose driving function is $\tilde{\lambda}_r = \lambda_{t+r} - \lambda_t$, see Figure 3.4. Since $f_s(z) = \sqrt{h_s(z^2)}$, by Corollary 3.20, $\tilde{\gamma} = \tilde{\eta}^2$, reparametrized by arclength, is a $C^{1,\beta}$ curve: thus for $r \le R_0$,

$$\begin{aligned} |\tilde{\gamma}'(r) + 1| &\le C_2 r^\beta, && \text{if } 0 < \beta < 1/2; \\ |\tilde{\gamma}'(r) + 1| &\le C_2 r^\beta \log(1/r), && \text{if } \beta = 1/2. \end{aligned}$$

Here R_0 and C_2 depend on $\beta, M, S, \|\gamma\|_{1,\beta}$, but are uniform in $s \in [0, S]$. By taking a perhaps smaller R_0, such that the modulus of continuity of $\tilde{\gamma}'$ at R_0 is less than $1/5$, inequality (3.10) in Lemma 3.26 applies again to $\tilde{\lambda}$. For $r \le R_0/5$,

$$\begin{aligned} |\lambda_{t+r} - \lambda_t| &\le cC_2(5r)^\beta r^{1/2} := Cr^{\beta+1/2} \text{ if } 0 < \beta < 1/2; \\ |\lambda_{t+r} - \lambda_t| &\le cC_2(5r)^{1/2} \log(1/5r) r^{1/2} \le Cr \log(1/r) \end{aligned}$$

if $\beta = 1/2$, where C depends only on the global parameters of γ and on $\|\gamma\|_{1,\beta}$.

• For $\beta > 1/2$: Since we expect that the curve has C^1 driving function, it is natural to compute directly the derivative of λ. Actually it is a multiple of L_s (defined in Proposition 3.19) which equals to the second derivative at 0 of the uniformizing map μ_s (Corollary 3.28). A similar result has been observed in [LT16] Lemma 6.1 (19) in a more general setting, with higher order of derivatives of λ. Here we reproduce a simple proof for the first derivative for the readers' convenience. We first prove a lemma, to see how the driving function changes under a conformal transformation. The proof is standard, the same computation appears also in the study of the conformal restriction property [LSW03] Sec. 5.

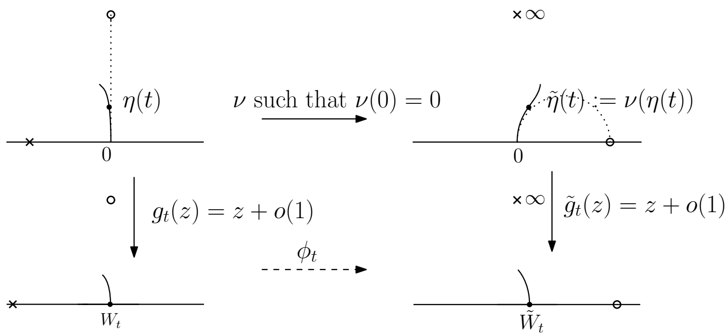

Figure 3.7: The conformal map ϕ_t factorizes the diagram.

Let ν be a conformal map on a neighborhood D of 0, such that $\nu(0) = 0$, $\nu(D \cap \mathbb{H}) \subset \mathbb{H}$ and $\nu(D \cap \mathbb{R}) \subset \mathbb{R}$. Let η be a curve in $\mathbb{H}$ driven by W such that η is contained in D. Define $\tilde\eta(t) := \nu(\eta(t))$. Let g_t and $\tilde g_t$ denote the mapping-out function of $\eta[0, t]$ and $\tilde\eta[0, t]$ respectively, and $\phi_t = \tilde g_t \circ \nu \circ g_t^{-1}$ denote the conformal map that factorizes the diagram (Figure 3.7). Note that $\phi_0 = \nu$, and define $\tilde W_t = \phi_t(W_t)$.

Lemma 3.27. *Assume that $|W_t/t|$ is bounded. Then we have*

$$\left|\tilde W_t - \nu'(0)W_t + 3\nu''(0)t\right| /t \xrightarrow{t\to 0} 0.$$

Proof. Notice that $\tilde\eta(t)$ is not capacity-parametrized. Let $2a(t)$ denote the capacity of $\tilde\eta[0, t]$. We have then $a'(t) = [\phi_t'(W_t)]^2$.

It is not hard to see that for any continuous driving function W, the map $t \mapsto \phi_t^{(n)}(z)$ is at least C^1 for all $n \ge 0$ and all $z \in \overline{\mathbb{H}}$ for which $\phi_t(z)$ is well-defined (when $z \in \mathbb{R}$, this follows from the Schwarz reflection principle). We deduce that $r \mapsto \phi_r'(W_r)$ and $r \mapsto \phi_r''(W_r)$ are both continuous as well as any higher order derivatives of ϕ_r evaluated at W_r (and differentiable if W is so).

From that, it is not hard to see that there exists $t_0, \delta > 0$, and $C > 0$, such that for all $t \le t_0$ and $|z| \le \delta$, we have $|R(z)| \le C\,|z|^3$

and $|R'(z)| \le C\,|z|^2$, where R is defined as

$$R(z) = \phi_t(W_t + z) - \tilde{W}_t - z\phi_t'(W_t) - z^2\phi_t''(W_t)/2,$$

and

$$R'(z) = \phi_t'(W_t + z) - \phi_t'(W_t) - z\phi_t''(W_t).$$

For $z \in \mathbb{H}$,

$$\begin{aligned}
&\partial_r\phi_r(z) = \partial_r\tilde{g}_r \circ \nu \circ g_r^{-1}(z) \\
&= a'(r)\partial_a\tilde{g}_r(\nu \circ g_r^{-1}(z)) + \tilde{g}_r'(\nu \circ g_r^{-1}(z))\nu'(g_r^{-1}(z))\partial_r g_r^{-1}(z) \\
&= \frac{2a'(r)}{\phi_r(z) - \tilde{W}_r} - \frac{2\phi_r'(z)}{z - W_r},
\end{aligned}$$

where we have used

$$\partial_r g_r^{-1}(z) = \frac{-2(g_r^{-1})'(z)}{z - W_t}.$$

For simplicity of notation, we will omit the argument W_t in the following computation.

$$\begin{aligned}
&\partial_r\phi_r(z + W_r) \\
&= \frac{2(\phi_r')^2}{z\phi_r' + z^2\phi_r''/2 + R(z)} - \frac{2(\phi_r' + z\phi_r'' + R'(z))}{z} \\
&= \frac{2\phi_r'}{z}\,\frac{1 - (1 + z\phi_r''/2\phi_r' + R(z)/z\phi_r')(1 + z\phi_r''/\phi_r' + R'(z)/\phi_r')}{1 + z\phi_r''/2\phi_r' + R(z)/z\phi_r'} \\
&= -3\phi_r''(W_r) + T_r(z),
\end{aligned}$$

with $T_r(z)/z$ bounded on $(z, r) \in \mathscr{O} \times [0, t_0]$, where $\mathscr{O}$ is a small neighborhood of 0. Thus $T_r(z) \to 0$ as $z \to 0$ uniformly in $r \in [0, t_0]$.

$$\begin{aligned}
&\tilde{W}_t - \nu'(0)W_t + 3\nu''(0)t \\
=& \lim_{z \to W_t} \phi_t(z) - \nu'(0)W_t + 3\nu''(0)t \\
=& -\nu'(0)W_t + \nu(W_t) + \lim_{z \to W_t} \int_0^t \partial_r \phi_r(z) dr + 3\nu''(0)t \\
=& \int_0^{W_t} (\nu'(s) - \nu'(0)) ds + \int_0^t 3(\nu''(0) - \phi_r''(W_r)) \\
&+ T_r(W_t - W_r) dr.
\end{aligned}$$

Since W_t/t is bounded, the first integral divided by t converges to 0 as $t \to 0$. The second integral divided by t converges to 0 since the integrand converges uniformly to 0 as $t \to 0$, which concludes the proof. □

In particular, if W is differentiable at 0, then the derivative with respect to the capacity of $\tilde{\eta}$ also exists at 0, and

$$\partial_a \tilde{W}|_{a=0} = \lim_{t \to 0} a'(0)^{-1} \partial_t \tilde{W}|_{t=0} = \dot{W}_0/\nu'(0) - 3\nu''(0)/\nu'(0)^2, \quad (3.11)$$

as $a'(0) = \nu'(0)^2$.

Corollary 3.28. *If $\beta > 1/2$, the driving function W is right differentiable. Moreover $\partial_{t+} W_t = 3L_s$, where $t(s) = t$ and L_s is defined in Proposition 3.19.*

Proof. (See Figure 3.4) We use the notation as in Corollary 3.21 and let $\nu = \mu_s^{-1}$. From Corollary 3.21, ν maps a Loewner chain driven by a certain function V to $\tilde{\eta}$. This Loewner chain is the square root of a $C^{1,\beta}$ curve. By inequality (3.10) and the same proof as for the case $\beta \le 1/2$, we have

$$|V_t| \le Ct^{\beta+1/2}$$

for small t, in particular $\dot{V}(0) = 0$ as $\beta > 1/2$. Recall that the driving function of $\tilde{\eta}$ is $\tilde{W}_h = W_{t+h} - W_t$. By Lemma 3.27 and equation (3.11),

we have
$$\partial_{t+}W_t = \dot{V}(0) - 3\nu''(0) = 3\mu_s''(0) = 3L_s,$$
where we have used $\nu'(0) = 1$. $\square$

In particular $\dot{W}_0 = 0$. Notice that the above corollary only deals with the right derivatives of W. In the following lemma, we will see that L is continuous. By elementary analysis, continuous right-derivative implies that W is C^1, with the actual derivative $3L$. See for example [Law08] Lemma 4.2 for a proof. Notice also that $3L_s$ depends only on $\gamma[0,s]$, it is then not surprising that it also gives the left derivative of W.

Lemma 3.29. *There exists C' and C'' such that for all $s \in [0, R]$,*
$$\begin{aligned} |L_s| &\le C' \left(\frac{\omega(s)}{\sqrt{s}} + \int_0^{\sqrt{s}} \frac{\omega(r^2)}{r^2} dr \right) \\ &\le C'' \begin{cases} s^{\beta-1/2}, & \text{if } \gamma \text{ is } C^{1,\beta}\ , \\ s^{1/2}\log(1/s) & \text{if } \gamma \text{ is weakly } C^{1,1}. \end{cases} \end{aligned}$$

Proof. We use the explicit expression for L_s. From equation (3.8) in Proposition 3.19,
$$L_s = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{w_s(r)}{r^2} dr,$$
where $w_s(r) = \operatorname{Im}\log(\varphi_s'(r)) - \operatorname{Im}\log(\varphi_s'(0))$. Since $s \le R \le 1/2$, from Lemma 3.18 and a similar proof of Lemma 3.17, we easily deduce that
$$|w_s(r)| \le C(\omega(r^2) \wedge \omega(s)).$$

This yields
$$\begin{aligned} |L_s| &\le \frac{2C}{\pi} \left(\omega(s) \int_{\sqrt{s}}^{\infty} \frac{1}{r^2} dr + \int_0^{\sqrt{s}} \frac{\omega(r^2)}{r^2} dr \right) \\ &= C' \left(\frac{\omega(s)}{\sqrt{s}} + \int_0^{\sqrt{s}} \frac{\omega(r^2)}{r^2} dr \right). \end{aligned}$$

In particular, when $\omega(\delta) = \|\gamma\|_{1,\beta}\delta^\beta$,

$$|L_s| \le C'\|\gamma\|_{1,\beta}\left(s^{\beta-1/2} + \frac{s^{\beta-1/2}}{2\beta-1}\right) = C''s^{\beta-1/2}.$$

When $\omega(\delta) = \delta\log(1/\delta)$,

$$\begin{aligned}|L_s| &\le C'\left(s^{1/2}\log(1/s) - 2\int_0^{\sqrt{s}}\log(r)dr\right)\\ &= C'\left(s^{1/2}\log(1/s) - 2[x\log(x) - x]_0^{\sqrt{s}}\right)\\ &\le C''s^{1/2}\log(1/s),\end{aligned}$$

where C'' does not depend on s but only on β, R, S and $\|\gamma\|_{1,\beta}$. $\square$

Now Theorem 3.5 for $1/2 < \beta \le 1$ follows directly from Corollary 3.21, Corollary 3.28 and Lemma 3.29.

3.4 Comments

3.4.1 The sharpness of Theorem 3.5

As we already argued in the introduction, as the converse of Theorem 3.2, Theorem 3.5 is sharp in the range $\beta \in (0, 1/2) \cup (1/2, 1)$. In fact, for those values of β, the regularity of the driving function implies (Theorem 3.2) capacity regularity of the generated curve which implies arclength regularity of the curve. Then by Theorem 3.5, it implies again the regularity of the driving function, where the regularities are taken accordingly with a shift of 1/2 as in both theorems.

The example in Section 7.2 of [LT16] shows that the driving function of a $C^{1,1/2}$- curve need not be in C^1 but may only be in $C^{0,1}$. Thus in the case $\beta = 1/2$, our theorem is sharp up to the logarithmic term. Similarly, Section 7.2 of [LT16] provides an example of a $C^{1,1}-$ curve whose driving function is $C^{1,1/2}$. We do not know if our result can be improved by removing the term "weakly" in the cases $\beta = 1/2$ and $\beta = 1$.

The case of higher regularity requires the consideration of higher angular derivatives of the uniformizing map φ_s at 0. Nevertheless, we believe that the proof of the natural generalization of Theorem 3.5 should be in the same spirit. Since the focus of this paper is on the Loewner energy, we refrain from discussing the converse of Theorem 3.3 in full generality.

3.4.2 Finite energy and slow spirals

Finite energy curves are rectifiable and therefore have tangents on a set of full length and full harmonic measure. However, we sketch an example showing that finite energy loops need not have tangents everywhere: Pick a sequence ε_k such that $\sum_k \varepsilon_k$ diverges but $\sum_k \varepsilon_k^2$ converges, and consider a sequence $r_k \to 0$ of scales. By Proposition 2.10 the chordal energy minimizing curve γ_k from 0 to $z_k = r_k e^{i(\pi/2+\varepsilon_k)}$ in $\mathbb{H}$ has energy $I_k = -8 \ln \sin(\pi/2 + \varepsilon_k) \sim 4\varepsilon_k^2$ so that the conformal concatenation Γ_k (whose mapping-out function is $G_k = g_k \circ g_{k-1} \circ ... \circ g_1$ and g_i is the mapping-out function of γ_i) has uniformly bounded energy. Denote α_k the tangent angle of the tip of Γ_k. Since G_k behaves like the square-root map near the tip of Γ_k, given $r_1, r_2, ..., r_k$ we have $\alpha_{k+1} = \alpha_k + 2\varepsilon_k + o(1)$ as $r_{k+1} \to 0$. Thus the sequence r_n can be chosen inductively in such a way that $\alpha_n \geq \alpha_1 + \sum_1^{n-1} \varepsilon_k$ for all n. Consequently, the limiting curve $\Gamma = \cup_k \Gamma_k$ has an infinite spiral at its tip and does not possess a tangent there.

3.4.3 Consequences of Theorem 3.1

Proposition 3.13 and Corollary 3.14 can be generalized as follows: fix a collection of distinct points $\overline{z} = (z_0, z_1, z_2, \cdots, z_n)$ and consider curves γ visiting these points in order. Figure 3.8 shows two such curves, visiting the same points in the same order, that cannot be continuously deformed into each other while fixing the points and keeping the curves simple. For three distinct points (the case $n = 2$)

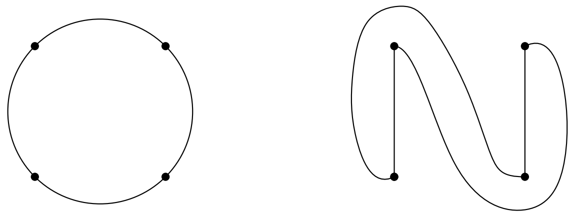

Figure 3.8: Two non-isotopic loops passing through four points in the same order.

there is only one isotopy class, and the minimal energy is 0. For four or more points, there are always countably infinite many classes. The proof of Proposition 3.13 can easily be modified to show that each of these isotopy classes of curves contain at least one loop energy minimizer. More precisely, fix a Jordan curve γ_0 compatible with $\overline{z}$, denote $\mathscr{L}(\overline{z}, \gamma_0)$ the set of all Jordan curves γ_1 for which there is a homotopy γ_t relative $\overline{z}$ through homeomorphisms (that is, in addition to the joint continuity of $\gamma_t(s)$, we require that each γ_t is a Jordan curve, and that $\gamma_t(\gamma_0^{-1}(z_j)) = z_j$ for all $j = 0, 1, ..., n$ and all $0 \leq t \leq 1$) and set

$$I^L(\{\overline{z}, \gamma_0\}) := \inf_{\gamma \in \mathscr{L}(\overline{z}, \gamma_0)} I^L(\gamma),$$

where we have dropped the root in the above expression since the loop energy is root-invariant.

Then we have:

Proposition 3.30. *There exists $\gamma \in \mathscr{L}(\overline{z}, \gamma_0)$ minimizing I^L in $\mathscr{L}(\overline{z}, \gamma_0)$, and every such γ is at least weakly $C^{1,1}$.*

It seems reasonable to believe that the minimizer in each class is unique. In any case, every minimizer has the property that the arc between consecutive points is a hyperbolic geodesic in the complement of the rest of the loop as in the proof of Proposition 3.13.

Chapter 4

Analytic descriptions of the loop energy

This chapter corresponds to the paper [Wan18a].

4.1 Introduction

The goal is to provide analytic descriptions of the loop Loewner energy. In fact, we will provide three equivalent expressions of the Loewner energy, that we briefly describe in the next three paragraphs.

Relation to the Dirichlet energy of the log-derivative of a uniformizing map

Let us introduce some notation: It turns out to be more convenient to work in a slit plane rather than in the upper half-plane (this just corresponds to conjugation of g_t by the square map) when studying the Loewner energy of chords. In other words, one looks at a chord from 0 to ∞ in the slit plane $\Sigma := \mathbb{C}\backslash\mathbb{R}_+$. Such a chord divides the slit plane into two connected components H_1 and H_2, and one can then define h_i to be a conformal map from H_i onto a half-plane which fixes ∞. See Figure 4.1 for a picturesque description of these two maps. Let h be the map defined on $\Sigma\backslash\gamma$, which coincides with h_i on H_i. Here and in the sequel $\mathrm{d}z^2$ denotes the Euclidean (area) measure on $\mathbb{C}$.

Theorem 4.1. *When γ is a chord from 0 to infinity in Σ with finite*

Loewner energy, then

$$I_{\Sigma,0,\infty}(\gamma) = \frac{1}{\pi}\int_{\Sigma\setminus\gamma} \left|\nabla \log |h'(z)|\right|^2 \, \mathrm{d}z^2 = \frac{1}{\pi}\int_{\Sigma\setminus\gamma} \left|\frac{h''(z)}{h'(z)}\right|^2 \, \mathrm{d}z^2.$$

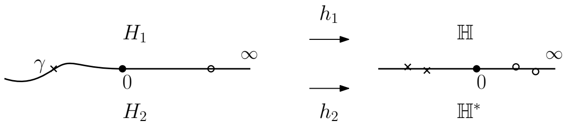

Figure 4.1: We often choose the half-planes to be $\mathbb{H}$ and the lower half-plane $\mathbb{H}^*$ as the image of h_1 and h_2 to fit into the Loewner setting. However, it is clear that the last two expressions of the equality in Theorem 4.1 is invariant under transformations $z \mapsto az + b$, for $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$.

The Loewner energy also has a natural generalization to oriented simple loops (Jordan curves) with a marked point (root) embedded in the Riemann sphere, such that if we identify the simple chord γ in Σ connecting 0 to ∞ with the loop $\gamma \cup \mathbb{R}_+$, then the loop energy of $\gamma \cup \mathbb{R}_+$ rooted at ∞ and oriented as γ is equal to the chordal Loewner energy of γ in $(\Sigma, 0, \infty)$. In Chapter 3, we have shown that this Loewner loop energy, denoted by I^L, depends only on the image (i.e. of the trace) of the loop. In particular, it does not depend on the root of the loop. The Loewner loop energy is therefore a non-negative and Möbius invariant quantity on the set of free loops, which vanishes only on circles. The proof in Chapter 3 relies on the reversibility of chordal Loewner energy (derived in Chapter 2 using SLE interpretation) and a certain type of surgeries on the loop to displace the root. This root-invariance suggests that the framework of loops seems to provide even more symmetries and invariance properties than the chordal case when one studies Loewner energy.

We will derive the counterpart of Theorem 4.1 for loops:

Theorem 4.2 (see Theorem 4.19)**.** *If γ is a loop passing through ∞*

with finite Loewner energy, then

$$I^L(\gamma) = \frac{1}{\pi}\int_{\mathbb{C}\setminus\gamma} \left|\nabla \log |h'(z)|\right|^2 \,\mathrm{d}z^2,$$

where h maps $\mathbb{C}\setminus\gamma$ conformally onto two half-planes and fixes ∞.

