On the optimal curve of SLE_{0+}

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Abstract

In this thesis, I show through the large deviation principle that the conditional law of the chordal SLE_κ in the upper half-plane $\mathbb{H}$ of passing to the right of a point $z_\theta \in \mathbb{H}$ with argument θ, converges to a deterministic curve when κ approaches 0. The limit is smooth and called the optimal curve to the right of z_θ. The point z_θ is on the curve if $\theta \le \frac{\pi}{2}$, and splits the curve into two parts, each of them is a geodesic in the domain left vacant by the other part. When we map $\mathbb{H}$ to the unit disc by sending z_θ to 0, and 0 to $e^{i\theta}$, the optimal curve is shown to be conjugate-invariant. The part with positive imaginary part is characterized as the radial $SLE_0(2)$, starting from $e^{i\theta}$ and forced by $e^{-i\theta}$. Then I give a second proof of the convergence, considering the conditioned driving function as the strong solution of a perturbed stochastic differential equation. Generalization to the multiple-constraint problem is discussed at the end of the article, where minimizing curves remain smooth, pass through some of its constraint-points, and satisfy the "geodesic" property.

Contents

1 Introduction

Let $(\Omega, \mathscr{F}, W)$ denote a completed standard Wiener space and B a Brownian motion. Let $(\mathscr{F}_t)_{t\geq 0}$ be the completed filtration generated by B, and write $X_t^\kappa = \sqrt{\kappa}B_t$. The chordal half-plane SLE_κ is the Loewner chain driven by X_t^κ.

It is well-known that for $0 < \kappa \leq 4$, the SLE_κ is almost surely a parameterized simple curve that never touches the real axis. Denoting the curve by $(\gamma_t^\kappa)_{t\geq 0}$, we know that $\gamma_t^\kappa \xrightarrow[t\to\infty]{a.s.} \infty$. Let $\gamma^\kappa = \bigcup_{t\geq 0} \gamma_t^\kappa$ be the trace of (γ_t^κ). For κ small, the driving function is a slight perturbation of 0 which generates the imaginary axis as Loewner trace. The SLE_κ is therefore expected to approach the imaginary axis. Hence we consider the following question: let z_θ be a point in $\mathbb{H}$ the open upper half-plane while $0 < \theta < \frac{\pi}{2}$, how does γ^κ behave when we condition it to pass to the right of z_θ?

The answer is, if we denote $\tilde{W}_\kappa$ as the conditional probability measure of X^κ on the rare event

$$A_{\kappa,z_\theta} = \{\gamma^\kappa \text{ passes to the right of } z_\theta\},$$

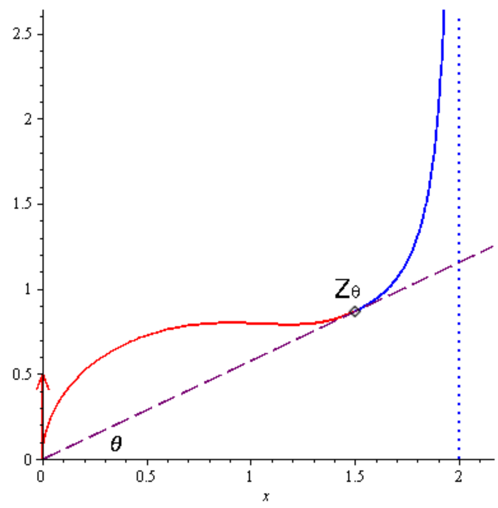

Figure 1: An illustration (not the exact computation result) of the optimal curve of SLE_{0+} in $\mathbb{H}$ to the right of z_θ with $\theta = \frac{\pi}{3}$.

then γ^κ under $\tilde{W}_\kappa$ converges to a deterministic curve as κ approaches 0, which is called *the optimal curve of* SLE_{0+} *to the right side of* z_θ.

More precisely, we prove in this work

Theorem 1

If we choose $z_\theta = e^{i\theta}$*, there exists a smooth curve* γ^* *passing through* z_θ *and is invariant under the map* $Rev : z \mapsto \frac{1}{\bar{z}}$*, such that*

$$\tilde{W}_\kappa(d(\gamma^\kappa, \gamma^*) \geq \delta) \xrightarrow[\kappa \to 0]{} 0, \quad \forall \delta > 0$$

where the metric d *is defined as following and induces the uniform convergence on compacts of driving function,*

$$d(\gamma_1, \gamma_2) := d(f_1, f_2) = \sum_{n=1}^{\infty} \frac{1 \wedge \|f_1 - f_2\|_{\infty,[0,n]}}{2^n},$$

and $\|\cdot\|_{\infty,[0,n]}$ *is the uniform norm on* $[0, n]$.