Actually, one can view Theorem 4.1 as a consequence of Theorem 4.19 (and this is the order in which we will derive things). Note that the identity also holds when $I^L(\gamma) = \infty$, which follows in fact from the characterization of Weil-Petersson quasicircles (see below) by its welding homeomorphism [ST18] that we will not enter into detail here.

Let us say a few words about the strategy of our proof of these two results, which will be the main purpose of the first part of the present paper (Sections 4.3 to 4.6). We will first derive the additivity (called *J-additivity*) of the integral on log-derivative of the uniformizing map when the curve is $C^{1,\alpha}$-regular (Section 4.3). Curves with piecewise linear driving function fall into this class of curves. Weak J-additivity suffices to obtain a version of Theorem 4.1 for piecewise linear driving functions of finite capacity chords using explicit infinitesimal computation in Section 4.4. It then provides the bound to deduce general J-additivity for all finite energy curves (Corollary 4.18) and the proof of the identity (Proposition 4.5) for finite capacity and finite energy chords is completed in Section 4.5. We prove Theorem 4.19 in Section 4.6 by passing the capacity to ∞ in Proposition 4.5 and generalize the identity to loops. It is worth emphasizing that already in the case of a linear driving function where the maps h is almost explicit, the proof is not immediate.

As briefly argued in the concluding section (Section 4.9) of the present paper, it is possible to heuristically interpret Theorem 4.19 as a $\kappa \to 0+$ limit of some relations between SLE_κ curves and Liouville Quantum gravity, as pioneered by Sheffield [She16]. This is actually the line of thought that led us to guessing the Theorem 4.19.

Theorem 4.19 then opens the door to a number of connections with other ideas, which we then investigate in Section 4.7 and Section 4.8

and that we now describe.

Relation to zeta-regularized determinants

The first approach involves zeta-regularized determinants of Laplacians for smooth loops. Our main result in this direction is Theorem 4.25, which can be summarized by:

Theorem 4.3. *For C^∞ loops, one has the identity*

$$\begin{aligned} I^L(\gamma) =& 12\log\det{}'_\zeta N(\gamma,g) - 12\log \mathrm{l}_g(\gamma) \\ & - (12\log\det{}'_\zeta N(S^1,g) - 12\log \mathrm{l}_g(S^1)), \end{aligned}$$

where g is any metric on the Riemann sphere conformally equivalent to the spherical metric, $\mathrm{l}_{\mathrm{g}}(\gamma)$ the arclength of γ, and $N(\gamma, g)$ the Neumann jump operator across γ.

Let us already note that the root-invariance (and also the reversibility) of the loop energy (for smooth loops) follows directly from this result, because there is no more parametrization involved in the right-hand side.

Notice that the regularization is usually well defined whenever the boundary of the bounded domain is regular enough (e.g. C^2), and that the variation formula was also derived under boundary regularity conditions. This is why to stay on the safe side in the present paper, we restrict ourselves to C^∞ loops whenever we consider these regularized determinants.

The zeta-regularized determinant of the Neumann jump operator $N(\gamma, g)$ that is referred to in Theorem 4.3 is closely related to such determinants of Laplacians via a Mayer-Vietoris type surgery formula [BFK92] that we will recall in Section 4.7.

Relation to Weil-Petersson Teichmüller space

Since finite energy loops are quasicircles, it is very natural to consider them as points in the universal Teichmüller space $T(1)$ which can be modeled by the homogeneous space $\text{Möb}(S^1)\backslash \text{QS}(S^1)$ that is the group $\text{QS}(S^1)$ of quasisymmetric homeomorphisms of the unit circle S^1 modulo Möbius transformations of S^1, via the welding function of the quasicircle (for basic material on quasiconformal maps and Teichmüller spaces, readers may consult e.g. [Leh12, Na88]). On the other hand, it is easy to see that quasicircles do not always have finite Loewner energy (for instance, quasicircles with corners have infinite energy). This raises the natural question to identify the subspace of finite energy loops in the Teichmüller space. The answer to this question is the main purpose of Section 4.8.

The class $\text{Möb}(S^1)\backslash\text{Diff}(S^1)$ of smooth diffeomorphisms of the circle is naturally embedded into $T(1)$ since they are clearly quasisymmetric. It carries a remarkable complex Kähler structure, and there is a unique (up to constant factor) homogeneous Kähler metric on it which has also been studied intensively by both physicists and mathematicians, see e.g. Bowick, Rajeev, Kirillov, Yur'ev, Witten [BR87a, BR87b, KY86, Wit88] as it plays an important role in the string theory. Nag and Verjovsky [Na88] showed that this metric coincides with the Weil-Petersson metric on $T(1)$ and Cui [Cui00] showed that the completion $T_0(1)$ (called the Weil-Petersson Teichmüller space) of $\text{Möb}(S^1)\backslash\text{Diff}(S^1)$ under the Weil-Petersson metric in $T(1)$ is the class of quasisymmetric functions whose quasiconformal extension has L^2-integrable complex dilation with respect to the hyperbolic metric.

The memoir by Takhtajan and Teo [TT06] studies systematically Weil-Petersson Teichmüller space. They proved that $T_0(1)$ is the connected component of the identity in $T(1)$ viewed as a complex Hilbert manifold (this is actually where the notation of $T_0(1)$ comes from) and established many other equivalent characterizations of Weil-Petersson Teichmüller space. They also introduced a quantity which is very relevant for the present paper: the universal Liouville action $\mathbf{S_1}$ (we will recall its definition in (4.16)) and showed that it is a Kähler potential for the Weil-Petersson metric on $T_0(1)$. Later, Shen [She18, ST18] did characterize $T_0(1)$ directly in terms of the homeomorphisms on the circle, see also [HS12].

The main result of Section 4.8 is Theorem 4.28 that loosely speaking says that:

Theorem 4.4. *A Jordan curve γ has finite Loewner energy if and only if $[\gamma] \in T_0(1)$ and*

$$I^L(\gamma) = \mathbf{S_1}([\gamma])/\pi,$$

where we identify γ with its welding function which lies in $\mathrm{QS}(S^1)$.

This provides therefore another characterization of $T_0(1)$ and a new viewpoint on its Kähler potential (or alternatively a way to look at the Loewner energy).

Again the root-invariance (and also the reversibility) of the loop energy can be viewed as a corollary of this result, because there is no more parametrization involved in the definition of $\mathbf{S_1}([\gamma])$.

The paper is structured as follows: Section 4.3 to Section 4.6 are devoted to the proof of Theorem 4.19 as we described above, from which we derive in Section 4.7 the identity with determinants (Theorem 4.25) for smooth loops. In Section 4.8, by choosing a particular metric in the identity of Theorem 4.25, we deduce Theorem 4.28 which relates Loewner energy to the Weil-Petersson Teichmüller space, via approximation of finite energy curves by smooth curves. In Section 4.9 we gather informal discussions on how we are led to these identities.

4.2 Preliminaries and notation

In this paper, a *domain* will mean a simply connected open subset of $\mathbb{C}$ whose boundary can be parametrized by a non self-intersecting continuous curve (not necessarily injective). We parametrize this boundary so that it winds anti-clockwise around the domain. When the boundary is a Jordan curve then we say that the domain is a *Jordan domain.*

We first recall that a real-valued function f defined on the compact interval $[a, b]$ is *absolutely continuous (AC)* if there exists a Lebesgue integrable function g on $[a, b]$, such that

$$f(x) = f(a) + \int_a^x g(t)\,\mathrm{d}t, \quad \text{for } x \in [a, b].$$

It is elementary to check that this is equivalent to any of the following two conditions (see [AL06] Sec. 4.4):

(AC1) For every $\varepsilon > 0$, there is $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of $[a, b]$ and $\sum_k (y_k - x_k) < \delta$, then

$$\sum_k |f(y_k) - f(x_k)| < \varepsilon.$$

(AC2) f has derivative almost everywhere, the derivative is Lebesgue integrable, and

$$f(x) = f(a) + \int_a^x f'(t)\,\mathrm{d}t, \quad \text{for } x \in [a, b].$$

A function f defined on a non-compact interval is said to be AC if f is AC on all the compact sub-intervals.

From now on in this section, we restrict ourselves in the domain $(D, a, b) = (\Sigma, 0, \infty)$ where $\Sigma = \mathbb{C}\backslash\mathbb{R}_+$. We will abbreviate $I_{(\Sigma,0,\infty)}$ as I. We choose $\sqrt{\cdot}$, the square root map taking values in the upper half-plane $\mathbb{H}$, to be the uniformizing conformal map of $(\Sigma, 0, \infty)$, so that the capacity of a bounded hull in Σ, as well as the driving function of Loewner chains in $(\Sigma, 0, \infty)$ are well-defined (and not up to scaling any more).

The following result is the counterpart of Theorem 4.1 for chords that do not make it all the way to infinity (i.e. $T < \infty$):

Proposition 4.5. *Let γ be a finite energy simple curve in $(\Sigma, 0, \infty)$,*

$$I(\gamma[0,T]) = \frac{1}{\pi}\int_{\Sigma\setminus\gamma[0,T]} \left|\frac{h_T''(z)}{h_T'(z)}\right|^2 \mathrm{d}z^2, \tag{4.1}$$

where $h_T : \Sigma\setminus\gamma \to \Sigma$ is the conformal mapping-out function of $\gamma[0,T]$, such that $h_T(\gamma_T) = 0$ and $h_T(z) = z + O(1)$ as $z \to \infty$.

Note that Proposition 4.5 is weaker than Theorem 4.1. Indeed, if we consider W as in Proposition 4.5 and then defines $\tilde{W}$ on all of $[0,\infty)$ by $\tilde{W}(t) := W(\min(t,T))$, then $\tilde{W}$ does generate the chord $\tilde{\gamma}$ from 0 to infinity in Σ that coincides with γ up to time T and then continues along the conformal geodesic from γ_T to infinity in $\Sigma\setminus\gamma[0,T]$ (see Figure 4.2).

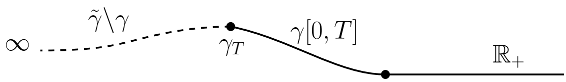

Figure 4.2: The infinite capacity curve $\tilde{\gamma}$ is the completion of γ by adding the conformal geodesic $\tilde{\gamma}\setminus\gamma = h_T^{-1}(\mathbb{R}_-)$ connecting γ_T to ∞ in $\Sigma\setminus\gamma[0,T]$.

It is easy to see that the restriction of h_T to $\Sigma\setminus\tilde{\gamma}$ is an admissible choice for $\tilde{h}$ (the conformal map in Theorem 4.1), which maps $\Sigma\setminus\tilde{\gamma}$ to two half-planes, so that Proposition 4.5 is a rephrasing of Theorem 4.1 for $\tilde{\gamma}$. However, we will explain how it is in fact possible to deduce Theorem 4.1 from Proposition 4.5 by letting $T \to \infty$ in Section 4.6 while proving the more general Theorem 4.19 for simple loops. We will therefore aim at establishing Proposition 4.5 which is completed in Section 4.5.

In the sequel we will denote the right-hand side of Proposition 4.5 by $J(h_T)$. Note that

$$J(h_T) = \frac{1}{\pi}\int_{\Sigma\setminus\gamma} \left|\frac{h_T''(z)}{h_T'(z)}\right|^2 \mathrm{d}z^2 = \frac{1}{\pi}\int_{\Sigma\setminus\gamma} |\nabla\sigma_{h_T}(z)|^2 \,\mathrm{d}z^2$$

is the Dirichlet energy of

$$\sigma_{h_T}(z) := \log |h_T'(z)| .$$

It is worthwhile noticing that this energy is the same for $h = h_T$ as for its inverse map $\varphi = h^{-1}$. More precisely, on has $\sigma_h \circ \varphi = -\sigma_\varphi$ and

$$\begin{aligned} \frac{1}{\pi}\int_\Sigma |\nabla \sigma_\varphi(z)|^2 \,\mathrm{d}z^2 &= \frac{1}{\pi}\int_\Sigma |\nabla(\sigma_h \circ \varphi(z))|^2 \,\mathrm{d}z^2 \\ &= \frac{1}{\pi}\int_\Sigma |\nabla \sigma_h|^2 (\varphi(z))\, |\varphi'(z)|^2 \,\mathrm{d}z^2 \\ &= \frac{1}{\pi}\int_{\Sigma\setminus\gamma} |\nabla\sigma_h|^2 (y)\,\mathrm{d}y^2. \end{aligned} \tag{4.2}$$

We will first consider regular enough curves in the proof of Proposition 4.5, the following theorem is useful which states that the regularity of the curve is characterized by the regularity of its driving function: recall that C^α is understood as the Hölder class $C^{k,\beta}$, where k is the integer part of α and $\beta = \alpha - k$, that are C^k functions with β-Hölder continuous k-th derivative.

Theorem 4.6 (see [RW17, Won14]). *For $0 < \alpha < 1$, $\alpha \neq 1/2$, A simple curve γ is $C^{1,\alpha}$ if and only if it is driven by $C^{\alpha+1/2}$ function.*

It allows us to deduce the regularity of the completed chord $\tilde\gamma$ from the regularity of γ.

Corollary 4.7. *If $T < \infty$, $0 < \alpha \le 1$ and $\gamma[0,T]$ is $C^{1,\alpha}$. Then $\tilde\gamma$ is $C^{1,\beta}$, where $\beta = \alpha$ if $\alpha < 1/2$, and β can take any value less than $1/2$ if $\alpha \ge 1/2$.*

Proof. From Theorem 4.6, the driving function W of γ is in $C^{\alpha+1/2}$ if $\alpha \neq 1/2$. The completion $\tilde\gamma$ of γ by conformal geodesic is driven by $\tilde W = W(\min(t,T))$, is then in $C^{\min(\alpha+1/2,1)}$. It in turn implies that $\tilde\gamma$ is in $C^{1,\beta}$. If $\alpha = 1/2$, it suffices to replace α by $1/2 - \varepsilon$ for small enough ε. □

4.3 Weak J-Additivity

Recall that I satisfies the additivity property. The first step in our proof of the fact that $J = I$ in Proposition 4.5 will be to show that J satisfies the same additivity property in the case of regular curves γ (this is our Proposition 4.12 which is the purpose of this section). More precisely, in this section, we deal mainly with a curve γ such that $\gamma \cup \mathbb{R}_+$ is $C^{1,\alpha}$ for some $\alpha > 0$. This is equivalent to say that the extended driving function $\overline{W} : (-\infty, T] \to \mathbb{R}$ of W, such that $\overline{W}(t) = 0$ for $t \leq 0$ has Hölder exponent larger than $1/2$. In fact, $\overline{W}$ is the driving function for the embedded arc $\gamma \cup \mathbb{R}_+$ rooted at ∞ (see Section 4.6 for more details on the extension of driving functions).

Let us first recall some classical analytical tools: Let D be a Jordan domain with boundary Γ and let φ be a conformal mapping from $\mathbb{D}$ onto D. From Carathéodory theorem (see e.g. [GM05] Thm. I.3.1), φ can be extended into a homeomorphism from $\overline{\mathbb{D}}$ to $\overline{D}$. Moreover, the regularity of φ is related to the regularity of Γ from Kellogg's theorem:

Theorem 4.8 (Kellogg's theorem, see e.g. [GM05] Thm. II.4.3)**.** *Let $n \in \mathbb{N}^*$, and $0 < \alpha < 1$. Then the following conditions are equivalent :*

(a) Γ *is of class* $C^{n,\alpha}$.
(b) $\arg(\varphi')$ *is in* $C^{n-1,\alpha}(\partial\mathbb{D})$.
(c) $\varphi \in C^{n,\alpha}(\overline{\mathbb{D}})$ *and* $\varphi' \neq 0$ *on* $\overline{\mathbb{D}}$.

If one of the above condition holds, we say that D is a $C^{n,\alpha}$ domain. When $\alpha = 0$, conditions (a) and (b) are still equivalent.

An unbounded domain is said to be $C^{n,\alpha}$ if there exists a Möbius map mapping it to a bounded $C^{n,\alpha}$ domain. Now let H be a $C^{1,\alpha}$ domain with $0 < \alpha < 1$ and $0, \infty \in \partial H$. We parametrize its boundary Γ by arclength $\Gamma : \mathbb{R} \to \partial H$, such that $\Gamma(0) = 0$. Let ϕ be a conformal map fixing ∞ from H onto a half-plane. By conjugating by Möbius transformations, Theorem 4.8 implies that ϕ^{-1} is $C^{1,\alpha}$ in all compacts of $\overline{\mathbb{H}}$. Since $(\phi^{-1})'$ is locally bounded away from 0, the inverse function theorem shows that ϕ is also $C^{1,\alpha}$ in all compacts of $\overline{H}$. In particular, both $\sigma_\phi = \log|\phi'|$ and its conjugate $\nu_\phi = \arg(\phi') = \operatorname{Im}\log(\phi')$ are C^α in all compacts of $\overline{H}$.

Lemma 4.9 (Extension of Stokes' formula)**.** *For $C^{1,\alpha}$ domain H as above and all smooth and compactly supported functions $g \in C_c^\infty(\overline{H})$,*

$$\int_H \nabla g(z) \cdot \nabla \sigma_\phi(z) \, \mathrm{d}z^2 = -\int_{\mathbb{R}} g(\Gamma(s)) \, \mathrm{d}\tau(s), \tag{4.3}$$

where $\tau(s) := \arg(\Gamma'(s))$ is chosen to be continuous, and the right-hand side is a Riemann-Stieljes integral.

The existence of the Riemann-Stieljes integral against $\mathrm{d}\tau(s)$ is due to a classical result of Young [You36]:

Theorem 4.10. *If $X \in C^\alpha([0,T],\mathbb{R})$ and $Y \in C^\beta([0,T],\mathbb{R})$, $\alpha+\beta > 1$, $\alpha, \beta \le 1$, then the limit below exists and we define*

$$\int_0^T Y(u) \, \mathrm{d}X(u) := \lim_{|P|\to 0} \sum_{(u,v)\in P} Y(u)(X(v) - X(u))$$

where P is a partition of $[0,T]$, $|P|$ the mesh size of P. The above limit is also equal to

$$\lim_{|P|\to 0} \sum_{(u,v)\in P} Y(v)(X(v) - X(u))$$

and the integration by parts holds:

$$\int_0^T Y(u) \, \mathrm{d}X(u) = Y(T)X(T) - Y(0)X(0) - \int_0^T X(u) \, \mathrm{d}Y(u).$$

Moreover, one has the bounds: for $0 \le s < t \le T$,

(a) $\left|\int_s^t Y(u) - Y(s) \, \mathrm{d}X(u)\right| \lesssim \|Y\|_\beta \|X\|_\alpha \, |t-s|^{\alpha+\beta}$.

(b) $\|\int_0^\cdot Y(u) \, \mathrm{d}X(u)\|_\alpha \lesssim (|Y(0)| + \|Y\|_\beta)\|X\|_\alpha$,

where $\lesssim$ means inequality up to a multiplicative constant depending on α, β and T.

Notice that when Γ is smooth, the outward normal derivative $\partial_n \sigma_\phi$ is well defined on the boundary, the above lemma is indeed the Stokes' formula

$$\begin{aligned}
&\int_H \nabla g(z) \cdot \nabla \sigma_\phi(z) \, \mathrm{d}z^2 \\
&= -\int_H g(z) \Delta \sigma_\phi(z) \, \mathrm{d}z^2 + \int_\Gamma g(z) \partial_n \sigma_\phi(z) \, \mathrm{dl(z)} \\
&= \int_\Gamma -g(z) k_0(z) \, \mathrm{dl}(z) = \int_\Gamma -g(z) \, \mathrm{d}\tau,
\end{aligned}$$

where $k_0(z)$ the geodesic curvature of ∂H at z and dl is integration w.r.t. the (Euclidean) arclength on the boundary. In this case, the first equality is due to the fact that g is compactly supported. The second equality follows from the fact that $\sigma_\phi(z)$ is harmonic and from Lemma 4.32,

$$\partial_n \sigma_\phi(z) = k(\phi(z)) e^{\sigma_\phi(z)} - k_0(z),$$

where $k(\phi(z))$ is the geodesic curvature of $\partial \mathbb{H}$ at $\phi(z)$ which is zero. Hence, the lemma's goal is to deal with the case where the boundary regularity is weaker (the geodesic curvature is not defined for $C^{1,\alpha}$ domains).

Lemma 4.9. Since (4.3) is unchanged if we replace ϕ by $a\phi + b$ for $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$, we assume without loss of generality that $\phi(H) = \mathbb{H}$.

Let $H_\varepsilon = \phi^{-1}(\mathbb{H} + i\varepsilon)$. We parametrize its boundary $\Gamma_\varepsilon = \phi^{-1}(\mathbb{R} + i\varepsilon)$ by arclength: $s \to \Gamma_\varepsilon(s)$. We may choose the parametrization such that $\Gamma_\varepsilon(0) \to \Gamma(0)$ as $\varepsilon \to 0$. Since Γ_ε is analytic, the remark above applies and one gets

$$\begin{aligned}
&\int_{H_\varepsilon} \nabla g(z) \cdot \nabla \sigma_\phi(z) \, \mathrm{d}z^2 = \int_{\Gamma_\varepsilon} g(z) \partial_n \sigma_\phi(z) \, \mathrm{d}z \\
&= \int_{\mathbb{R}} g(\Gamma_\varepsilon(s)) \partial_s \nu_\phi(\Gamma_\varepsilon(s)) \, \mathrm{d}s = \int_{\mathbb{R}} -\partial_s g(\Gamma_\varepsilon(s)) \nu_\phi(\Gamma_\varepsilon(s)) \, \mathrm{d}s
\end{aligned}$$

by integration by parts. Since ϕ^{-1} is $C^{1,\alpha}$ in all compacts of $\overline{\mathbb{H}}$, the bijective map ψ from $\overline{\mathbb{H}}$ to itself $(x, y) \mapsto (s, y)$ such that $\Gamma_y(s) =$

$\phi^{-1}(x+iy)$ is continuous. The inverse of ψ is continuous therefore uniformly continuous on compacts. Since $\Gamma_\varepsilon(\cdot) = \phi^{-1}\circ\psi^{-1}(\cdot,\varepsilon)$, we have that $\Gamma_\varepsilon(\cdot)$ converges to $\Gamma(\cdot)$ uniformly on compacts. Hence on compacts, $\nu_\phi(\Gamma_\varepsilon(\cdot))$ converges uniformly to $\nu_\phi(\Gamma(\cdot))$. The above integral converges as $\varepsilon \to 0$ to

$$\begin{aligned}\int_{\mathbb{R}} -\partial_s g(\Gamma(s))\nu_\phi(\Gamma(s))\,\mathrm{d}s &= \int_{\mathbb{R}} g(\Gamma(s))\,\mathrm{d}\nu_\phi(\Gamma(s)) \\ &= \int_{\mathbb{R}} -g(\Gamma(s))\,\mathrm{d}\tau(s),\end{aligned}$$

since $g(\Gamma(\cdot))$ is at least C^1 and $\nu_\phi(\Gamma(\cdot))$ is C^α in the support of $g(\Gamma(\cdot))$, the integration by parts in the first equality holds. In the second equality, we use $\mathrm{d}\nu_\phi(\Gamma(s)) = -\,\mathrm{d}\tau(s)$. □

Now we would like to apply Lemma 4.9 to the special case of the slit domain $\Sigma\backslash\gamma$ where $\gamma\cup\mathbb{R}_+$ is at least $C^{1,\alpha}$, $\alpha > 0$. A little bit of caution is needed because this is not a $C^{1,\alpha}$ domain. However, Corollary 4.7 shows that the completion of γ by conformal geodesic connecting $\gamma(T)$ and ∞ in $\Sigma\backslash\gamma$ is $C^{1,\beta}$ for some $0<\beta<1/2$. The complement of $\tilde\gamma\cup\mathbb{R}_+$ has two connected components H_1 and H_2, both are unbounded $C^{1,\beta}$ domains. In fact, the regularity of $\tilde\gamma\cup\mathbb{R}_+$ at ∞ (after being mapped to a finite point via Möbius transformation) can be easily computed and is at least $C^{1,1/2}$ ([MR07] Proposition 3.12). And the mapping-out function $h = h_T$ maps both domains to $\mathbb{H}$ and the lower-half plane $\mathbb{H}^*$ respectively.

We parametrize $\Gamma = \tilde\gamma\cup\mathbb{R}_+$ by arclength such that $\Gamma(0)=0$ and consider it as the boundary of H_1 (so that H_1 is on the left-hand side of Γ), we denote by $\tilde\Gamma(s) = \Gamma(-s)$ the arclength-parametrized boundary of H_2 (see Figure 4.1).

For a domain $D\subset\mathbb{C}$, we introduce the space of smooth functions with finite Dirichlet energy:

$$\mathscr{D}^\infty(D) := \left\{ g\in C^\infty(D),\ \int_D |\nabla g(z)|^2\,\mathrm{d}z^2 < \infty \right\}.$$

Proposition 4.11. *If a finite capacity curve γ in $(\Sigma, 0, \infty)$ satisfies:*

- $\gamma \cup \mathbb{R}_+$ *is* $C^{1,\alpha}$ *for some* $\alpha > 0$*,*
- σ_h *is in* $\mathscr{D}^\infty(\Sigma \backslash \gamma)$*.*

Then for all $g \in \mathscr{D}^\infty(\Sigma)$*,*

$$\int_{\Sigma\backslash\gamma} \nabla g(z) \cdot \nabla \sigma_h(z) \, \mathrm{d}z^2 = 0.$$

Proof. We have already seen that H_1 and H_2 are $C^{1,\beta}$ domains for some $\beta > 0$.

Assume first that $g \in \mathscr{D}^\infty(\Sigma)$ is compactly supported (in $\mathbb{C}$) and that both $g|_{H_1}$ and $g|_{H_2}$ can be extended to $C^\infty(\overline{H_1})$ and $C^\infty(\overline{H_2})$ (with possibly different values along $\mathbb{R}_+$), then Lemma 4.9 applies:

$$\begin{aligned} \int_{\Sigma\backslash\gamma} \nabla g(z) \cdot \nabla \sigma_h(z) \, \mathrm{d}z^2 &= (\int_{H_1} + \int_{H_2}) \nabla g(z) \cdot \nabla \sigma_h(z) \, \mathrm{d}z^2 \\ &= -\int_{\mathbb{R}} g(\Gamma(s)) \, \mathrm{d}\tau(s) - \int_{\mathbb{R}} g(\tilde{\Gamma}(s)) \, \mathrm{d}\tilde{\tau}(s) \end{aligned}$$

where $\tau(s) = \arg(\Gamma'(s))$, and $\tilde{\tau}(s) = \arg(\tilde{\Gamma}'(s)) = \tau(-s) + \pi$.

Since $\Gamma(s) \in \Sigma$ for $s < 0$, and $\mathrm{d}\tau(s) = 0$ for $s \geq 0$, it follows that this quantity is also equal to

$$\begin{aligned} &-\int_{-\infty}^{0} g(\Gamma(s)) \, \mathrm{d}\tau(s) - \int_{0}^{+\infty} g(\Gamma(-s)) \, \mathrm{d}\tau(-s) \\ =& -\int_{-\infty}^{0} g(\Gamma(s)) \, \mathrm{d}\tau(s) - \int_{0}^{-\infty} g(\Gamma(t)) \, \mathrm{d}\tau(t) = 0. \end{aligned}$$

The conclusion then follows from the density of compactly supported functions in $\mathscr{D}^\infty(\Sigma)$ and the assumption $\sigma_h \in \mathscr{D}^\infty(\Sigma \backslash \gamma)$. □

We are now ready to state and prove the J-additivity for sufficiently smooth curves: Let h_t be the mapping-out function of $\gamma[0,t]$ as in the proof of Proposition 4.11. We denote for $s < t$, $h_{t,s} = h_t \circ h_s^{-1}$, the mapping-out function of $h_s(\gamma[s,t])$.

Proposition 4.12 (Weak J-Additivity)**.** *If γ is a simple curve in $(\Sigma, 0, \infty)$ such that $\gamma \cup \mathbb{R}_+$ is $C^{1,\alpha}$. For $0 \le s < t \le T$, if both $J(h_s)$ and $J(h_{t,s})$ are finite, then $J(h_t) = J(h_s) + J(h_{t,s})$.*

Proof. Let $\gamma := \gamma[0,t]$, $\hat{\gamma} := h_s(\gamma[s,t])$. We write $\sigma_r(z) = \log|h_r'(z)|$ and $\sigma_{t,s}(z) = \log\left|h_{t,s}'(z)\right|$. From

$$\sigma_t(z) = \log|h_t'(z)| = \log|(h_{t,s} \circ h_s)'(z)| = \sigma_{t,s}(h_s(z)) + \sigma_s(z),$$

we deduce

$$\begin{aligned}\pi J(h_t) = \pi J(h_s) &+ \int_{\Sigma\setminus\gamma} \left|\nabla\sigma_{t,s}(h_s(z))\right|^2 \mathrm{d}z^2 \\ &+ 2\int_{\Sigma\setminus\gamma} \nabla\sigma_s(z)\cdot\nabla\sigma_{t,s}(h_s(z))\,\mathrm{d}z^2.\end{aligned}$$

The second term on the right-hand side equals to $\pi J(h_{t,s})$ by the conformal invariance of the Dirichlet energy. Now we show that the third term vanishes. We write it in a slightly different way: it is equal to

$$\begin{aligned}&\int_{\Sigma\setminus\gamma} -\nabla\sigma_{h_s^{-1}}(h_s(z))\cdot\nabla\sigma_{t,s}(h_s(z))\,\mathrm{d}z^2 \\ =&\int_{\Sigma\setminus\hat{\gamma}} -\nabla\sigma_{h_s^{-1}}(y)\cdot\nabla\sigma_{t,s}(y)\,\mathrm{d}y^2.\end{aligned}$$

Since $J(h_s)$ equals to the Dirichlet energy of σ_s, also of $\sigma_{h_s^{-1}}$, from the assumption, $\sigma_{h_s^{-1}} \in \mathscr{D}^\infty(\Sigma)$, $\sigma_{t,s} \in \mathscr{D}^\infty(\Sigma\setminus\hat{\gamma})$, and $\hat{\gamma}\cup\mathbb{R}_+$ is at least $C^{1,\beta}$ with the same β as in Corollary 4.7, the result then follows from Proposition 4.11. ☐

4.4 Piecewise linear driving functions

Let us first prove the identity of the Loewner energy of γ with Dirichlet energy of σ_h in the special case of a curve driven by a linear function: Let γ be the Loewner chain in $(\Sigma, 0, \infty)$ driven by the function W :

$[0,T] \to \mathbb{R}$, where $W(t) = \lambda t$ for some $\lambda \in \mathbb{R}$. We denote (f_t) the centered Loewner flow in $\mathbb{H}$ driven by W and (h_t) the Loewner flow in Σ. They are related by

$$h(t) = f_t^2(\sqrt{z}).$$

In particular the mapping-out function $h = h_T$. For $z = \Gamma(s)$ on the boundary Γ of the domain, we put $\tau(z) := \arg(\Gamma'(s))$.

Proposition 4.13. *Proposition 4.5 holds when γ is driven by a linear function.*

First notice that the function $W(t) = \lambda t$ for $t \geq 0$ and $W(t) = 0$ for $t \leq 0$ is $C^{0,1}$. Therefore, $\gamma \cup \mathbb{R}_+$ is $C^{1,\alpha}$ for $\alpha < 1/2$ by Theorem 4.6. Once we have shown that $J(h_\varepsilon) < \infty$ for some $\varepsilon > 0$, the weak J-additivity (Proposition 4.12) applies. We can note that the J-additivity and the I-additivity imply that $J(h_T)$ and $I(\gamma[0,T])$ are both linear with respect to T, so that it will suffice to check that $I(\gamma[0,T]) \sim J(h_T)$ as $T \to 0$.

Proof. Notice that γ is in fact C^∞ curve and it is only in the neighborhood of 0 the regularity of $\gamma \cup \mathbb{R}_+$ is $C^{1,\alpha}$. Hence, σ_h is C^∞ up to the boundary apart from 0. First we show that the Stokes' formula applies and $J(h)$ equals to the integral on the boundary:

$$J(h) = -\frac{1}{\pi} \int_{\partial(\Sigma \setminus \gamma)} \sigma_h(z) \,\mathrm{d}\tau(z), \tag{4.4}$$

where the slit $\gamma \cup \mathbb{R}_+$ is counted twice in $\partial(\Sigma\setminus\gamma)$ with opposite orientation. The above integral is interpreted as the limit when $\varepsilon \to 0$ of $\int_{\partial(\Sigma\setminus\gamma)\setminus B(0,\varepsilon)} \sigma_h(z) \,\mathrm{d}\tau(z)$. And away from 0, both τ and σ_h are C^∞ so that the integral is well-defined.

We need to be careful at γ_T and 0 where the boundary is not regular enough to apply the Stokes' formula. The singularity at γ_T is actually simple to deal with: We extend γ to a C^∞ curve $\overline{\gamma}$ going to ∞, since σ_h is continuous across $\overline{\gamma}\setminus\gamma$, and $\mathrm{d}\tau(z)$ has opposite sign on both side of $\overline{\gamma}$, the sum of the integrals on both copies of $\overline{\gamma}\setminus\gamma$ cancels out.

Moreover, $J(h) = J(h|_{\Sigma\setminus\overline{\gamma}})$, so it suffices to check that the singularity at 0 does not affect the application of the Stokes' formula.

We will use the Loewner flow to control the behavior of $\nabla\sigma_h$ near 0. The centered forward Loewner flow $f_t(\cdot) := g_t(\cdot) - W(t)$ of the simple curve $\sqrt{\gamma}$ in $\mathbb{H}$ driven by $W(t) = \lambda t$ satisfies for $z \in \Sigma$,

$$\partial_t f_t(z) = 2/f_t(z) - W'(t) = 2/f_t(z) - \lambda.$$

The mapping-out function $h_t = f_t^2(\sqrt{z})$ for $\gamma[0,t]$ satisfies

$$\partial_t h_t(z) = 2f_t(\sqrt{z})(2/f_t(z) - \lambda) = 4 - 2\lambda f_t(\sqrt{z}).$$

Taking derivatives in z,

$$h_t'(z) = f_t(\sqrt{z})f_t'(\sqrt{z})/\sqrt{z} \text{ and } \partial_t h_t'(z) = -\lambda f_t'(\sqrt{z})/\sqrt{z}.$$

Using the short-hand σ_t for σ_{h_t} and σ_T for σ_h, one gets

$$\partial_t \sigma_t(z) = \operatorname{Re}\left(\partial_t h_t'(z)/h_t'(z)\right) = -\lambda \operatorname{Re}(1/f_t(\sqrt{z})).$$

Therefore for $z \in \mathbb{H}$,

$$\begin{aligned}
\sigma_t(z) &= -\lambda \operatorname{Re}\left(\int_0^t \frac{1}{f_r(\sqrt{z})}\right) \mathrm{d}r \\
&= \frac{-\lambda}{2} \operatorname{Re}\left(\int_0^t \partial_r f_r(\sqrt{z}) + \partial_r W_r \,\mathrm{d}r\right) \\
&= -\frac{\lambda}{2}\Big(\lambda t + \operatorname{Re}(f_t(\sqrt{z})) - \operatorname{Re}(\sqrt{z})\Big).
\end{aligned}$$

In particular as $z \to 0$,

$$\begin{aligned}
|\nabla\sigma_T(z)| &= \left|\frac{\lambda}{2}\left(\frac{f_T'(\sqrt{z})}{2\sqrt{z}} - \frac{1}{2\sqrt{z}}\right)\right| \\
&= \left|\frac{\lambda}{2}\left(\frac{h_T'(\sqrt{z})}{2f_T(\sqrt{z})} - \frac{1}{2\sqrt{z}}\right)\right| = O\left(\frac{1}{|\sqrt{z}|}\right)
\end{aligned}$$

since h' is bounded on the closure of $C^{1,\alpha}$ domain and $f_T(\sqrt{z})$ is bounded away from 0 as $z \to 0$. It shows that $\|\nabla \sigma_T\|_{L^2(B(0,\varepsilon))} \to 0$ and the integral of $\sigma_T \partial_n \sigma_T$ along a smooth arc of length ε inside $B(0,\varepsilon)$ go to 0 as $\varepsilon \to 0$. Hence for every $\delta > 0$, there exists $\varepsilon > 0$ and a sub-domain Σ_ε of $\Sigma \backslash \gamma$ with smooth boundary which coincides with $\Sigma \backslash \gamma$ outside of $B_\varepsilon(0)$, such that $\varepsilon \to 0$ when $\delta \to 0$,

$$\left| J(h_T) - \frac{1}{\pi} \int_{\Sigma_\varepsilon} |\nabla \sigma_T|^2 \, \mathrm{d}z^2 \right| \leq \delta,$$

and

$$\left| \int_{\partial \Sigma_\varepsilon} \sigma_T(z) \partial_n \sigma_T(z) \, \mathrm{dl}(z) - \int_{\partial(\Sigma \backslash \gamma) \backslash B(0,\varepsilon)} \sigma_T(z) \partial_n \sigma_T(z) \, \mathrm{dl}(z) \right|$$

is also bounded by $\pi\delta$. It then suffices to apply Stokes' formula on Σ_ε. We control the decay of $\nabla \sigma_T$ as $z \to \infty$: take the gradient of the expression of $\partial_t \sigma_t$, one gets:

$$|\partial_t \nabla \sigma_t(z)| = \left| \frac{\lambda f_t'(\sqrt{z})}{2 f_t^2(\sqrt{z})\sqrt{z}} \right| = O\left(|z|^{-3/2} \right)$$

which implies

$$|\nabla \sigma_T(z)| = O(|z|^{-3/2}). \tag{4.5}$$

It allows us to apply the Stokes' formula (one can look at the integral on $\Sigma_\varepsilon \cap B(0,R)$ and see that the contribution of the contour integral on $\partial B(0,R)$ goes to 0 as $R \to \infty$) together with the harmonicity of σ_T:

$$\int_{\Sigma_\varepsilon} |\nabla \sigma_T|^2 \, \mathrm{d}z^2 = \int_{\partial \Sigma_\varepsilon} \sigma_T(z) \partial_n \sigma_T(z) \, \mathrm{dl}(z),$$

which yields

$$\left| J(h_T) - \int_{\partial(\Sigma \backslash \gamma) \backslash B(0,\varepsilon)} \sigma_T(z) \partial_n \sigma_T(z) \, \mathrm{dl}(z) \right| \leq 2\delta.$$

Using $\partial_n \sigma_T(z) = -\partial_s \tau(z)$ on the smooth boundary of Σ_ε, then let

$\varepsilon \to 0$ and $\delta \to 0$, we obtain (4.4).

Now we prove the identity

$$I(\gamma) = -\frac{1}{\pi} \int_{\partial(\Sigma\setminus\gamma)} \sigma_h(z) \, \mathrm{d}\tau(z).$$

Similar to the computation of $\sigma_t(z)$, $\nu_t(z) := \operatorname{Im} \log(h_t'(z))$ satisfies

$$\nu_t(z) = -\frac{\lambda}{2} \operatorname{Im} \int_0^t \partial_r f_r(\sqrt{z}) \, \mathrm{d}r = -\frac{\lambda}{2} \Big(\operatorname{Im}(f_t(\sqrt{z})) - \operatorname{Im}(\sqrt{z}) \Big).$$

We use the notations of Γ and $\tilde{\Gamma}$ as in the description prior to Proposition 4.11 to distinguish the two copies of $\gamma \cup \mathbb{R}_+$ as parts of the boundary. We also keep in mind that γ is capacity parametrized and Γ is arclength-parametrized. Let S be the total length of $\gamma[0, T]$. A point γ_t on γ can be considered as a point in both Γ and $\tilde{\Gamma}$, and there is $s \geq 0$, such that $\gamma_t = \Gamma(-s) = \tilde{\Gamma}(s)$. We deduce from the expression of ν_t, that for $0 \leq s \leq S$,

$$\tau(\Gamma(-s)) = -\nu_T(\gamma_t) = -\frac{\lambda}{2} \operatorname{Im}(\sqrt{\gamma_t}),$$

$$\mathrm{d}\tau(\Gamma(-s)) = \frac{\lambda}{2} \operatorname{Im}(\partial_t \sqrt{\gamma_t}) \, \mathrm{d}t.$$

Similarly,

$$\tau(\tilde{\Gamma}(s)) = -\nu_T(\gamma_t) + \pi = -\frac{\lambda}{2} \operatorname{Im}(\sqrt{\gamma_t}) + \pi,$$

$$\mathrm{d}\tau(\tilde{\Gamma}(s)) = -\frac{\lambda}{2} \operatorname{Im}(\partial_t \sqrt{\gamma_t}) \, \mathrm{d}t.$$

Hence the integral in (4.4) equals to

$$\begin{aligned} J(h) =& -\frac{1}{\pi}\int_{\gamma_t\in\Gamma}\Big(-\frac{\lambda}{2}\operatorname{Re}(f_T(\sqrt{\gamma_t}))\Big)\frac{\lambda}{2}\operatorname{Im}(\partial_t\sqrt{\gamma_t})\,\mathrm{d}t \\ & -\frac{1}{\pi}\int_{\gamma_t\in\tilde\Gamma}\Big(-\frac{\lambda}{2}\operatorname{Re}(f_T(\sqrt{\gamma}_t))\Big)\frac{\lambda}{2}\operatorname{Im}(-\partial_t\sqrt{\gamma_t})\,\mathrm{d}t \\ =& \frac{\lambda^2}{4\pi}\int_0^T\Big(f_{T-t}(0^+)-f_{T-t}(0^-)\Big)\operatorname{Im}\left(\partial_t\sqrt{\gamma_t}\right)\,\mathrm{d}t. \end{aligned}$$

The second equality holds because of the linearity of the driving function, and $s\mapsto f_s(0^+)>0$ and $s\mapsto f_s(0^-)<0$ are respectively the two Loewner flows starting from 0. We also know that $\sqrt{\gamma_t}$ satisfies the backward Loewner equation, that is for $t\in(0,T]$,

$$\partial_t\sqrt{\gamma_t} = -2/\sqrt{\gamma_t}+\lambda. \tag{4.6}$$

In fact, for a fixed $t\in[0,T]$, $\sqrt{\gamma_t}$ can be computed as follows. Consider the reversed driving function $\beta^t:[0,t]\to\mathbb{R}$ defined as

$$\beta^t(s) := W(t)-W(t-s).$$

The reversed Loewner flow starting from $z\in\mathbb{H}$ is the solution $[0,t]\to\mathbb{H}$ to the differential equation:

$$\partial_s Z_s^t(z) = -2/Z_s^t(z)+\dot\beta^t(s) \quad \text{for } s\in[0,t], \tag{4.7}$$

with initial condition $Z_0^t(z)=z$ and we have $\lim_{y\downarrow 0} Z_t^t(iy)=\sqrt{\gamma_t}$ (see [RS05]). Since $\dot\beta^t(s)=\lambda$ for all $0\le s\le t\le T$, we have for any $t\le T$, $Z_\cdot^t(iy)=Z_\cdot^T(iy)$ on $[0,t]$. In particular

$$\sqrt{\gamma_t} = \lim_{y\downarrow 0} Z_t^t(iy) = \lim_{y\downarrow 0} Z_t^T(iy).$$

For $t\in(0,T]$, the limit commutes with differentiation in (4.7) which then gives (4.6). (See also [STW17] for the approach considering the singular differential equation (4.7) starting directly from 0 when W is sufficiently regular.)