We will use the same notation for metrics on Loewner chains and on $C([0,\infty[)$ functions unless specified. If z_θ is replaced by $\tilde{z}_\theta = \lambda z_\theta$, $\lambda > 0$, then the optimal curve $\tilde{\gamma}^* = \lambda\gamma^*$ due to the scaling invariance of SLE_κ. Therefore without loss of generality, the target point z_θ is chosen to be $e^{i\theta}$ in the previous statement. Since the modulus of z_θ plays *a priori* no crucial role in the study of the optimal curve, the optimal curve is better seen in the unit disk $\mathbb{D}$ by applying the conformal mapping ψ_θ sending z_θ to 0, 0 to θ, which in turn sends ∞ to $e^{-i\theta}$:

$$\psi_\theta : z \mapsto e^{-i\theta}\frac{z - z_\theta}{z - \overline{z_\theta}}.$$

Note that the image of trace is invariant when we apply the previous scaling.

Hence we obtain an equivalent version of the previous theorem:

Theorem 2

The conditional law of the chordal SLE_κ in $\mathbb{D}$ from $e^{i\theta}$ to $e^{-i\theta}$ by the rare event of passing to the left of 0, converges in the previous sense when $\kappa \to 0$ to a deterministic curve containing 0, which is symmetric about the real axis.

Moreover, the optimal curve can be characterized and then identified as:

Theorem 3

The following statements are equivalent:

1. *$\gamma \subset \mathbb{H}$ is the optimal curve of SLE_{0+} passing to the right of z_θ.*
2. *γ is a simple curve in $\mathbb{H}$ from 0 to ∞ parameterized as Loewner chain, passing through z_θ at time T, and has a tangent on z_θ, such that the trace $\hat{g}_t(\gamma_{[t,\infty[}) = g_t(\gamma_{[t,\infty[}) - f_t$ is invariant under a multiple of Rev which preserves $z_t = \hat{g}_(z_\theta)$ for all $t \in [0, T]$.*
3. *γ passes through z_θ and the image of γ under $\psi(z) = e^{-i\theta}\frac{z-z_\theta}{z-\overline{z_\theta}}$ is symmetric about the real axis. The part above the real axis is described as radial Loewner chain starting from $e^{i\theta}$ driven by the function $\xi(t)$ up to time reparameterization, such that the associated radial Loewner flow $h_t : \mathbb{D}\backslash\gamma_{[0,t]} \to \mathbb{D}$ satisfies $h_t(e^{-i\theta}) = e^{-i\xi(t)}$. Consequently $\xi(t)$ is the solution of the differential equation:*

$$\begin{cases} \xi'(t) = \cot \xi(t) \\ \xi(0) = \theta. \end{cases}$$

The part of the optimal curve above real axis (resp. stopped when it hits z_θ) in the radial (resp. half-plane) characterization is the $SLE_0(2)$ starting at $(e^{i\theta}, e^{-i\theta})$ (resp. the $SLE_0(-8)$ starting at $(0, z_\theta)$).

The proof consists of the large deviation principle on Brownian paths and we have defined the energy on absolutely continuous functions. We prove that there exists a single minimizing function (see 3.5) among all functions whose Loewner transform is a simple curve which passes to the right of z_θ in $\mathbb{H}$. Then it must be the optimal curve (see 3.3).

Section 4 gives a direct proof of the convergence of conditional SLE_κ to the optimal curve on any compact time interval before the hitting time of the optimal curve at the target point. Under the conditional law, the driving function X_t and the flow of z_θ form

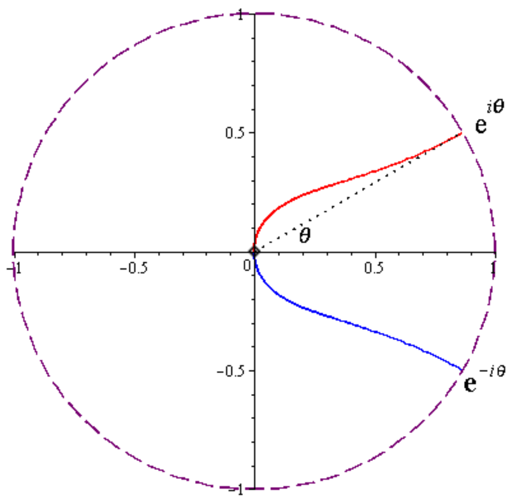

Figure 2: An illustration of the optimal curve of SLE_{0+} in $\mathbb{D}$ from $e^{i\theta}$ to $e^{-i\theta}$ passing to the left of 0 with $\theta = \frac{\pi}{3}$.

a solution to a perturbed stochastic differential equation,

$$(E_\kappa): \quad \begin{cases} dX_t = \sqrt{\kappa}d\beta_t + \frac{F(\cot(z_t))}{y_t}dt + \frac{\varepsilon_\kappa(\cot(z_t))}{y_t}dt \\ dz_t = \frac{2}{z_t}dt - dX_t \end{cases} \qquad \text{for } t < T$$

where β is a Brownian motion, T the survival time of the solution to (E_0) and

$$y_t = \operatorname{Im}(z_t), \quad F(w) = \frac{8w}{w^2+1} \wedge 0 \text{ and } \varepsilon_\kappa(w) \xrightarrow[\kappa \to 0]{\text{unif. in } w} 0.$$

We show that the solution converges to the unperturbed one, similar to the Freidlin-Wentzell theorem (see [FSW12]).