From the explicit computation of the Loewner flow driven by a linear function in [KNK04], we have the asymptotic expansions as $t \to 0$:

$$f_t(0^+) = 2\sqrt{t} + O(t), \quad \sqrt{\gamma_t} = 2i\sqrt{t} + O(t).$$

Hence as $T \to 0$,

$$\begin{aligned}
&\Big(f_{T-t}(0^+) - f_{T-t}(0^-)\Big)\,\mathrm{Im}\,(\partial_t\sqrt{\gamma_t}) \\
=&\Big(f_{T-t}(0^+) - f_{T-t}(0^-)\Big)\,\mathrm{Im}\,(-2/\sqrt{\gamma_t}) \\
=&\frac{4\sqrt{T-t}}{\sqrt{t}}(1+O(\sqrt{T})),
\end{aligned}$$

which yields

$$\begin{aligned}
J(h_T) &= (1+O(\sqrt{T}))\frac{\lambda^2}{\pi}\int_0^T \sqrt{T-t}/\sqrt{t}\,\mathrm{d}t \\
&= (1+O(\sqrt{T}))\frac{\lambda^2 T}{\pi}\int_0^1 \sqrt{1-t}/\sqrt{t}\,\mathrm{d}t \\
&= \frac{\lambda^2}{2}(T+O(T^{3/2})).
\end{aligned}$$

By the weak J-additivity and again the linearity of W, one gets $J(h_T) = \lambda^2 T/2$ for any $T \geq 0$. In fact,

$$J(h_T) = TnJ(h_{1/n}) = Tn\frac{\lambda^2}{2}\left(n^{-1} + O(n^{-3/2})\right)$$

converges to $\frac{\lambda^2}{2}T = I(W)$ as $n \to \infty$. □

The weak J-additivity, the I-additivity and Proposition 4.13 do immediately imply the following fact:

Corollary 4.14. *Proposition 4.5 holds when γ is driven by a piecewise linear function.*

4.5 Proof of Proposition 4.5

We now want to deduce Proposition 4.5 from Corollary 4.14 the result for general curves approximations by curves. We give first the following lemma on the lower semi-continuity which is the key tool here:

Lemma 4.15. *If $T < \infty$, $(W^{(n)})_{n\geq 1}$ is a sequence of driving functions defined on $[0,T]$, that converges uniformly to W. Then*

$$J(h) \leq \liminf_{n\to\infty} J(h^{(n)}),$$

where $h^{(n)}$ is the mapping-out function of $\gamma^{(n)}$, the Loewner chain in $(\Sigma, 0, \infty)$ driven by $W^{(n)}$ and h the mapping-out function of γ, driven by W.

Proof. Let $\varphi = h^{-1}$ be the inverse map of the mapping-out function of γ and $\varphi^{(n)} = (h^{(n)})^{-1}$. Since $W^{(n)}$ converges uniformly to W, $\gamma^{(n)}$ converges to γ in the Carathéodory topology, that is the uniform convergence on compacts of $\varphi^{(n)}$ to φ. We have also that

$$\left|\nabla\sigma_{\varphi^{(n)}}(z)\right|^2 = \left|\frac{\varphi^{(n)}(z)''}{\varphi^{(n)}(z)'}\right|^2$$

converges uniformly on compacts to $\left|\nabla\sigma_{\varphi}(z)\right|^2$. Hence

$$\begin{aligned}\liminf_{n\to\infty} J(\varphi^{(n)}) &= \liminf_{n\to\infty} \sup_{K\subset\Sigma} \frac{1}{\pi}\int_K \left|\nabla\sigma_{\varphi^{(n)}}(z)\right|^2 \,\mathrm{d}z^2 \\ &\geq \sup_{K\subset\Sigma} \liminf_{n\to\infty} \frac{1}{\pi}\int_K \left|\nabla\sigma_{\varphi^{(n)}}(z)\right|^2 \,\mathrm{d}z^2 \\ &= \sup_{K\subset\Sigma} \frac{1}{\pi}\int_K \left|\nabla\sigma_{\varphi}(z)\right|^2 \,\mathrm{d}z^2 = J(\varphi),\end{aligned}$$

where the supremum is taken over all compacts in Σ. Then we conclude with (4.2). □

We have the following corollary which gives us the finiteness of J-energy when the Loewner energy is finite.

Corollary 4.16. *If γ driven by W has finite Loewner energy in $(\Sigma, 0, \infty)$ and finite total capacity T, then $J(h) \leq I(\gamma)$. In particular, $\sigma_h \in \mathscr{D}^\infty(\Sigma \backslash \gamma)$.*

Proof. We find a sequence of piecewise linear functions $W^{(n)}$ such that $W^{(n)}$ converges to W uniformly and

$$I(W^{(n)} - W) = \frac{1}{2} \int_0^T \left| W'^{(n)}(t) - W'(t) \right|^2 \mathrm{d}t \xrightarrow{n \to \infty} 0.$$

This is possible since the family of step functions is dense in $L^2([0,T])$. Thus we can find a sequence of step functions Y_n which converges to W' in L^2, and define $W^{(n)}(t) = \int_0^t Y_n(t)\, \mathrm{d}t$. The convergence is also uniform since

$$\begin{aligned} \left| W^{(n)}(t) - W(t) \right| &\leq \int_0^t \left| W'^{(n)}(s) - W'(s) \right| \mathrm{d}s \\ &\leq \sqrt{T} \sqrt{2I(W^{(n)} - W)} \end{aligned}$$

by Cauchy-Schwarz inequality. Lemma 4.15 and Corollary 4.14 imply that

$$I(\gamma) = I(W) = \lim_{n \to \infty} I(W^{(n)}) = \lim_{n \to \infty} J(h^{(n)}) \geq J(h)$$

as desired. □

Given the finiteness of the J-energy, one can improve the J-additivity proposition by dropping the regularity condition on γ. The following lemma is a stronger version of Proposition 4.11 by assuming only the finiteness of Loewner energy of γ.

Lemma 4.17. *If γ is a Loewner chain in $(\Sigma, 0, \infty)$ with finite Loewner energy and finite total capacity. Then for all $g \in \mathscr{D}^\infty(\Sigma)$,*

$$\int_{\Sigma \backslash \gamma} \nabla g(z) \cdot \nabla \sigma_h(z)\, \mathrm{d}z^2 = 0. \tag{4.8}$$

Proof. Take the same approximation of the driving function W of γ by a family of piecewise linear driving functions $W^{(n)}$ as in Corollary 4.16. Let $\gamma^{(n)}$ be the curve driven by $W^{(n)}$. Let $A = \sup_{n\geq 1} I(\gamma^{(n)}) \geq I(\gamma)$. We may assume that $A < \infty$. Corollary 4.16 implies that $J(h) \leq A$. Moreover, from Corollary 2.4, every subsequence of $\gamma^{(n)}$ has a subsequence that converges uniformly to γ as capacity-parametrized curves, due to the fact that they are all k-quasiconformal curve with k depending only on A. Hence, the uniform convergence of $\gamma^{(n)}$ is on the whole sequence.

Since $\gamma^{(n)}$ are all $C^{1,\alpha}$ for $\alpha < 1/2$, let $h^{(n)}$ be the mapping-out function of $\gamma^{(n)}$, one has

$$\int_{\Sigma\setminus\gamma^{(n)}} \nabla g(z) \cdot \nabla \sigma_{h^{(n)}}(z)\, \mathrm{d}z^2 = 0,$$

by Proposition 4.11. Since g and σ_h are in $\mathscr{D}^\infty(\Sigma\setminus\gamma)$, for every $\varepsilon > 0$, there exists a compact set $K \subset \Sigma\setminus\gamma$, such that

$$\int_{(\Sigma\setminus\gamma)\setminus K} |\nabla g(z)|^2 \, \mathrm{d}z^2 \leq \varepsilon,$$

which implies

$$\int_{(\Sigma\setminus\gamma)\setminus K} \nabla g(z) \cdot \nabla \sigma_h(z)\, \mathrm{d}z^2 \leq \sqrt{\pi A \varepsilon}$$

by Cauchy-Schwarz inequality. It holds also for $\sigma_{h^{(n)}}$. As $\gamma^{(n)}$ converges uniformly to γ, for n large enough, $\gamma^{(n)} \cap K = \emptyset$ and $h^{(n)}$ converges uniformly to h on K (Carathéodory convergence [Dur83] Thm. 3.1). It yields that

$$|\nabla\sigma_h(z) - \nabla\sigma_{h^{(n)}}(z)| = \left|\frac{h''}{h'}(z) - \frac{(h^{(n)})''}{(h^{(n)})'}(z)\right| \xrightarrow[n\to\infty]{\text{unif. on } K} 0.$$

Hence,

$$\begin{aligned}
&\left|\int_{\Sigma\setminus\gamma} \nabla g(z)\cdot\nabla\sigma_h(z)\,\mathrm{d}z^2\right| \\
&=\left|\int_{\Sigma\setminus\gamma} \nabla g(z)\cdot\nabla\sigma_h(z)\,\mathrm{d}z^2 - \int_{\Sigma\setminus\gamma^{(n)}} \nabla g(z)\cdot\nabla\sigma_{h^{(n)}}(z)\,\mathrm{d}z^2\right| \\
&\le\left|\int_{K} \nabla g(z)\cdot\nabla\sigma_h(z)\,\mathrm{d}z^2 - \int_{K} \nabla g(z)\cdot\nabla\sigma_{h^{(n)}}(z)\,\mathrm{d}z^2\right| \\
&\quad + 2\sqrt{\pi A\varepsilon} \xrightarrow[n\to\infty]{} 2\sqrt{\pi A\varepsilon}.
\end{aligned}$$

Letting $\varepsilon \to 0$, we get (4.8). $\square$

We then deduce the strong J-additivity from Lemma 4.17 using the same notation as in Proposition 4.12:

Corollary 4.18 (Strong J-additivity)**.** *If γ has finite Loewner energy, then $J(h_t) = J(h_s) + J(h_{t,s})$ for $0 \le s \le t \le T$.*

Proof. From Corollary 4.16, we have $J(h_s) \le \int_0^s W'(r)^2/2\,\mathrm{d}r$ and $J(h_{t,s}) \le \int_s^t W'(r)^2/2\,\mathrm{d}r$. They are in particular finite when $I(\gamma)$ is finite. The proof then follows exactly the same line as the weak J-additivity, by applying Lemma 4.17 with $g = \sigma_{h_s^{-1}}$. $\square$

Now we have all the ingredients for proving Proposition 4.5.

Proof. Given Corollary 4.16, we only need to prove $J(h) \ge I(\gamma)$.

Consider two functions

$$a(t) := J(h_t) \text{ and } b(t) := \frac{1}{2}\int_0^t W'(s)^2\,\mathrm{d}s = I(\gamma[0,t]).$$

Both of them satisfy the respective additivity. From the definition of absolutely continuous function, $b(\cdot)$ is AC on $[0,T]$. By the additivity, Corollary 4.16 and (AC1), $a(\cdot)$ is also AC function. Thus (AC2) implies that on a full Lebesgue measure set S, the functions $a(\cdot)$, $b(\cdot)$ and

$W(\cdot)$ are differentiable and $b'(t) = W'(t)^2/2$. Corollary 4.16 shows in particular $a'(t) \le b'(t)$. Now it suffices to show that $b'(t) \le a'(t)$ for $t \in S$.

By additivity, without loss of generality, we assume that $t = 0$ and $T = 1$. Consider $W^{(n)}$ obtained by concatenating n copies of $W[0, 1/n]$, that is

$$W^{(n)}(t) = \lfloor tn \rfloor W(1/n) + W(t - \lfloor tn \rfloor /n), \quad \forall t \in [0, 1].$$

We show that $I(W^{(n)})$ converges to $I(W^\infty)$, where W^∞ is the linear function $t \mapsto tW'(0)$. In fact,

$$I(W^{(n)}) = nb(1/n) \xrightarrow[n\to\infty]{} b'(0) = W'(0)^2/2 = I(W^\infty).$$

We have also $W^{(n)}$ converges uniformly to W^∞. In fact, since W is differentiable at 0, for every $\varepsilon > 0$, there exists n_0, such that for all $n \ge n_0$, for all $t \le 1/n$,

$$|W(t) - W'(0)t| \le \varepsilon/n.$$

Hence for $t \in [0, 1]$,

$$\begin{aligned}
&\left|W^{(n)}(t) - tW'(0)\right| \\
&\le \left|W^{(n)}(\lfloor tn \rfloor/n) - W'(0)\lfloor tn \rfloor/n\right| + |W(\delta) - \delta W'(0)| \\
&= \lfloor tn \rfloor \left|W^{(n)}(1/n) - (1/n)W'(0)\right| + |W(\delta) - \delta W'(0)| \\
&\le \varepsilon(tn + 1)/n \le 2\varepsilon,
\end{aligned}$$

where $\delta = t - \lfloor tn \rfloor/n$.

The uniform convergence of driving function and Lemma 4.15 imply that

$$J(h^\infty) \le \liminf_{n\to\infty} J(h^{(n)}) = \liminf_{n\to\infty} na(1/n) = a'(0),$$

where h^∞ is the mapping-out function generated by W^∞, $h^{(n)}$ is generated by $W^{(n)}$. The first equality comes from the J-additivity.

From Proposition 4.13,

$$J(h^\infty) = I(W^\infty) = |W'(0)|^2 /2 = b'(0)$$

which yields $b'(0) \le a'(0)$ and concludes the proof. $\square$

4.6 The Loop Loewner Energy

The generalization of the chordal Loewner energy to loops is first studied in [RW17] and the goal in this section is to derive the loop energy identity Theorem 4.19. Let γ be a Jordan curve on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, that is parametrized by a continuous 1-periodic function that is injective on $[0,1)$. The *Loewner loop energy* of γ rooted at $\gamma(0)$ is given by

$$I^L(\gamma, \gamma(0)) := \lim_{\varepsilon \to 0} I_{\hat{\mathbb{C}} \setminus \gamma[0,\varepsilon], \gamma(\varepsilon), \gamma(0)}(\gamma[\varepsilon, 1]).$$

We use the abbreviation $I_{\gamma[0,\varepsilon]}$ in the future for $I_{\hat{\mathbb{C}} \setminus \gamma[0,\varepsilon], \gamma(\varepsilon), \gamma(0)}$. From the definition, the loop energy is conformally invariant (i.e. invariant under Möbius transformations): if $\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius function, then

$$I^L(\gamma, \gamma(0)) = I^L(\mu(\gamma), \mu(\gamma)(0)).$$

Moreover, the loop energy vanishes only on circles ([RW17] Section 2.2).

The loop energy can be expressed in terms of the driving function as well: we first define the driving function of an embedded arc in $\hat{\mathbb{C}}$ rooted at one tip of the arc. *An embedded arc* is the image of an injective continuous function $\gamma : [0,1] \to \hat{\mathbb{C}}$. We parametrize the arc by the capacity seen from $\gamma(0)$ as follows (and the capacity parametrized arc is denoted as $t \mapsto \Gamma(t)$):

- Choose first a point $\gamma(s_0)$ on γ, for some $s_0 \in (0,1]$. Define $\Gamma(0)$ to be $\gamma(s_0)$.
- Choose a uniformizing conformal mapping ψ_{s_0} from the complement of $\gamma[0, s_0]$ onto $\mathbb{H}$, such that $\psi_{s_0}(\gamma(s_0)) = 0$ and $\psi_{s_0}(\gamma(0)) = \infty$.

- Define the conformal mapping ψ_s from the complement of $\gamma[0,s]$ onto $\mathbb{H}$ to be the unique mapping such that the tip $\gamma(s)$ is mapped to 0, $\gamma(0)$ to ∞, and $\psi_s \circ \psi_{s_0}^{-1}(z) = z + O(1)$ as $z \to \infty$.
- Set $\gamma(s) = \Gamma(t)$ if the development of $\psi_s \circ \psi_{s_0}^{-1}$ at ∞ is actually

$$\psi_s \circ \psi_{s_0}^{-1}(z) = z - W(t) + 2t/z + o(1/z),$$

  for some $W(t) \in \mathbb{R}$ and $2t$ is called the *capacity* of $\gamma[0,s]$ seen from $\gamma(0)$, relatively to $\gamma(s_0)$ and ψ_{s_0}. The capacity parametrization $s \mapsto t$ is increasing and has image $(-\infty, T]$ for some $T \in \mathbb{R}_+$. We set $\Gamma(-\infty) = \gamma(0)$.
- We define $h_t := \psi_s^2$ to be the *mapping-out function* of $\gamma[0,s]$, which maps the complement of $\gamma[0,s]$ to the complement of $\mathbb{R}_+$ such that $h_t(\gamma(0)) = \infty$ and $h_t(\gamma(s)) = 0$.
- The continuous function W defined on $(-\infty, T]$ is called again the *driving function* of the arc γ.
- The *Loewner arc energy* of γ is the Dirichlet energy of W which is

$$I^A(\gamma, \gamma(0)) = \int_{-\infty}^{T} W'(t)^2/2 \,\mathrm{d}t = \lim_{\varepsilon \to 0} I_{\gamma[0,\varepsilon]}(\gamma[\varepsilon, 1]).$$

Notice that the capacity parametrization t, h_t and $W(t)$ depend on the choice of s_0 and ψ_{s_0}. A different choice of s_0 and ψ_{s_0} changes the driving function to

$$\tilde{W}(t) = W(\lambda^2(t+a))/\lambda - W(\lambda^2 a)/\lambda, \tag{4.9}$$

for some $\lambda > 0$ and $a \in \mathbb{R}$. However, the Dirichlet energy of W is invariant under such transformations.

From the definition, as $T \to \infty$, the arc targets at its root to form a loop which allows us to define the driving function of a simple loop γ embedded in $\hat{\mathbb{C}}$: we parametrize and define the arc driving function of $\gamma[0, 1-\varepsilon]$ seen from $\gamma(0)$ for every $0 < \varepsilon < 1/2$. With the same choice of s_0 and ψ_{s_0}, the driving functions of $\gamma[0, 1-\varepsilon]$ are consistent with respect to restrictions for all $\varepsilon > 0$. Hence as $\varepsilon \to 0$, $T \to \infty$

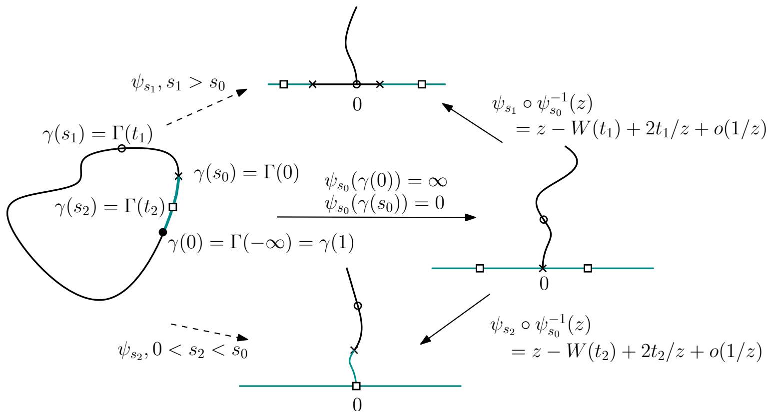

Figure 4.3: Illustration of the definition of loop driving function $W : \mathbb{R} \to \mathbb{R}$ and capacity reparameterization $\Gamma(t)$, where $0 < s_2 < s_0 < s_1$ correspond to the capacities $-\infty < t_2 < 0 < t_1$.

and we obtain the driving function $W : \mathbb{R} \to \mathbb{R}$ of the loop rooted at $\gamma(0)$. Given the root $\gamma(0)$ and the orientation of the parametrization, the driving function is defined modulo transformations in (4.9). The loop energy is therefore the Dirichlet energy of the driving function W which is invariant under those transformations.

It is clear that the loop energy depends *a priori* on the root $\gamma(0)$ and the orientation of the parametrization, since the change of root or orientation induces non-trivial changes on the driving function. However, the main result of [RW17] shows that the Loewner loop energy of γ only depends on the image of γ. In this section we prove of the identity (Theorem 4.19) that will give other approaches to the parametrization independence of the loop energy in Section 4.7 and 4.8. Although we do not presume the root-invariance of the loop energy, we sometimes omit the root in the expression of Loewner loop energy. In this case, the root is taken to be $\gamma(0)$.

From the conformal invariance of the Loewner energy, we may assume that γ is a simple loop on $\hat{\mathbb{C}}$ such that $\gamma(0) = \infty$ and passes through 0 and 1. The complement of γ has two unbounded connected components H_1 and H_2.

Theorem 4.19. *If γ has finite Loewner energy, then*

$$I^L(\gamma, \infty) = \frac{1}{\pi} \left(\int_{\mathbb{C} \setminus \gamma} |\nabla \sigma_h(z)|^2 \, \mathrm{d}z^2 \right),$$

where $h|_{H_1}$ (resp. $h|_{H_2}$) maps H_1 (resp. H_2) conformally onto a half-plane and fixes ∞.

Notice that the expression $J(h)$ on the right-hand side already does not depend on the orientation of the loop, but does *a priori* depend on the special point ∞ which is the root of γ.

We have mentioned in the introduction that the loop energy is a generalization of the chordal energy. In fact, consider the loop $\gamma = \mathbb{R}_+ \cup \eta$, where η is a simple chord in $(\Sigma, 0, \infty)$ from 0 to ∞, and we choose $\gamma(0) = \infty$, $\gamma(s_0) = 0$, $\psi_{s_0}(\cdot)$ to be $\sqrt{\cdot}$, the orientation such that $\gamma[0, s_0] = \mathbb{R}_+$. Then from the definition, the driving function of γ coincides with the driving function of η in $\mathbb{R}_+$ and is 0 in $\mathbb{R}_-$. Hence

$$I(\eta) = I^L(\eta \cup \mathbb{R}_+, \infty).$$

Theorem 4.1 follows immediately from Theorem 4.19.

As we described above, loops can be understood as embedded arcs with $T = +\infty$. For arcs which do not make it all the way back to its root ($T < \infty$), the mapping-out function h_T is a natural choice for the uniformizing function h. Let us first prove the analogous identity for an embedded arc.

Lemma 4.20. *If γ is a simple arc in $\hat{\mathbb{C}}$ such that $\gamma(0) = \infty$ with finite arc energy. Then*

$$J(h) = I^A(\gamma, \infty),$$

where $h = h_T$ is a mapping-out function of γ.

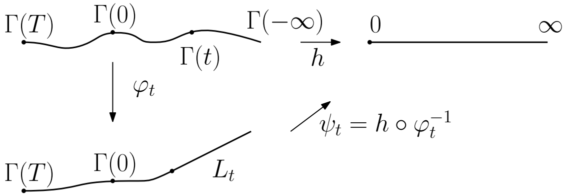

Figure 4.4: Conformal mappings in the proof of Lemma 4.20 where φ_t is defined in the complement of $\Gamma[-\infty, t]$ and h in the complement of $\Gamma[-\infty, T]$. Both of them map the tips to tips.

Proof. We will use the "blowing-up at the root" procedure to bring it back to the case of a finite capacity chord attached to $\mathbb{R}_+$. Let $\Gamma[-\infty, T] \to \hat{\mathbb{C}}$ be capacity reparametrization of γ and $\Gamma(-\infty) = \infty$ as described in the beginning of the section, and we choose a point on γ different from the tip $\gamma(1)$ to be $\gamma(s_0)$ so that $T > 0$.

For every $t \in (-\infty, 0]$, there exists a conformal mapping φ_t fixing ∞, the tip $\Gamma(T)$ and $\Gamma(0)$ that maps the complement of $\Gamma[-\infty, t]$ to a simply connected domain which is the complement of a half-line L_t. In fact, the mapping-out function of $\Gamma[-\infty, t]$ maps the complement of $\Gamma[-\infty, t]$ to the complement of $\mathbb{R}_+$. Then we use a Möbius transformation of $\hat{\mathbb{C}}$ which sends the image of $\Gamma(0)$ and $\Gamma(T)$ back to $\Gamma(0)$ and $\Gamma(T)$ while fixing ∞.

We prove first

$$J(h) \le I^A(\Gamma[-\infty, T], \infty). \tag{4.10}$$

For $n \in \mathbb{N}$, the family $(\varphi_t|_{\hat{\mathbb{C}} \setminus \Gamma[-\infty, -n]})_{t \le -n}$ is a normal family, and by diagonal extraction, there exists a subsequence that converges uniformly on compacts in $\mathbb{C}$ to a conformal map φ that can be continuously extended to $\hat{\mathbb{C}}$. Since φ fixes three points on $\hat{\mathbb{C}}$, it is the identity map.

Let Γ^t be the curve which consists of the image of $\Gamma[t, T]$ under φ_t attached to the half-line L_t. The map $\psi_t := h \circ \varphi_t^{-1}$ maps the complement of Γ^t to the complement of $\mathbb{R}_+$, that fixes ∞. From

Proposition 4.5 and the invariance of J under affine transformations,

$$J(h \circ \varphi_t^{-1}) = I_{L_t}(\varphi_t(\Gamma[t,T])) = I_{\Gamma[-\infty,t]}(\Gamma[t,T]).$$

Hence, it follows from the lower-semicontinuity of J and the definition of arc Loewner energy that

$$J(h) \le \liminf_{t\to-\infty} J(h \circ \varphi_t^{-1}) = I^A(\Gamma[-\infty,T], \Gamma(-\infty)).$$

For the other inequality, it suffices to show that

$$J(h) = J(\varphi_t) + J(\psi_t) \tag{4.11}$$

as it implies that as $t \to -\infty$,

$$J(h) \ge J(\psi_t) = I_{\Gamma[-\infty,t]}(G[t,T]) \to I^A(\Gamma[-\infty,T], \Gamma(-\infty)).$$

In fact, (4.11) is equivalent to

$$\int_{\hat{\mathbb{C}}\setminus\Gamma^t} -\nabla\sigma_{\psi_t}(y) \cdot \nabla\sigma_{\varphi_t^{-1}}(y)\, \mathrm{d}y^2 = 0.$$

Notice that φ_t^{-1} is conformal in the complement of L_t. From (4.10), $\sigma_{\varphi_t^{-1}} \in \mathscr{D}^\infty(\hat{\mathbb{C}}\setminus L_t)$ and the curve attached to L_t has finite chordal energy which is equal to $I_{\Gamma[-\infty,t]}(\Gamma[t,T])$. Hence we conclude with Lemma 4.17 by replacing $\mathbb{R}_+$ by L_t. □

The proof of Theorem 4.19 consists of making $T \to \infty$. The strategy is the same as the proof of Lemma 4.20. As we assume (without loss of generality) that γ passes through 0, 1 and ∞, we can choose the uniformizing mappings $h|_{H_1}$ and $h|_{H_2}$ that fix 0, 1 and ∞ on the boundary.

Proof of Theorem 4.19. We prove first that $J(h) \le I^L(\gamma, \gamma(0))$. Fix a point $\gamma(s_0)$ on γ and the conformal map ψ_{s_0}, let $(h_t)_{t\in\mathbb{R}}$ be the mapping-out functions, W the driving function and Γ the capacity reparametrized loop with $\Gamma(-\infty) = \infty$.

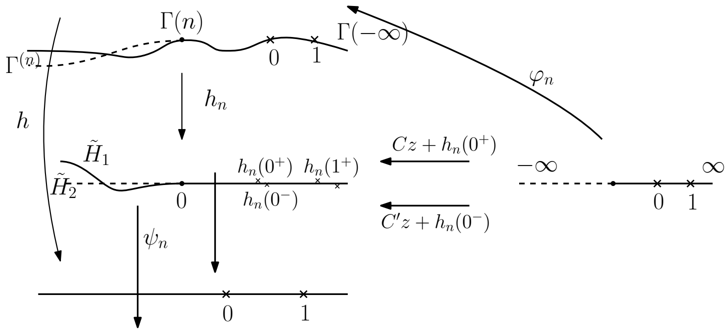

Figure 4.5: Conformal mappings in the proof of Theorem 4.19. We define $\varphi_n(z) = (h_n)^{-1}(Cz + h_n(0^+))$ on $\mathbb{H}$ and $\varphi_n(z) = (h_n)^{-1}(C'z + h_n(0^-))$, where C and C' are chosen such that φ_n fixes $0, 1$ and ∞.

For $n \geq 0$, we consider $W^{(n)}(\cdot) := W(\cdot \wedge n)$, and $\Gamma^{(n)}$ the loop generated by $W^{(n)}$ which coincide with Γ on $[-\infty, n]$, that is the simple arc $\Gamma[-\infty, n]$ followed by the hyperbolic geodesic in $\mathbb{C}\backslash\Gamma[-\infty, n]$. The mapping-out function h_n of $\Gamma[-\infty, n]$ maps both connected components $H_1^{(n)}$ and $H_2^{(n)}$ in the complement of $\Gamma^{(n)}$ to half-planes. From Lemma 4.20,

$$I^L(\Gamma^{(n)}) = I^A(\Gamma[-\infty, n], \infty) = J(h_n).$$

Notice that h_n is not continuous on $\Gamma[-\infty, n]$, we denote by $h_n(0^+)$ (resp. $h_n(0^-)$) the image of 0 by $h_n|_{H_1}$ (resp. $h_n|_{H_2}$). Since Γ passes through 0, 1 and ∞ by assumption, we define φ_n such that it maps respectively $\mathbb{H}$ and $\mathbb{H}^*$ to $H_1^{(n)}$ and $H_2^{(n)}$ while fixing 0, 1 and ∞. Let $\varphi = h^{-1}$. Since $(\varphi_n)_{n\geq 1}$ is a normal family, there exists a subsequence that converges uniformly on compacts, by Carathéodory kernel theorem, the limit is φ. Hence

$$I^L(\gamma) = \lim_{n\to\infty} J(h_n) = \lim_{n\to\infty} J(\varphi_n) \geq J(\varphi).$$

Now we prove that $J(h) \geq I^L(\gamma)$. Let $\psi_n := h \circ h_n^{-1}$ which maps each connected component $\tilde{H}_i := h_n(H_i)$ of $\Sigma \backslash h_n(\Gamma[n,\infty])$ to a half-plane, we have then

$$J(h) = J(\psi_n) + J(h_n) + \frac{2}{\pi} \int_{\tilde{H}_1 \cup \tilde{H}_2} \nabla \sigma_{h_n^{-1}} \cdot \nabla \sigma_{\psi_n}. \tag{4.12}$$

Lemma 4.20 shows that $\sigma_{h_n^{-1}}$ has finite Dirichlet energy bounded by the arc Loewner energy of $\Gamma[-\infty, n]$ hence by $I^L(\Gamma, \infty)$. On the other hand, the inequality $J(h) \leq I^L(\gamma)$ that we have proved above gives us the finiteness of the Dirichlet energy of σ_{ψ_n}: For every $\varepsilon > 0$, there exists n_0 large enough, such that $\forall n \geq n_0$,

$$J(\psi_n) \leq I_{\mathbb{R}_+}(h_n(\Gamma[n,\infty])) = \int_n^\infty W'^2(t)/2 \,\mathrm{d}t \leq \varepsilon.$$

By the Cauchy-Schwarz inequality, the cross terms in (4.12) converges to 0 as $n \to \infty$, and $J(h_n)$ converges to $I^L(\Gamma, \infty)$. Hence $J(h) \geq I^L(\gamma)$. $\square$

4.7 Zeta-regularized Determinants

In this section we prove the identity of the Loewner loop energy with a functional of zeta-regularized determinants of Laplacians (i.e., Theorem 4.25 which is the complete version of Theorem 4.3). This functional has also been studied by Burghelea, et al. [BFK94].

The zeta-regularized determinant of Laplacian depends on both the conformal structure and the metric of the surface. Within a conformal class of metrics (two metrics g and g' are *conformally equivalent* if g' is a *Weyl-scaling* of g, i.e. $g' = e^{2\sigma} g$ for some $\sigma \in C^\infty(M)$), the variation of determinants is given by the so-called Polyakov-Alvarez conformal anomaly formula that we now recall (a proof of the formula can be found in [OPS88]).

Let (M, g_0) be a surface without boundary, and with the same notation for the metric, (D, g_0) a compact surface with boundary. If $g = e^{2\sigma} g_0$ is a metric conformally equivalent to g_0, with the obvious notation associated to either g_0 or g, we denote by

- Δ_0 and Δ_g the Laplace-Beltrami operator (with Dirichlet boundary condition for D),
- vol_0 and vol_g the area measure,
- l_0 and l_g the arclength measure on the boundary,
- K_0 and K_g the scalar curvature in the bulk,
- k_0 and k_g the geodesic curvature on the boundary.

Theorem 4.21 (Polyakov-Alvarez Conformal Anomaly [OPS88])**.** *For a compact surface M without boundary,*

$$\begin{aligned}\log\det{}'_\zeta(-\Delta_g) = &-\frac{1}{6\pi}\left[\frac{1}{2}\int_M |\nabla_0\sigma|^2\,\mathrm{dvol}_0 + \int_M K_0\sigma\,\mathrm{dvol}_0\right]\\ &+\log\mathrm{vol}_g(M)+\log\det{}'_\zeta(-\Delta_0)-\log\mathrm{vol}_0(M).\end{aligned}$$

The analogue for a compact surface D with smooth boundary is:

$$\begin{aligned}&\log\det_\zeta(-\Delta_g)\\ &= -\frac{1}{6\pi}\left[\frac{1}{2}\int_D |\nabla_0\sigma|^2\,\mathrm{dvol}_0 + \int_D K_0\sigma\,\mathrm{dvol}_0 + \int_{\partial D} k_0\sigma\,\mathrm{dl}_0\right]\\ &-\frac{1}{4\pi}\int_{\partial D}\partial_n\sigma\,\mathrm{dl}_0 + \log\det_\zeta(-\Delta_0),\end{aligned}$$

where ∂_n is the outward normal derivative.

Let $M = S^2$ be the sphere equipped with a Riemannian metric g, $\gamma\subset S^2$ a smooth Jordan curve dividing S^2 into two components D_1 and D_2. Denote by $\Delta_{D_i,g}$ the Laplacian with Dirichlet boundary condition on (D_i, g). We introduce the functional $\mathscr{H}(\cdot, g)$ on the space of smooth Jordan curves:

$$\begin{aligned}\mathscr{H}(\gamma, g) :=& \log\det{}'_\zeta(-\Delta_{S^2,g}) - \log\mathrm{vol}_g(S^2)\\ &-\log\det_\zeta(-\Delta_{D_1,g}) - \log\det_\zeta(-\Delta_{D_2,g}).\end{aligned} \tag{4.13}$$

As a side remark, Burghelea, et al. [BFK92] (see also Lee [Lee97]) proved a Mayer-Vietoris type surgery formula for determinants of elliptic differential operators. In our case, it allows to express $\mathscr{H}$ by determinants of Neumann jump operators. However, we will not use it in our proof.

Theorem 4.22 (Mayer-Vietoris Surgery Formula [BFK92]). *We have*

$$\mathscr{H}(\gamma, g) = \log \det{}'_\zeta(N(\gamma, g)) - \log \mathrm{l}_g(\gamma),$$

where $N(\gamma, g)$ denotes the Neumann jump operator through the Jordan curve γ: for $f \in C^\infty(\gamma, \mathbb{R})$,

$$N(\gamma, g)f = \partial_{n_1} u_1 + \partial_{n_2} u_2,$$

where n_i is the outer unit normal vector on the boundary of the domain D_i, u_i is the harmonic extension of f in D_i.

The choice of outer normal derivatives makes $N(\gamma, g)$ a non-negative, essentially self-adjoint operator. Its zeta-regularized determinant is defined similarly as for $-\Delta$: we use its positive spectrum to define the zeta function then take $-\log \det_\zeta N(\gamma, g)$ to be the derivative of zeta function's analytic continuation at 0. Notice that the harmonic extension u_i depends on the metric only by its conformal class and the normal derivatives depend on the data of g only in a neighborhood of γ.

Now we will see how $\mathscr{H}$ is related to Loewner energy. First, by simply applying the Polyakov-Alvarez formula, we obtain:

Proposition 4.23. *The functional $\mathscr{H}(\cdot, g)$ is invariant under Weyl-scalings.*

Proof. Let $\sigma \in C^\infty(S^2, \mathbb{R})$ and $g = e^{2\sigma} g_0$,

$$
\begin{aligned}
&\mathscr{H}(\gamma, g) - \mathscr{H}(\gamma, g_0) \\
&= -\frac{1}{6\pi}\left[\frac{1}{2}\int_{S^2} |\nabla_0 \sigma|^2 \,\mathrm{dvol}_0 + \int_S^2 K_0 \sigma \,\mathrm{dvol}_0\right] \\
&\quad - \sum_{i=1}^2 \Big(-\frac{1}{6\pi}\left[\frac{1}{2}\int_{D_i} |\nabla_0 \sigma|^2 \,\mathrm{dvol}_0 + \int_{D_i} K_0 \sigma \,\mathrm{dvol}_0\right] \\
&\qquad\qquad - \frac{1}{6\pi}\int_{\partial D_i} k_{i,0}\sigma \,\mathrm{dl}_0 - \frac{1}{4\pi}\int_{\partial D_i} \partial_{n_i}\sigma \,\mathrm{dl}_0\Big),
\end{aligned}
$$

where $k_{i,0}$ is the geodesic curvature on the boundary of D_i under the metric g_0. The domain integrals cancel out. And for $z \in \gamma$, we have $k_{1,0}(z) = -k_{2,0}(z)$, thus the terms $\int_{\partial D_i} k_{i,0}\sigma \,\mathrm{dl}_0$ sum up to 0. We have also the relation (Lemma 4.32)

$$\partial_{n_i}\sigma = k_{i,g}e^\sigma - k_{i,0},$$

which yields

$$
\begin{aligned}
\int_{\partial D_i} \partial_{n_i}\sigma \,\mathrm{dl}_0 &= \int_{\partial D_i} k_{i,g}e^\sigma - k_{i,0} \,\mathrm{dl}_0 \\
&= \int_{\partial D_i} k_{i,g} \,\mathrm{dl}_g - \int_{\partial D_i} k_{i,0} \,\mathrm{dl}_0
\end{aligned}
$$

that sum up to zero as well. ☐

Corollary 4.24. *$\mathscr{H}(\cdot, g)$ is conformally invariant: let μ be a conformal map from S^2 onto S^2, then*

$$\mathscr{H}(\gamma, g) = \mathscr{H}(\mu(\gamma), g).$$

Proof. We have

$$\mathscr{H}(\mu(\gamma), g) = \mathscr{H}(\gamma, \mu^* g) = \mathscr{H}(\gamma, g)$$

where $\mu^* g$ is the pull-back of g, that is conformally equivalent to g.

The second equality follows from Proposition 4.23. ☐

We are now ready to state the main result of this section:

Theorem 4.25. *If $g = e^{2\varphi} g_0$ is a metric conformally equivalent to the spherical metric g_0 on the 2-sphere S^2, then:*

(i) Circles minimize $\mathscr{H}(\cdot, g)$ among all smooth Jordan curves.
(ii) Let γ be a smooth Jordan curve on S^2. We have the identity

$$\begin{aligned} I^L(\gamma, \gamma(0)) &= 12\mathscr{H}(\gamma, g) - 12\mathscr{H}(S^1, g) \\ &= 12 \log \frac{\det_\zeta(-\Delta_{\mathbb{D}_1,g}) \det_\zeta(-\Delta_{\mathbb{D}_2,g})}{\det_\zeta(-\Delta_{D_1,g}) \det_\zeta(-\Delta_{D_2,g})}, \end{aligned}$$

where $\mathbb{D}_1$ and $\mathbb{D}_2$ are two connected components of the complement of S^1.

Let us make the following two comments:

- The right-hand side in (ii) does not depend on the root, so that the root-invariance of the loop energy for smooth loops follows.
- We also recognize the functional introduced in [BFK94], where they defined

$$h_g(\gamma) := \log \det_\zeta(-\Delta_{D_1,g}) + \log \det_\zeta(-\Delta_{D_2,g}),$$

so that our identity above can be expressed as

$$I^L(\gamma) = 12 h_g(S^1) - 12 h_g(\gamma).$$

Proof. The second equality in (ii) follows directly from the definition. Since $I^L(\gamma)$ is non-negative, (ii) implies that S^1 minimizes $\mathscr{H}(\cdot, g)$. Corollary 4.24 implies that $\mathscr{H}(C, g) = \mathscr{H}(S^1, g)$ for any circle C and we get (i).

Therefore it suffices to prove the first equality in (ii) for $g = g_0$ by Proposition 4.23. We also assume that S^1 is a geodesic circle and both γ and S^1 pass through a point $\infty \in S^2$. We use the stereographic projection $S^2 \backslash \{\infty\} \to \mathbb{C}$ from ∞ and the image of D_1, D_2, $\mathbb{D}_1$ and $\mathbb{D}_2$

are H_1, H_2, $\mathbb{H}$ and $\mathbb{H}^*$. With a slight abuse we use the same notation for the induced metric in $\mathbb{C}$:

$$g_0(z) = \frac{4\,\mathrm{d}z^2}{(1+|z|^2)^2} =: e^{2\psi(z)}\,\mathrm{d}z^2,$$

and $\langle\cdot,\cdot\rangle_0 := g_0(\cdot,\cdot)$. Let h be a conformal map that maps respectively from H_1 and H_2 to $\mathbb{H}$ and $\mathbb{H}^*$ fixing ∞ as in previous sections and we put $f = h^{-1}$. Let g_1 be the pull-back of g_0 by f:

$$\begin{aligned} g_1(z) =& f^* g_0(z) = e^{2\psi(f(z))}\,|f'(z)|^2\ \mathrm{d}z^2 \\ =& e^{2\psi(f(z))-2\psi(z)+2\sigma_f(z)} g_0(z) := e^{2\sigma(z)} g_0(z), \end{aligned}$$

where $\sigma_f(z) = \log|f'(z)|$ and we set

$$\theta(z) = \psi(f(z)) - \psi(z)$$

so that

$$\sigma(z) = \theta(z) + \sigma_f(z).$$

From the Polyakov-Alvarez conformal anomaly formula:

$$\begin{aligned} &\log\det{}_\zeta(-\Delta_{H_1,g_0}) - \log\det{}_\zeta(-\Delta_{\mathbb{H},g_0}) \\ &\quad= \log\det{}_\zeta(-\Delta_{\mathbb{H},g_1}) - \log\det{}_\zeta(-\Delta_{\mathbb{H},g_0}) \\ &\quad= -\frac{1}{6\pi}\left[\frac{1}{2}\int_{\mathbb{H}} |\nabla_0\sigma|^2\ \mathrm{dvol}_0 + \int_{\mathbb{H}} K_0\sigma\,\mathrm{dvol}_0 + \int_{\mathbb{R}} k_0\sigma\,\mathrm{dl}_0\right] \\ &\qquad - \frac{1}{4\pi}\int_{\mathbb{R}} \partial_{n_0}\sigma\,\mathrm{dl}_0. \end{aligned}$$

As in the proof of Proposition 4.23, the last term above cancels out when we add both the variation in $\mathbb{H}$ and $\mathbb{H}^*$. We have $K_0 \equiv 1$, $k_0 \equiv 0$, but as we will reuse the proof in Section 4.8, we keep first K_0 and k_0

in the expressions. The right-hand side in (ii) equals to

$$\begin{aligned}
&\frac{1}{\pi}\int_{\mathbb{H}\cup\mathbb{H}^*} |\nabla_0(\sigma_f+\theta)|^2 + 2K_0\sigma_f + 2K_0\theta \,\mathrm{dvol}_0 + \frac{2}{\pi}\int_{\mathbb{R}} k_0\sigma \,\mathrm{dl}_0 \\
=&\frac{1}{\pi}\int |\nabla_0\sigma_f|^2 \,\mathrm{dvol}_0 + \frac{2}{\pi}\int \left(\langle\nabla_0\sigma_f, \nabla_0\theta\rangle_0 + K_0\sigma_f\right) \mathrm{dvol}_0 \\
&+\frac{1}{\pi}\int \left(|\nabla_0\theta|^2 + 2K_0\theta\right) \mathrm{dvol}_0 + \frac{2}{\pi}\int_{\mathbb{R}} k_0\sigma \,\mathrm{dl}_0.
\end{aligned} \tag{4.14}$$

Since the Dirichlet energy is invariant under Weyl-scalings of the metric, the first term on the right-hand side of the equality equals $J(f)$, which is also equal to $I^L(\gamma,\infty)$ by Theorem 4.19. As $k_0 \equiv 0$, we only need to prove that the sum of the second and the third terms vanishes.

We denote the quantities/operators/measures with respect to the Euclidean metric in $\mathbb{C}$ without subscript, then we have

$$\begin{aligned}
\Delta_0 &= e^{-2\psi}\Delta; & \partial_{n_0} &= e^{-\psi}\partial_n; \\
\mathrm{dvol}_0 &= e^{2\psi}\,\mathrm{d}z^2; & \mathrm{dl}_0 &= e^{\psi}\,\mathrm{dl}; \\
\partial_n\sigma_f(z) &= k(f(z))e^{\sigma_f(z)} - k(z); \\
\Delta_0\psi &= e^{-2\psi}\Delta\psi = e^{-2\psi}(K - e^{2\psi}K_0) = -K_0; \\
\partial_{n_0}\psi &= e^{-\psi}\partial_n\psi = e^{-\psi}(e^{\psi}k_0 - k) = k_0 - e^{-\psi}k.
\end{aligned}$$

For the second term in (4.14), from Stokes' formula:

$$
\begin{aligned}
&\int_{\mathbb{H}} \langle \nabla_0 \sigma_f, \nabla_0(\psi \circ f)\rangle_0 \,\mathrm{dvol}_0 \\
&= \int_{\mathbb{R}} \psi(f)\partial_{n_0}\sigma_f \,\mathrm{dl}_0 - \int_{\mathbb{H}} \psi(f)\Delta_0\sigma_f \,\mathrm{dvol}_0 \\
&= \int_{\mathbb{R}} \psi(f)\partial_n \sigma_f \,\mathrm{dl} \\
&= \int_{\mathbb{R}} k(f)e^{\sigma_f}\psi(f)\,\mathrm{dl} - \int_{\mathbb{R}} k\psi(f)\,\mathrm{dl} \\
&= \int_{\gamma} k\psi \,\mathrm{dl}(z) - \int_{\mathbb{R}} k\psi(f)\,\mathrm{dl},
\end{aligned}
$$

the contributions from the first term in the above expression cancels out when we sum up both sides. Similarly we have

$$
\begin{aligned}
\int_{\mathbb{H}} \langle \nabla_0\sigma_f, \nabla_0\psi\rangle_0 \,\mathrm{dvol}_0 &= \int_{\mathbb{R}} \sigma_f \partial_{n_0}\psi \,\mathrm{dl}_0 - \int_{\mathbb{H}} \sigma_f \Delta_0 \psi \,\mathrm{dvol}_0 \\
&= \int_{\mathbb{R}} \sigma_f \partial_{n_0}\psi \,\mathrm{dl}_0 + \int_{\mathbb{H}} K_0 \sigma_f \,\mathrm{dvol}_0.
\end{aligned}
$$

Hence the second term in (4.14) equals to

$$
-\frac{2}{\pi}\int_{\mathbb{R}\sqcup\mathbb{R}} \left(k\psi(f) + \sigma_f \partial_n \psi\right)\,\mathrm{dl}.
$$

For the third term in (4.14), notice that

$$
\begin{aligned}
&\int_{\mathbb{H}} \langle \nabla_0(\psi\circ f), \nabla_0\psi\rangle_0 \,\mathrm{dvol}_0 \\
&= \int_{\mathbb{R}} \psi(f)\partial_{n_0}\psi \,\mathrm{dl}_0 - \int_{\mathbb{H}} \psi(f)\Delta_0\psi \,\mathrm{dvol}_0 \\
&= \int_{\mathbb{R}} \psi(f)\partial_n\psi \,\mathrm{dl} + \int_{\mathbb{H}} \psi(f)K_0 \,\mathrm{dvol}_0.
\end{aligned}
$$

Similarly,

$$\int_{\mathbb{H}} \langle \nabla_0(\psi \circ f), \nabla_0(\psi \circ f)\rangle_0 \,\mathrm{dvol}_0 = \int_{\mathbb{H}} \langle \nabla_0 \psi, \nabla_0 \psi\rangle_0 \,\mathrm{dvol}_0$$
$$= \int_{\mathbb{R}} \psi \partial_n \psi \,\mathrm{dl} + \int_{\mathbb{H}} \psi K_0 \,\mathrm{dvol}_0.$$

Hence the third term equals to

$$\begin{aligned} &\frac{1}{\pi}\int_{\mathbb{H}\cup\mathbb{H}^*} \langle \nabla_0\theta, \nabla_0\theta\rangle_0 + 2K_0\theta \,\mathrm{dvol}_0 \\ &= \frac{2}{\pi}\left(\int_{\mathbb{R}\sqcup\mathbb{R}} \psi\partial_n\psi \,\mathrm{dl} - \psi(f)\partial_n\psi \,\mathrm{dl}\right) \\ &= -\frac{2}{\pi}\int_{\mathbb{R}\sqcup\mathbb{R}} \theta\partial_n\psi \,\mathrm{dl}. \end{aligned}$$

Therefore the sum of the second and the third terms of (4.15) equals

$$\frac{2}{\pi}\int_{\mathbb{R}\sqcup\mathbb{R}} -k\psi(f) - \sigma\partial_n\psi \,\mathrm{dl}, \tag{4.15}$$

which vanishes because $k, k_0 \equiv 0$ on $\mathbb{R}$ and $\partial_n\psi = e^{\psi}k_0 - k \equiv 0$ as well. $\square$

4.8 Weil-Petersson class of loops

In this section we establish the link between finite Loewner energy class and the Weil-Petersson class as explained in the introduction (we will prove Theorem 4.28, which is the precise version of Theorem 4.4).

Let us start with some background material on the universal Teichmüller space $T(1)$ and the Weil-Petersson Teichmüller space $T_0(1)$. We follow here the notations of [TT06]. We define

$$\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}, \quad \mathbb{D}^* = \{z \in \mathbb{C}, |z| > 1\},$$

and let $S^1 = \partial\mathbb{D}$ be the unit circle. Let $\mathrm{QS}(S^1)$ be the group of sense-preserving quasisymmetric homeomorphisms of the unit circle (see e.g. [LV73]), $\text{Möb}(S^1) \simeq \text{PSL}(2,\mathbb{R})$ the group of Möbius transformations of S^1 and $\text{Rot}(S^1)$ the rotation group of S^1. The *universal Teichmüller space* is defined as the right cosets

$$\begin{aligned} T(1) &:= \text{Möb}(S^1)\backslash\, \text{QS}(S^1) \\ &\simeq \{\varphi \in \text{QS}(S^1),\, \varphi \text{ fixes } -1, -i \text{ and } 1\}. \end{aligned}$$

We write $[\varphi]$ for the class of φ. From the Beurling-Ahlfors extension theorem, for every $\varphi \in \text{QS}(S^1)$ fixing $-1, -i$ and 1, there exists a unique $\alpha \in \text{Möb}(S^1)$ such that $\alpha(1) = 1$, and conformal maps f and g on $\mathbb{D}$ and $\mathbb{D}^*$ satisfying:

CW1. f and g admit quasiconformal extensions to $\mathbb{C}$.
CW2. $\alpha \circ \varphi = g^{-1} \circ f|_{S^1}$.
CW3. $f(0) = 0,\ f'(0) = 1,\ f''(0) = 0$.
CW4. $g(\infty) = \infty$.

Let $\mathscr{U}$ denote the set of conformal maps (univalent functions) on $\mathbb{D}$, $T(1)$ can be identified as

$$\{f \in \mathscr{U} \mid f(0) = 0, f'(0) = 1, f''(0) = 0, f \text{ q.c. extends to } \mathbb{C}\}.$$

We say that (f, g) are *canonical conformal mappings associated to* $[\varphi] \in T(1)$.

Takhtajan and Teo have proved that $T(1)$ carries a natural structure of complex Hilbert manifold and that the connected component of the identity $T_0(1)$ is characterized by:

Theorem 4.26 ([TT06] Theorem 2.1.12)**.** *A point $[\varphi]$ is in $T_0(1)$ if the associated canonical conformal maps f and g satisfy one of the following equivalent conditions:*

(i) $\int_{\mathbb{D}} |f''(z)/f'(z)|^2 \,\mathrm{d}z^2 < \infty$;
(ii) $\int_{\mathbb{D}^*} |g''(z)/g'(z)|^2 \,\mathrm{d}z^2 < \infty$;
(iii) $\int_{\mathbb{D}} |S(f)|^2 \rho^{-1}(z) \,\mathrm{d}z^2 < \infty$;
(iv) $\int_{\mathbb{D}^*} |S(g)|^2 \rho^{-1}(z) \,\mathrm{d}z^2 < \infty$,

where $\rho(z)\,\mathrm{d}z^2 = 1/(1-|z|^2)^2\,\mathrm{d}z^2$ *is the hyperbolic metric on* $\mathbb{D}$ *or* $\mathbb{D}^*$ *and*

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is the Schwarzian derivative of f.

Theorem 4.27 ([TT06] Theorem 2.4.1)**.** *The universal Liouville action* $\mathbf{S_1} : T_0(1) \to \mathbb{R}$ *defined by*

$$\mathbf{S_1}([\varphi]) := \int_{\mathbb{D}} \left|\frac{f''}{f'}(z)\right|^2 \mathrm{d}z^2 + \int_{\mathbb{D}^*} \left|\frac{g''}{g'}(z)\right|^2 \mathrm{d}z^2 - 4\pi \log|g'(\infty)|\,, \tag{4.16}$$

where $g'(\infty) = \lim_{z\to\infty} g'(z) = \tilde{g}'(0)^{-1}$ *and* $\tilde{g}(z) = 1/g(1/z)$*, is a Kähler potential for the Weil-Petersson metric on* $T_0(1)$*.*

Notice that from Theorem 4.26, the right-hand side in (4.16) is finite if and only if $[\varphi] \in T_0(1)$.

We define similarly the *universal Liouville action for quasicircles.* If γ is a bounded quasicircle, we denote (and in the sequel) the bounded connected component of $\mathbb{C}\backslash\gamma$ by D, and the unbounded connected component by D^*. Let f be any conformal map from $\mathbb{D}$ onto D, and g from $\mathbb{D}^*$ onto D^* and fixes ∞. Conformal maps from $\mathbb{D}$ onto a quasidisk always admit a quasiconformal extension to $\mathbb{C}$. We denote again by f and g their quasiconformal extension. We say that $\varphi := g^{-1} \circ f|_{S^1}$ is

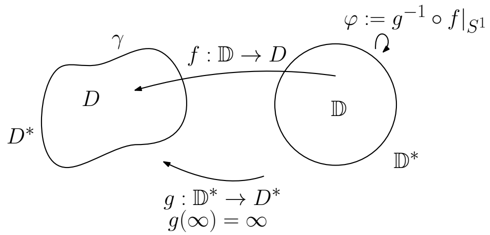

Figure 4.6: Welding function φ of a simple loop γ.

a *welding function* of γ (see Figure 4.6), which lies in $\mathrm{QS}(S^1)$ as it is the boundary value of the quasiconformal map $g^{-1} \circ f$ on $\mathbb{D}$ and does not depend on the extensions.

We say $\varphi \in \mathrm{QS}(S^1)$ is in the *Weil-Petersson class* if $[\varphi] \in T_0(1)$, and γ is a *Weil-Petersson quasicircle* if its welding function φ is in the Weil-Petersson class. We define $\mathbf{S_1}(\gamma)$ to be

$$\mathbf{S}(f,g) := \int_{\mathbb{D}} \left|\frac{f''}{f'}(z)\right|^2 \mathrm{d}z^2 + \int_{\mathbb{D}^*} \left|\frac{g''}{g'}(z)\right|^2 \mathrm{d}z^2 \\ + 4\pi \log|f'(0)| - 4\pi \log|g'(\infty)|,$$

which is finite if and only if γ is a Weil-Petersson quasicircle and the value does not depend on the choice of f and g. In fact, for any other choice of conformal maps $\tilde{f}$ and $\tilde{g}$ for γ, there exists $\mu \in \text{Möb}(S^1)$ and $\nu \in \mathrm{Rot}(S^1)$ such that $\tilde{f} = f \circ \mu$ and $\tilde{g} = g \circ \nu$. It follows from explicit computations ([TT06, Lem. 2.3.4]) that

$$\mathbf{S}(f,g) = \mathbf{S}(\tilde{f}, \tilde{g})$$

which is also equal to $\mathbf{S_1}([\varphi])$, see [TT06, Lem. 2.3.4, Thm. 2.3.8].

Now we can state the main theorem of this section:

Theorem 4.28. *Let γ be a (bounded) Jordan curve, γ has finite Loewner energy if and only if γ is Weil-Petersson quasicircle. Moreover,*

$$I^L(\gamma) = \mathbf{S_1}(\gamma)/\pi. \tag{4.17}$$

It is worth mentioning other characterizations of $T_0(1)$ due to Cui, Shen, Takhtajan and Teo, from which one obtains immediately other analytic characterizations of finite energy loops given Theorem 4.28:

Theorem 4.29 ([Cui00, She18, TT06])**.** *With the same notation as in Theorem 4.26, φ is in Weil-Petersson class if and only if one of the following equivalent condition holds:*

(i) φ has quasiconformal extension to $\mathbb{D}$, whose complex dilation

$\mu = \partial_{\overline{z}}\varphi / \partial_z \varphi$ satisfies

$$\int_{\mathbb{D}} |\mu(z)|^2 \rho(z) \,\mathrm{d}z^2 < \infty;$$

(ii) φ is AC with respect to arclength measure, such that $\log \varphi'$ belongs to the Sobolev space $H^{1/2}(S^1)$;

(iii) the Grunsky operator associated to f or g is Hilbert-Schmidt.

Now we proceed to the proof of Theorem 4.28. We first prove it for smooth loops using results from Section 4.7.

Proof for smooth loops. Let γ be a smooth (bounded) Jordan curve. It is clear from the definition that $\mathbf{S_1}(\gamma)$ is invariant under affine transformation of $\mathbb{C}$. By Möbius invariance of the Loewner loop energy, we may also assume that γ is inside the Euclidean ball of radius 2 and of center 0.

Let $g_0 = e^{2\psi} dz^2$ be a metric conformally equivalent to the Euclidean metric (or the spherical metric), such that $\psi \equiv 0$ on $B(0,2)$ and $e^{2\psi(z)} = 4/(1+|z|^2)^2$ in a neighborhood of ∞ which makes g_0 coincide with the spherical metric near ∞. We compute the quotient on the right hand side of the expression in Theorem 4.25 (ii) by taking $g = g_0$.

The same computation (and the same notations) as in the proof of Theorem 4.25 shows that

$$\begin{aligned}
&12\pi \log \frac{\det_\zeta(-\Delta_{\mathbb{D},g_0})\det_\zeta(-\Delta_{\mathbb{D}^*,g_0})}{\det_\zeta(-\Delta_{D,g_0})\det_\zeta(-\Delta_{D^*,g_0})} \\
&= \int_{\mathbb{D}\cup\mathbb{D}^*} |\nabla_0 \sigma|^2 + 2K_0\sigma \,\mathrm{dvol}_0 + \int_{S^1 \sqcup S^1} 2k_0 \sigma \,\mathrm{dl}_0 + 3\partial_{n_0}\sigma \,\mathrm{dl}_0 \\
&= \int_{\mathbb{D}} |\nabla \sigma_f|^2 \,\mathrm{d}z^2 + \int_{\mathbb{D}^*} |\nabla \sigma_g|^2 \,\mathrm{d}z^2 + 2\int_{S^1 \sqcup S^1} k_0 \sigma \,\mathrm{dl}_0,
\end{aligned}$$

where $\sigma = \sigma_f + \psi(f) - \psi$ for $z \in \overline{\mathbb{D}}$, and $\sigma = \sigma_g + \psi(g) - \psi$ for $z \in \overline{\mathbb{D}}^*$, $S^1 \sqcup S^1$ denotes the two copies of S^1 as the boundary of $\mathbb{D}$ and of $\mathbb{D}^*$, the value of σ on the boundary depends on the copy accordingly. In fact, the analogous sum (4.15) of the second and the third term in

(4.14)
$$\frac{2}{\pi}\int_{S^1\sqcup S^1} -k\psi(f) - \sigma\partial_n\psi \,\mathrm{dl}$$

also vanishes here since ψ is identically 0 in a neighborhood of S^1 and of γ. The only difference with the proof of Theorem 4.25 is that we have an extra term (analogous to the last term in (4.14)): that is $\int_{S^1\sqcup S^1} k_0\sigma\,\mathrm{dl}_0$ since k_0 is not vanishing: $k_0(z) = 1$ for $z \in \partial\mathbb{D}$ and $k_0(z) = -1$ for $z \in \partial\mathbb{D}^*$. Using again the fact that $\psi(f(z)) = \psi(z) = 0$ for $z \in S^1$, the smoothness up to boundary and the harmonicity of σ_f and σ_g, we get:

$$\frac{2}{\pi}\int_{S^1\sqcup S^1} k_0\sigma\,\mathrm{dl}_0 = 4\log|f'(0)| - 4\log|g'(\infty)|\,.$$

Hence,
$$I^L(\gamma) = \mathbf{S_1}(\gamma)/\pi,$$
for the smooth loop γ by Theorem 4.25. □

In particular, for a bounded smooth loop $\gamma \subset \mathbb{C}$, we have the identity

$$J(h) = \frac{1}{\pi}\int_{\mathbb{C}\setminus\mu(\gamma)} \left|\frac{h''}{h'}(z)\right|^2 \mathrm{d}z^2 = \frac{1}{\pi}\mathbf{S_1}(\gamma) \tag{4.18}$$

where μ is a Möbius function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\mu(\gamma(0)) = \infty$, and h is a conformal map from the complement of $\mu(\gamma)$ onto $\mathbb{H}\cup\mathbb{H}^*$ that fixes ∞, as defined in Theorem 4.19. The identity (4.18) of two domain integrals has *a priori* no reason to depend on the boundary regularity, which then implies Theorem 4.28 for general loops by an approximation argument.

To make the approximation precise, we will use the following lemma which characterizes the convergence in the *universal Teichmüller curve*

$\mathscr{T}(1)$ which is a complex fibration over $T(1)$, given by

$$\begin{aligned}&\mathrm{Rot}(S^1)\backslash \mathrm{QS}(S^1)\\ &\simeq \{\varphi \in \mathrm{QS}(S^1), \varphi(1) = 1\}\\ &\simeq \{f \in \mathscr{U}, f(0) = 0, f'(0) = 1, f \text{ admits q.c. extension to } \mathbb{C}\}.\end{aligned}$$

The second identification is obtained from solving the conformal welding problem as for $T(1)$: for each $\varphi \in \mathrm{QS}(S^1)$ that fixes 1, there exist unique conformal maps f and g on $\mathbb{D}$ and $\mathbb{D}^*$ (*canonically associated to* $\varphi \in \mathscr{T}(1)$), which satisfy CW1. and CW4. and

CW'2. $\varphi = g^{-1} \circ f|_{S^1}$.
CW'3. $f(0) = 0$, $f'(0) = 1$.

Let $\pi : \mathscr{T}(1) \to T(1)$ be the projection and $\mathscr{T}_0(1) := \pi^{-1}(T_0(1))$ which is also equipped with a Hilbert manifold structure such that π is a fibration of Hilbert manifolds, see [TT06, Appx. A].

Lemma 4.30 ([TT06] Corollary A.4 and Corollary A.6)**.** *Let* $\{\varphi_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathscr{T}_0(1)$*, let* f_n *and* g_n *be the conformal maps canonically associated to* φ_n *such that* $\varphi_n = g_n^{-1} \circ f_n$*, and similarly let* $\varphi = g^{-1} \circ f \in \mathscr{T}_0(1)$*. Then the following conditions are equivalent:*

1. *In* $\mathscr{T}_0(1)$ *topology,*
$$\lim_{n\to\infty} \varphi_n = \varphi.$$
2. *We have*
$$\lim_{n\to\infty} \int_{\mathbb{D}} \left| \frac{f_n''}{f_n'}(z) - \frac{f''}{f'}(z) \right|^2 \mathrm{d}z^2 = 0.$$
3. *Let* $\tilde{g}(z) := 1/g(1/z)$*,* $\tilde{g}_n(z) := 1/g_n(1/z)$ *for all* $n \geq 1$*,*
$$\lim_{n\to\infty} \int_{\mathbb{D}} \left| \frac{\tilde{g}_n''}{\tilde{g}_n'}(z) - \frac{\tilde{g}''}{\tilde{g}'}(z) \right|^2 \mathrm{d}z^2 = 0.$$

If above conditions are satisfied, then we have also

$$\lim_{n\to\infty} \int_{\mathbb{D}^*} \left| \frac{g_n''}{g_n'}(z) - \frac{g''}{g'}(z) \right|^2 \mathrm{d}z^2 = 0,$$

and

$$\lim_{n\to\infty} \mathbf{S_1}([\varphi_n]) = \lim_{n\to\infty} \mathbf{S}(f_n, g_n) = \mathbf{S}(f,g) = \mathbf{S_1}([\varphi]).$$

We will also use the lower-semicontinuity of $\mathbf{S_1}$:

Lemma 4.31. *If a sequence $(\gamma_n : [0,1] \to \hat{\mathbb{C}})_{n\geq 0}$ of simple loops converges uniformly to a bounded loop γ, then*

$$\liminf_{n\to\infty} \mathbf{S_1}(\gamma_n) \geq \mathbf{S_1}(\gamma).$$

Proof. There is n_0 large enough, such that $(\gamma_n)_{n\geq n_0}$ are bounded and $\cap_{n\geq n_0} D_n \neq \emptyset$ where D_n denotes the bounded connected component of $\mathbb{C}\backslash\gamma_n$. Let $z_0 \in \cap_{n\geq n_0} D_n$, and for $n \geq n_0$, $f_n : \mathbb{D} \to D_n$ a conformal map such that $f_n(0) = z_0$ and $f_n'(0) > 0$.

From the Carathéodory kernel theorem, f_n converges uniformly on compacts to $f : \mathbb{D} \to D$, where D is the bounded connected component of $\hat{\mathbb{C}}\backslash\gamma$. It yields that for $K \subset \mathbb{D}$ compact set,

$$\begin{aligned}\liminf_{n\to\infty} \int_{\mathbb{D}} \left|\frac{f_n''}{f_n'}(z)\right|^2 \mathrm{d}z^2 &\geq \liminf_{n\to\infty} \int_K \left|\frac{f_n''}{f_n'}(z)\right|^2 \mathrm{d}z^2 \\ &= \int_K \left|\frac{f''}{f'}(z)\right|^2 \mathrm{d}z^2.\end{aligned}$$

Since K is arbitrary,

$$\liminf_{n\to\infty} \int_{\mathbb{D}} \left|\frac{f_n''}{f_n'}(z)\right|^2 \mathrm{d}z^2 \geq \int_{\mathbb{D}} \left|\frac{f''}{f'}(z)\right|^2 \mathrm{d}z^2.$$

Similarly, let g_n be the conformal map from $\mathbb{D}^*$ onto the unbounded connected component D_n^* of $\mathbb{C}\backslash\gamma_n$ and $g : \mathbb{D}^* \to D^*$ that fix ∞. We have also that g_n converges locally uniformly on compacts to g, and

$$\liminf_{n\to\infty} \int_{\mathbb{D}^*} \left|\frac{g_n''}{g_n'}(z)\right|^2 \mathrm{d}z^2 \geq \int_{\mathbb{D}^*} \left|\frac{g''}{g'}(z)\right|^2 \mathrm{d}z^2.$$

And we have also $g_n'(\infty) \to g'(\infty)$, $f_n'(0) \to f'(0)$.

Hence

$$\liminf_{n\to\infty} \mathbf{S_1}(\gamma_n) = \liminf_{n\to\infty} \mathbf{S}(f_n, g_n) \geq \mathbf{S}(f, g) = \mathbf{S_1}(\gamma)$$

as we claimed. □

We also cite the similar lower-semicontinuity of the Loewner loop energy from [RW17]: with the same condition,

$$\liminf_{n\to\infty} I^L(\gamma_n, \gamma_n(0)) \geq I^L(\gamma, \gamma(0)).$$

We can now finally prove Theorem 4.28 in the general case using approximations by smooth loops.

Proof for general loops. Assume that $\mathbf{S_1}(\gamma) < \infty$. Let $f : \mathbb{D} \to D$ and $g : \mathbb{D}^* \to D^*$ be conformal maps associated to γ, without loss of generality we may assume that $f(0) = 0$, $f'(0) = 1$ and $g(\infty) = \infty$, so that (f, g) is canonically associated to $g^{-1} \circ f \in \mathscr{T}_0(1)$. Consider the sequence $\gamma^n := f(c_n S^1)$ of smooth loops that converges uniformly as parametrized loop (by S^1) to γ, where $c_n \uparrow 1$. Let $f_n(z) := c_n^{-1} f(c_n z)$ such that $f_n(0) = 0$ and $f_n'(0) = 1$. It is not hard to see that

$$\lim_{n\to\infty} \int_{\mathbb{D}} \left| \frac{f_n''}{f_n'}(z) - \frac{f''}{f'}(z) \right|^2 \mathrm{d}z^2 = 0.$$

In fact, f_n converges uniformly to f on $(1-\varepsilon)\mathbb{D}$ for $\varepsilon > 0$. And the above integral on the annulus $\mathbb{D} \backslash (1-\varepsilon)\mathbb{D}$ is arbitrarily small as $\varepsilon \to 0$ since $\mathbf{S_1}(\gamma)$ is finite.

Hence by Lemma 4.30, $\mathbf{S_1}(\gamma^n)$ converges to $\mathbf{S_1}(\gamma)$. Since γ^n converges uniformly to γ, from the lower-semicontinuity of Loewner energy and Theorem 4.28 for smooth loops,

$$\mathbf{S_1}(\gamma)/\pi = \liminf_{n\to\infty} \mathbf{S_1}(\gamma^n)/\pi = \liminf_{n\to\infty} I^L(\gamma^n) \geq I^L(\gamma). \tag{4.19}$$

Similarly, if $I^L(\gamma) < \infty$, with driving function $W : \mathbb{R} \to \mathbb{R}$. Assume without loss of generality that γ passes through $-1, -i, 1$ and is bounded.

Let $W_n \in C_0^\infty(\mathbb{R})$ be a sequence of compactly supported smooth function, such that

$$\int_{-\infty}^{\infty} |W'(t) - W_n'(t)|^2 \, \mathrm{d}t \xrightarrow{n\to\infty} 0.$$

Let γ_n be a loop in Σ with driving function W_n, by [LT16], any such loop is C^∞ loops. We may assume that $\sup_{n\geq 1} I^L(\gamma_n) < \infty$ and γ_n passes through $-1, -i, 1$ as well. By [RW17] Proposition 2.6, there exists $K > 1$ such that γ and γ_n are K-quasicircles. The compactness of K-quasiconformal maps allows us to subtract a subsequence γ_{n_k} that converges uniformly to γ.

From Theorem 4.28 for smooth loops γ_n and Lemma 4.31, we have

$$I^L(\gamma) = \liminf_{k\to\infty} I^L(\gamma_{n_k}) = \liminf_{k\to\infty} \mathbf{S_1}(\gamma_{n_k})/\pi \geq \mathbf{S_1}(\gamma)/\pi.$$

We conclude that $I^L(\gamma) < \infty$ if and only if $\mathbf{S_1}(\gamma) < \infty$ and $I(\gamma) = \mathbf{S_1}(\gamma)/\pi$ as claimed in Theorem 4.28. ☐

4.9 An informal discussion

Let us conclude with some very loose comments on the relation between our Theorem 4.19 and the theory of SLE and Liouville quantum gravity (LQG). Recall first that the Loewner energy was shown in Chapter 2 to be a large deviation rate function of SLE_κ as κ goes to 0. Heuristically,

$$I(\gamma) = \lim_{\varepsilon\to 0} \lim_{\kappa\to 0} -\kappa \log \mathrm{P}(\mathrm{SLE}_\kappa \text{ stays } \varepsilon\text{-close to } \gamma).$$

Given a sufficiently smooth simple curve γ, the mapping-out function h from the complement of γ to a standard domain ($\mathbb{H} \cup \mathbb{H}^*$), induces a metric on the standard domain that is the push-forward of the Euclidean metric of the initial domain. The exponential exponent of the conformal factor is given by $\sigma_{h^{-1}}(\cdot) := \log\left|h^{-1}(\cdot)'\right|$. It prescribes in turn the welding function of the curve γ by identifying boundary points according to arclength of this metric (see Figure 4.7).

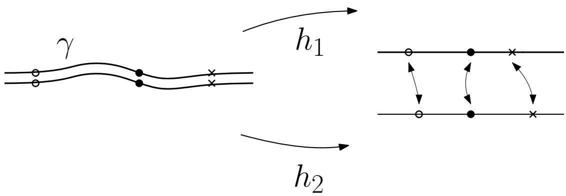

Figure 4.7: Welding of a simple loop γ passing through ∞.

On the other hand, the LQG approach to SLE pioneered by Sheffield in [She16] provides an interpretation of SLE curves via welding of structures defined using the exponential of the Gaussian Free Field (GFF). More specifically, let Γ be a free boundary Gaussian free field on the standard domain. That is the random field that can be described in loose term as having a "density" proportional to

$$\exp\left(-\frac{1}{4\pi}\int |\nabla\Gamma(z)|^2 \,\mathrm{d}z^2\right).$$

One takes formally $\exp(\sqrt{\kappa}\Gamma)$ times the Lebesgue measure (modulo some appropriate renormalization procedure) to define a random measure (LQG) on the standard domain (which corresponds in fact to $\sqrt{\kappa}$-quantum wedges with opening angle θ which converges to π when $\kappa \to 0$). It also induces a random boundary length which can be viewed as $\exp((\sqrt{\kappa}/2)\Gamma)$ times the Euclidean arclength (again modulo some appropriate renormalization procedure). Intuitively, the quantum zipper then states that welding two independent free boundary GFFs up according to their random boundary length gives an SLE_κ curve.

We can note that Dirichlet energy of $\sigma_{h^{-1}}$ is the action functional that is naturally associated to the Gaussian free field, so that in a certain sense, one has a large deviation principle of the type

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{\kappa\to 0} -\kappa\log \mathrm{P}((\sqrt{\kappa}/2)\Gamma \text{ stays } \varepsilon\text{-close to } \sigma_{h^{-1}}) \\
&= \lim_{\kappa\to 0} -\kappa \log\exp\left(-\frac{1}{4\pi}\int\left|\frac{2\nabla\sigma_{h^{-1}}}{\sqrt{\kappa}}\right|^2 \mathrm{d}z^2\right) \\
&= \frac{1}{\pi}\int\left|\nabla\sigma_{h^{-1}}(z)\right|^2 \mathrm{d}z^2.
\end{aligned}$$

Hence, our identity between the Loewner energy and the Dirichlet energy of σ_h (which is the same as the Dirichlet energy of $\sigma_{h^{-1}}$) is loosely speaking equivalent to the fact that (in some sense) as $\kappa\to 0$ (and then $\varepsilon\to 0$), the decay rates of

$$\mathrm{P}((\sqrt{\kappa}/2)\Gamma \text{ stays } \varepsilon\text{-close to } \sigma_{h^{-1}})$$

and

$$\mathrm{P}(\mathrm{SLE}_\kappa \text{ stays } \varepsilon\text{-close to } \gamma)$$

are comparable. However, the above argument is not even close to be rigorous (it would be interesting to explore it though) and the proof in this paper follows a completely different route and does not use any knowledge about SLE, LQG or the quantum zipper.

4.A Geodesic curvature formula

As many of our proofs rely on the following formula of change of geodesic curvature under Weyl-scaling of metric, we sketch a short proof for readers' convenience.

Lemma 4.32. *Let (D, g_0) be a surface with smooth boundary $\gamma = \partial D$. If $\sigma \in C^\infty(D,\mathbb{R})$ and $g = e^{2\sigma}g_0$, the geodesic curvature of ∂D under the metric g satisfies*

$$k_g = e^{-\sigma}\left(k_0 + \partial_{n_0}\sigma\right),$$

where k_0 is the geodesic curvature under the metric g_0, and ∂_{n_0} the *outer-normal derivative w.r.t. to g_0.*

Proof. We parameterize γ by arclength in g and let N be the outer normal vector field on γ, namely $g(\dot\gamma,\dot\gamma) = g(N,N) \equiv 1$. We have that $\dot\gamma_0 := e^\sigma \dot\gamma$ and $N_0 := e^\sigma N$ are unit vectors under g_0. The geodesic curvature of ∂D is given by

$$k_g = g\left(\nabla_{g,\dot\gamma}\dot\gamma, -N\right).$$

The covariant derivative ∇_g is related to the covariant derivative ∇_0 under g_0 by

$$\nabla_{g,X}Y = \nabla_{0,X}Y + X(\sigma)Y + Y(\sigma)X - g_0(X,Y)\nabla_0\sigma.$$

Therefore,

$$\begin{aligned}\nabla_{g,\dot\gamma}\dot\gamma &= \nabla_{0,\dot\gamma}\dot\gamma + 2g_0(\dot\gamma,\nabla_0\sigma)\dot\gamma - g_0(\dot\gamma,\dot\gamma)\nabla_0\sigma \\ &= e^{-2\sigma}\nabla_{0,\dot\gamma_0}\dot\gamma_0 + 2g_0(\dot\gamma,\nabla_0\sigma)\dot\gamma - e^{-2\sigma}\nabla_0\sigma.\end{aligned}$$

Since $g(\dot\gamma,N) = 0$, we have

$$\begin{aligned}g\left(\nabla_{g,\dot\gamma}\dot\gamma, -N\right) &= e^{2\sigma} g_0\left(e^{-2\sigma}(\nabla_{0,\dot\gamma_0}\dot\gamma_0 - \nabla_0\sigma), -e^{-\sigma}N_0\right) \\ &= e^{-\sigma}(k_0 + \partial_{n_0}\sigma)\end{aligned}$$

as claimed. $\square$

Chapter 5

Loop measure description

This chapter corresponds to the paper [Wan18b].

We establish an expression (Theorem 5.7) of the Loewner energy of a Jordan curve on the Riemann sphere in terms of the Brownian loop measure (or Werner's measure on simple loops of $\mathrm{SLE}_{8/3}$ type). The proof is based on a formula for the change of the Loewner energy under a conformal map that is reminiscent of the restriction properties derived for SLE processes.

5.1 Conformal restriction for chords

We first recall the variation formula of the chordal Loewner energy under conformal restriction, first appeared in [Dub07] and [Wan16]: Let K be a compact hull in $\mathbb{H}$ at positive distance to 0. The simply connected domain $H_K := \mathbb{H}\backslash K$ coincides with $\mathbb{H}$ in the neighborhoods of 0 and ∞. Let Γ be a simple chord contained in H_K connecting 0 to ∞ with finite Loewner energy in $(\mathbb{H}, 0, \infty)$.

Proposition 5.1 ([Wan16] Proposition 4.1)**.** *The energy of* Γ *in* $(\mathbb{H}, 0, \infty)$ *and in* $(H_K, 0, \infty)$ *differ by*

$$\begin{aligned} I_{H_K,0,\infty}(\Gamma) - I_{\mathbb{H},0,\infty}(\Gamma) &= 3\log|\psi'(0)\psi'(\infty)| + 12\mathscr{B}(\Gamma, K; \mathbb{H}) \\ &= 3\ln|\psi'(0)\psi'(\infty)| + 12\mathscr{W}(\Gamma, K; \mathbb{H}), \end{aligned}$$

where ψ *is a conformal map* $H_K \to \mathbb{H}$ *fixing* $0, \infty$.

Notice that the derivatives of ψ at boundary points 0 and ∞ are well-defined by Schwarz reflection principle since H_K coincides with $\mathbb{H}$ in their neighborhood. The first equality is the analogy of the conformal restriction property of SLE derived in [LSW03]. The second equality is due to the fact that $\mathbb{H}$ is simply connected domain with non-polar boundary and both K and Γ are attached to the boundary, so that the Brownian loop hits both K and Γ if and only if the outer-boundary hits them. For readers' convenience, we include the derivation of the first equality below.

Without loss of generality, we choose the conformal map $\psi : H_K \to H$ as in Proposition 5.1 such that $\psi'(\infty) = 1$. Let K_t be the image of K under the flow g_t associated to Γ, $\tilde{g}_t$ the mapping-out function of $\psi(\Gamma[0,t])$, and $\psi_t = \tilde{g}_t \circ \psi \circ g_t^{-1} : \mathbb{H} \backslash K_t \to \mathbb{H}$ making the diagram commute (see figure 5.1). It suffices to show that for $T < \infty$,

$$\begin{aligned} I_{\mathbb{H},0,\infty}(\psi(\Gamma[0,T])) - I_{\mathbb{H},0,\infty}(\Gamma[0,T]) \\ = 3 \ln |\psi'(0)| + 12\mathscr{B}(\Gamma[0,T], K; \mathbb{H}) - 3 \ln |\psi_T'(0)| \end{aligned} \tag{5.1}$$

which implies Proposition 5.1 since the last term $3 \ln |\psi_T'(0)| \to 0$ when $T \to \infty$ and $I_{\mathbb{H},0,\infty}(\psi(\Gamma[0,T])) = I_{H_K,0,\infty}(\Gamma[0,T])$.

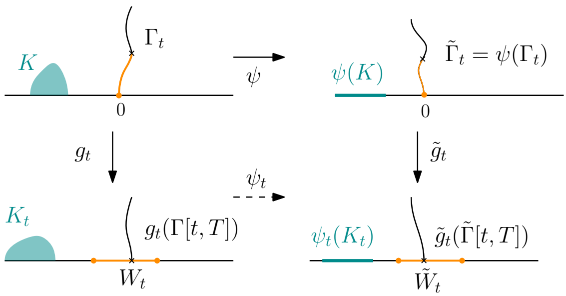

Figure 5.1: Maps in the proof of Proposition 5.1, $\tilde{W}_t = \psi_t(W_t)$.

Proof. We write W_t for $W(t)$ to shorten the notation. We show first

$$I_{\mathbb{H},0,\infty}(\psi(\Gamma[0,T])) = \frac{1}{2} \int_0^T \left[\partial_t W_t - \frac{3\psi_t''(0)}{\psi_t'(0)} \right]^2 \mathrm{d}t. \tag{5.2}$$

Notice that $(\tilde{\Gamma}_t := \psi(\Gamma_t))_{t \geq 0}$ is not capacity-parametrized. We denote

$a(t)$ the capacity of $\tilde{\Gamma}[0,t]$, such that

$$\tilde{g}_t(z) = z + 2a(t)/z + o(1/z), \text{ as } z \to \infty.$$

By a scaling consideration, we have $\partial_t a(t) = [\psi_t'(W_t)]^2$. The family of conformal maps $\tilde{g}_t$ satisfies the Loewner differential equation: for $z \in \mathbb{H}$,

$$\partial_t \tilde{g}_t(z) = \partial_a \tilde{g}_t(z) \partial_t a(t) = \frac{2[\psi_t'(W_t)]^2}{\tilde{g}_t(z) - \tilde{W}_t}.$$

Now we compute the variation of $\tilde{W}$. Since ψ_t is defined by $\tilde{g}_t \circ \psi \circ g_t^{-1}$, we have

$$\begin{aligned}
\partial_t \psi_t(z) &= \partial_t \tilde{g}_t(\psi \circ g_t^{-1}(z)) + (\tilde{g}_t \circ \psi)'(g_t^{-1}(z)) \partial_t (g_t^{-1}(z)) \\
&= \frac{2[\psi_t'(W_t)]^2}{\tilde{g}_t \circ \psi \circ g_t^{-1}(z) - \tilde{W}_t} + (\tilde{g}_t \circ \psi)'(g_t^{-1}(z)) \frac{-2(g_t^{-1})'(z)}{z - W_t} \\
&= \frac{2[\psi_t'(W_t)]^2}{\psi_t(z) - \tilde{W}_t} - \frac{2\psi_t'(z)}{z - W_t}.
\end{aligned} \tag{5.3}$$

Expanding ψ_t in the neighborhood of W_t (this is possible since ψ_t is analytic by Schwarz reflection principle), we obtain

$$\partial_t \psi_t(z) = -3\psi_t''(W_t) + O(z - W_t).$$

Therefore

$$\begin{aligned}
\partial_t \tilde{W}_t = \partial_t(\psi_t(W_t)) &= (\partial_t \psi_t)(W_t) + \psi_t(W_t) \partial_t W_t \\
&= \left(-3\frac{\psi_t''(W_t)}{\psi_t'(W_t)} + \partial_t W_t\right) \psi_t'(W_t).
\end{aligned}$$

Notice that since we assumed that Γ has finite Loewner energy in $\mathbb{H}$, it implies that W is in $W^{1,2}$. In particular, W is absolutely continuous. It is not hard to see that it implies that $\tilde{W}$ is also absolutely continuous and the above computation of $\partial_t \tilde{W}_t$ makes sense for almost every t.

The Loewner energy of $\psi(\Gamma[0,T])$ is given by

$$
\begin{aligned}
\frac{1}{2}\int_0^{a(T)} \left|\partial_a \tilde{W}(t(a))\right|^2 da &= \frac{1}{2}\int_0^T \left|\partial_t \tilde{W}(t)\right|^2 (a'(t))^{-1} dt \\
&= \frac{1}{2}\int_0^T \left[-3\frac{\psi_t''(W_t)}{\psi_t'(W_t)} + \partial_t W_t\right]^2 dt
\end{aligned}
$$

as we claimed.

Now we relate the right-hand side of (5.1) with the mass of Brownian loop measure attached to both Γ and K. Differentiating (5.3) in z and taking $z \to W_t$, we obtain

$$
(\partial_t \psi_t')(W_t) = \frac{\psi_t''(W_t)^2}{2\psi_t'(W_t)} - \frac{4\psi_t'''(W_t)}{3}.
$$

We have also

$$
\partial_t[\ln \psi_t'(W_t)] = \frac{1}{2}\left(\frac{\psi_t''(W_t)}{\psi_t'(W_t)}\right)^2 - \frac{4}{3}\frac{\psi_t'''(W_t)}{\psi_t'(W_t)} + \frac{\psi_t''(W_t)}{\psi_t'(W_t)}\partial_t W_t.
$$

Therefore

$$
\begin{aligned}
&\frac{1}{2}\left[\partial_t W_t - 3\frac{\psi_t''(W_t)}{\psi_t'(W_t)}\right]^2 - \frac{1}{2}(\partial_t W_t)^2 \\
&= \frac{9}{2}\left(\frac{\psi_t''(W_t)}{\psi_t'(W_t)}\right)^2 - 3\frac{\psi_t''(W_t)}{\psi_t'(W_t)}\partial_t W_t \\
&= -3\partial_t[\ln \psi_t'(W_t)] - 4S\psi_t(W_t),
\end{aligned} \tag{5.4}
$$

where

$$
S\psi_t = \frac{\psi_t'''}{\psi_t'} - \frac{3}{2}\left(\frac{\psi_t''}{\psi_t'}\right)^2
$$

is the Schwarzian derivative of ψ_t. Intergrating (5.4) over $[0,T]$, we obtain the identity (5.1) by identifying the Schwarzian derivative term using the path decomposition of the Brownian loop measure (see

[LSW03, LW04])

$$-4 \int_0^T S\psi_t(W_t)dt = 12\mathscr{B}(\Gamma[0,T], K; \mathbb{H}).$$

□

From Proposition 5.1, we deduce a more general relation of Loewner energy of the same chord in two domains D and D' which coincide in a neighborhood of both marked boundary points, by comparing to the Riemann surface $D \sqcup D'$ identified along the connected component of $D \cap D'$ containing Γ:

Corollary 5.2. *Let (D, a, b) and (D', a, b) be two simply connected domains in $\mathbb{C}$ coinciding in a neighborhood of a and b, and Γ a simple curve in both (D, a, b) and (D', a, b). Then we have*

$$\begin{aligned} I_{D',a,b}(\Gamma) - I_{D,a,b}(\Gamma) =& 3 \log |\psi'(a)\psi'(b)| \\ &+ 12\mathscr{W}(\Gamma, D \backslash D'; D) - 12\mathscr{W}(\Gamma, D' \backslash D; D'), \end{aligned}$$

where $\psi : D' \to D$ is a conformal map fixing a and b.

5.2 Conformal restriction for loops

We prove in this section the following conformal restriction formula for the loop energy. The loop version has the advantage compared to the chordal case of no longer having the boundary terms.

Theorem 5.3. *If η is a Jordan curve with finite energy and $\Gamma = f(\eta)$, where $f : A \to \tilde{A}$ is conformal on a neighborhood A of η, then*

$$I^L(\Gamma) - I^L(\eta) = 12\mathscr{W}(\eta, A^c; \mathbb{C}) - 12\mathscr{W}(\Gamma, \tilde{A}^c; \mathbb{C}).$$

The loop terms are finite since we can replace A^c by its boundary. Both ∂A^c and η are compact and are at positive distance.

Remark 5.4. *The right-hand side of the above identity remains the same if we replace A by a subset B such that $\eta \subset B \subset A$. In fact, since $A \backslash B$ is at positive distance with η, we have $\mathscr{W}(\eta, A\backslash B; A) < \infty$. We then decompose the loop measure*

$$\mathscr{W}(\eta, B^c; \mathbb{C}) = \mathscr{W}(\eta, A^c; \mathbb{C}) + \mathscr{W}(\eta, A\backslash B; A).$$

The conformal invariance of Werner's measure provides that

$$\mathscr{W}(\eta, A\backslash B; A) = \mathscr{W}(f(\eta), f(A\backslash B); f(A)) = \mathscr{W}(\Gamma, \tilde{A}\backslash f(B); \tilde{A}).$$

Hence

$$I^L(\Gamma) - I^L(\eta) = 12\mathscr{W}(\eta, B^c; \mathbb{C}) - 12\mathscr{W}(\Gamma, f(B)^c; \mathbb{C})$$

given that the formula holds for A.

Now we can prove Theorem 5.3:

Proof. From the remark, we assume that A is an annulus without loss of generality. Since the loop energy is a limit of chordal Loewner energies, the idea is to bring back to the conformal restriction in the chordal framework.

More precisely, let a, b be two points on the curve η, $\tilde{a} = f(a)$ and $\tilde{b} = f(b)$. Let $D := \hat{\mathbb{C}}\backslash(ab)_\eta$ and $\tilde{D} := \hat{\mathbb{C}}\backslash(\tilde{a}\tilde{b})_\Gamma$. We take a "stick" T attached to the arc $(ab)_\eta$ of the curve η, such that $D\backslash K$ is simply connected, where $K = A^c \cup T$ is the union of two "lollipops". Define $\tilde{T} := f(T) \subset \tilde{A}$ the conformal image of T and similarly $\tilde{K} = \tilde{A}^c \cup \tilde{T}$ (see Figure 5.2).

Now we compare the chordal Loewner energy of $(ba)_\eta$ (the complement of $(ab)_\eta$ in the curve η) in D and $(\tilde{b}\tilde{a})_\Gamma$ in $\tilde{D}$. Notice that D and $D\backslash K$ coincide in a neighborhood of both a and b. Let ψ and $\tilde{\psi}$ be a choice of conformal maps as in Figure 5.2, and g factorizes the

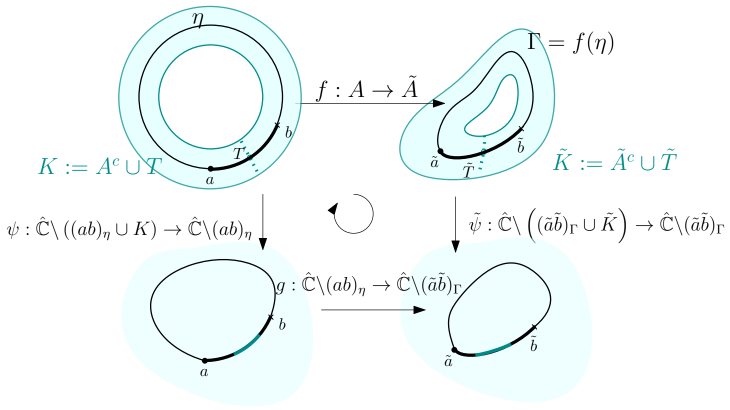

Figure 5.2: Maps in the proof of Theorem 5.3.

diagram. Applying Corollary 5.2 to $(ba)_\eta$ in D, we have

$$\begin{aligned} &I_{D\setminus K}((ba)_\eta) - I_D((ba)_\eta) \\ &\qquad = 3\log|\psi'(a)\psi'(b)| + 12\mathscr{W}((ba)_\eta, K; D), \end{aligned} \tag{5.5}$$

and similarly,

$$\begin{aligned} &I_{\tilde{D}\setminus \tilde{K}}((\tilde{b}\tilde{a})_\Gamma) - I_{\tilde{D}}((\tilde{b}\tilde{a})_\Gamma) \\ &\qquad = 3\log\left|\tilde{\psi}'(\tilde{a})\tilde{\psi}'(\tilde{b})\right| + 12\mathscr{W}((\tilde{b}\tilde{a})_\Gamma, \tilde{K}; \tilde{D}). \end{aligned} \tag{5.6}$$

From the construction,

$$\begin{aligned} I_{D\setminus K}((ba)_\eta) &= I_D(\psi[(ba)_\eta]) = I_{\tilde{D}}(g \circ \psi([(ba)_\eta]) \\ &= I_{\tilde{D}}(\tilde{\psi}[(\tilde{b}\tilde{a})_\Gamma]) = I_{\tilde{D}\setminus\tilde{K}}((\tilde{b}\tilde{a})_\Gamma), \end{aligned}$$

where the second equality follows from the conformal invariance of the chordal Loewner energy.

We write $H(a, b; D)$ for the Poisson excursion kernel between two boundary points a, b of the domain D (relatively to the local analytic coordinates). Choosing the same analytic coordinates near a, b in the above four pictures, then we have

$$\begin{aligned}\frac{\psi'(a)\psi'(b)}{\tilde{\psi}'(\tilde{a})\tilde{\psi}'(\tilde{b})} &= \frac{H(a,b;A\backslash(ab)_\eta)}{H(a,b;D)}\frac{H(\tilde{a},\tilde{b};\tilde{D})}{H(\tilde{a},\tilde{b};\tilde{A}\backslash(\tilde{a}\tilde{b})_\Gamma)} \\ &= \frac{H(a,b;A\backslash(ab)_\eta)}{H(\tilde{a},\tilde{b};\tilde{A}\backslash(\tilde{a}\tilde{b})_\Gamma)}\frac{H(\tilde{a},\tilde{b};\tilde{D})}{H(a,b;D)} = \frac{f'(a)f'(b)}{g'(a)g'(b)}\end{aligned}$$

which no longer depends on the stick T chosen.

Also notice that we can decompose the loop measure term as in the remark:

$$\mathscr{W}((ba)_\eta, K; D) = \mathscr{W}((ba)_\eta, A^c; D) + \mathscr{W}((ba)_\eta, T; A\backslash(ba)_\eta).$$

Since the Werner's measure is conformally invariant, we have in particular

$$\mathscr{W}((ba)_\eta, T; A\backslash(ba)_\eta) = \mathscr{W}((\tilde{b}\tilde{a})_\Gamma, \tilde{T}; \tilde{A}\backslash(\tilde{b}\tilde{a})_\Gamma).$$

Taking the difference (5.5) - (5.6) combining the above observations, we get

$$\begin{aligned}&I_{\tilde{D}}((\tilde{b}\tilde{a})_\Gamma) - I_D((ba)_\eta) \\ =&3\log\left|\frac{f'(a)f'(b)}{g'(a)g'(b)}\right| + 12\mathscr{W}((ba)_\eta, A^c; D) \\ &-12\mathscr{W}((\tilde{b}\tilde{a})_\Gamma, \tilde{A}^c; \tilde{D}).\end{aligned} \tag{5.7}$$

We conclude the proof by taking $b \to a$ on η, using the definition of loop energy

$$I_D((ba)_\eta) \xrightarrow{b\to a} I^L(\eta, a) = I^L(\eta)$$

and the fact that

$$\mathscr{W}((ba)_\eta, A^c; D) \xrightarrow{b\to a} \mathscr{W}(\eta, A^c; \mathbb{C}).$$

The log-derivative terms goes 0 thanks to the following lemma and concludes the proof of the theorem. $\square$

Lemma 5.5. *With the same notations as in the proof of Theorem 5.3* *(see Figure 5.2),*

$$\lim_{b \to a} \left| \frac{f'(a)f'(b)}{g'(a)g'(b)} \right| = 1.$$

Proof. Without loss of generality, we may assume that $A = \mathbb{D}$ the unit disk, $a = 0$, $\tilde{a} = f(a) = 0$. Let ψ be the conformal map $\hat{\mathbb{C}} \backslash (ab)_\eta \to \mathbb{D}$,

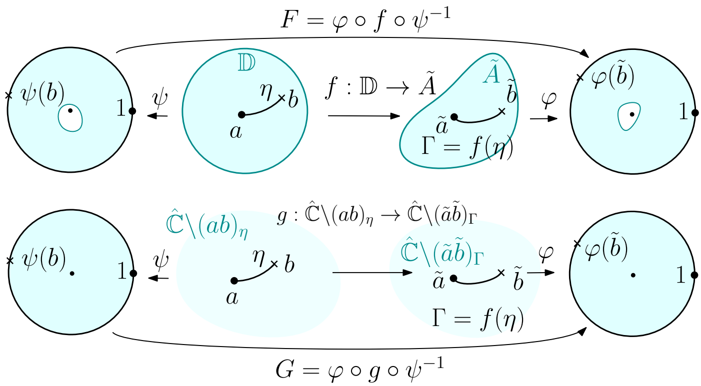

Figure 5.3: Maps in the proof of Lemma 5.5.

such that $\psi(\infty) = 0$, $\psi(0) = 1$. Similarly let φ be the conformal map $\hat{\mathbb{C}} \backslash (\tilde{a}\tilde{b})_\Gamma \to \mathbb{D}$, such that $\varphi(\infty) = 0$, $\varphi(0) = 1$. Define $F = \varphi \circ f \circ \psi^{-1}$ and $G = \varphi \circ f \circ \psi^{-1}$ between the blue-shaded area in Figure 5.3.

It is not hard to see that the diameter of $\psi(\mathbb{D}^c)$ and $\varphi(\mathbb{D}^c)$ shrinks to 0 as $b \to a$. Therefore

$$\frac{F'(1)F'(\psi(b))}{G'(1)G'(\psi(b))} \xrightarrow{b \to a} 1.$$

On the other hand,

$$\frac{F'(1)F'(\psi(b))}{G'(1)G'(\psi(b))} = \frac{H(a,b;A\backslash(ab)_\eta)}{H(\tilde{a},\tilde{b};\tilde{A}\backslash(\tilde{a}\tilde{b})_\Gamma)} \frac{H(\tilde{a},\tilde{b};\tilde{D})}{H(a,b;D)} = \frac{f'(a)f'(b)}{g'(a)g'(b)}$$

which concludes the proof. $\square$

By taking $\eta = S^1$, we deduce immediately the interpretation of the Loewner energy of an analytic Jordan curve:

Corollary 5.6. *If $\Gamma = f(S^1)$ is an analytic curve, then*

$$I^L(\Gamma) = 12\mathscr{W}(S^1, A^c; \mathbb{C}) - 12\mathscr{W}(\Gamma, \tilde{A}^c; \mathbb{C}),$$

where $f : A \to \tilde{A}$ maps conformally a neighborhood A of S^1 to a neighborhood $\tilde{A}$ of Γ.

Proof. This follows immediately from Theorem 5.3 and that $I^L(S^1) = 0$. $\square$

5.3 Cut-off by equipotentials

Let Γ be a Jordan curve in $\mathbb{C}$, D the bounded connected component of $\mathbb{C}\backslash\Gamma$ and f a conformal map from the unit disk $\mathbb{D}$ to D. For $1 > \varepsilon > 0$, let $S^{(1-\varepsilon)}$ denote the circle of radius $1-\varepsilon$, centered at 0, and $\Gamma^{(1-\varepsilon)} := f(S^{(1-\varepsilon)})$ the equi-potential.

Theorem 5.7. *We have*

$$I^L(\Gamma) = \lim_{\varepsilon \to 0} 12\mathscr{W}(S^1, S^{(1-\varepsilon)}; \mathbb{C}) - 12\mathscr{W}(\Gamma, \Gamma^{(1-\varepsilon)}; \mathbb{C}).$$

Proof. For each ε, we apply Corollary 5.6 to the analytic curve $\Gamma^{(1-\varepsilon)}$ with $A := D$ which gives

$$I^L(\Gamma^{(1-\varepsilon)}) = 12\mathscr{W}(S^1, S^{(1-\varepsilon)}; \mathbb{C}) - 12\mathscr{W}(\Gamma, \Gamma^{(1-\varepsilon)}; \mathbb{C}).$$

Now it suffices to see that $I^L(\Gamma^{(1-\varepsilon)})$ converges to $I^L(\Gamma)$.

If $I^L(\Gamma) < \infty$, from the geometric description of Loewner energy (see [Wan18a] Section 8) Γ is a quasi-circle of the Weil-Petersson class, which is equivalent to

$$\int_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 dz^2 < \infty.$$

Let $f_\varepsilon(z) := f((1-\varepsilon)z)$ denote the uniformizing conformal map from $\mathbb{D}$ to the bounded connected component $\mathbb{C}\backslash\Gamma^{(1-\varepsilon)}$. We have for $\varepsilon < \varepsilon_0 < 1/2$,

$$\begin{aligned}
&\int_{\mathbb{D}} \left|\frac{f''}{f'} - \frac{f''_\varepsilon}{f'_\varepsilon}\right|^2 dz^2 \\
&= \int_{|z|<1-\varepsilon_0} \left|\frac{f''}{f'} - \frac{f''_\varepsilon}{f'_\varepsilon}\right|^2 dz^2 + \int_{1-\varepsilon_0\le|z|<1} \left|\frac{f''}{f'} - \frac{f''_\varepsilon}{f'_\varepsilon}\right|^2 dz^2 \\
&\le \int_{|z|<1-\varepsilon_0} \left|\frac{f''}{f'} - \frac{f''_\varepsilon}{f'_\varepsilon}\right|^2 dz^2 + 4\int_{1-2\varepsilon_0\le|z|<1} \left|\frac{f''}{f'}\right|^2 dz^2 \\
&\xrightarrow{\varepsilon\to 0} 4\int_{1-2\varepsilon_0\le|z|<1} \left|\frac{f''}{f'}\right|^2 dz^2,
\end{aligned}$$

the convergence is due to the fact that $f''_\varepsilon/f'_\varepsilon$ converges uniformly on compacts to f''/f'. As $\varepsilon_0 \to 0$, the above integral converges to 0, and we conclude that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{D}} \left|\frac{f''(z)}{f'(z)} - \frac{f''_\varepsilon(z)}{f'_\varepsilon(z)}\right|^2 dz^2 = 0.$$

It yields that $\Gamma^{(1-\varepsilon)}$ converges in the Weil-Petersson metric to Γ (see [TT06] Corollary A.4 or [Wan18a] Lemma H) and therefore $I^L(\Gamma^{(1-\varepsilon)})$ converges as well to $I^L(\Gamma)$.

If $I^L(\Gamma) = \infty$, from the lower-semicontinuity of the Loewner loop energy ([RW17], Lemma 2.9) and the fact that $\Gamma^{(1-\varepsilon)}$ converges uniformly (parametrized by S^1 via f_ε) to Γ, we have

$$\liminf_{\varepsilon\to 0} I^L(\Gamma^{(1-\varepsilon)}) \ge I^L(\Gamma) = \infty.$$

Hence $I^L(\Gamma^{(1-\varepsilon)})$ converges to ∞ as $\varepsilon \to 0$. $\square$

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