The last part of this work deals with the generalization to multiple-point constraints instead of one single point. We show that a minimizing curve passes through some of the constraint points, it is also a local minimizing curve if we omit constraint points not on the curve. Any part of the curve between two consecutive points is a geodesic in the simply connected domain left vacant by other parts of the curve. This property, called (P), is stated more precisely in the last section.

2 Schramm-Loewner Evolution

We now very briefly review the definition and relevant properties of chordal and radial SLE. The presentation is minimal and these properties are stated without proof. For proofs and further descriptions, readers can refer to excellent textbooks such as [BN11], [Wer03], and [SR05].

2.1 Loewner chain and the Loewner-Kufarev theorem

We say that a subset K of $\mathbb{H}$ is a *Compact* $\mathbb{H}$*-Hull of size* $a(K)$ *seen from* ∞, if K is bounded and $H_K := \mathbb{H} \backslash K$ is simply connected, the unique conformal transformation $g_K : H_K \to \mathbb{H}$ such that $g_K(z)$ around ∞ is of form

$$g_K(z) = z + \frac{2a(K)}{z} + o(\frac{1}{z}).$$

We call g_K the *mapping-out function* of K.

Let $\mathscr{K}$ denote the set of compact $\mathbb{H}$-hulls, endowed with the Caratheodory metric

$$d_K(K_1, K_2) = \|g_{K_1}^{-1} - g_{K_2}^{-1}\|_\infty.$$

Note that it is quite different from the Hausdorff distance. For instance, the arc $\{e^{i\theta}, \theta \in [0, \pi - \varepsilon]\}$ converges to the half-disc of radius 1 and center 0 for the Caratheodory metric, but not for the Hausdorff metric.

Let $(K_t)_{t>0} \subset \mathscr{K}$ be an (strictly) increasing family of compact $\mathbb{H}$-hulls for inclusion. For $s < t$, define $K_{s,t} = g_{K_s}(K_t \backslash K_s)$. We say that the sequence $(K_t)_{t>0}$ has *local growth property* if

$$\mathrm{rad}(K_{t,t+h}) \xrightarrow[h \downarrow 0]{\text{unif. on compacts in t}} 0.$$

where *rad* is the radius of the Euclidean metric centering on the real axis:

$$\mathrm{rad}(K) = \inf\{r \geq 0 : \ K \subset r\overline{\mathbb{D}} + x \text{ for some } x \in \mathbb{R}\}.$$

Fix $f \in C([0, \infty[)$, consider the Loewner differential equation

$$\begin{cases} \partial_t g_t(z) & = \frac{2}{g_t(z) - f(t)} \\ g_0(z) & = z. \end{cases} \tag{1}$$

The *chordal Loewner chain in* $\mathbb{H}$ *driven by the function* f (or *the Loewner transform of* f), is an increasing sequence $(K_t) \in \mathscr{L}$ such that

$$K_t = \{z \in \mathbb{H} \,|\, \tau(z) \leq t\}$$

Where $\tau(z)$ is the maximum surviving time of the solution, *i.e.*,

$$\tau(z) = \sup\{t \geq 0 \,|\, \inf_{0 \leq s \leq t} |g_s(z) - f(s)| > 0\}.$$

Moreover, when equation (1) can be extended to $\overline{\mathbb{H}}$, there is a non-trivial fact (see eg. sec. 8.2 [BN11]) that

$$\{z \in \overline{\mathbb{H}} \,|\, \tau(z) \leq t\} = \overline{K_t},$$

where K_t is a compact $\mathbb{H}$-hull in $\mathscr{K}$ of size t, and the family $(K_t)_{t\geq 0}$ satisfies local growth property. The map $g_t : \mathbb{H}\backslash K_t \to \mathbb{H}$ is the mapping-out function of K_t, and the family (g_t) is called its *Loewner flow.*

Denote $\mathscr{L}$ as the set of all increasing sequences of compact $\mathbb{H}$-hulls $(K_t)_{t\geq 0}$, parameterized with $a(K_t) = t$. We endow $\mathscr{L}$ with the topology of uniform convergence on compact intervals of t.

Hence the Loewner equation provides a way to generate a family of compact $\mathbb{H}$-hulls in $\mathscr{L}$ from a continuous function. The following theorem gives the explicit inverse and claims that both procedures are continuous when we equip $C([0,\infty[,\mathbb{R})$ with the uniform convergence on compacts.

Theorem 4 (*Loewner-Kufarev*)

The Loewner transform $L : C([0,\infty[,\mathbb{R}) \to \mathscr{L}$ *is a bi-adapted homeomorphism. The inverse function is given by,*

$$\xi_t = \bigcap_{h>0} \overline{K_{t,t+h}}, \forall t \geq 0$$

where $K_{s,s'} = g_s(K_{s'}\backslash K_s)$ *for* $s < s'$.

2.2 Some properties of SLE

Definition 1. Fix $\kappa > 0$, the Loewner transform of $(\sqrt{\kappa}B_t)_{t\geq 0}$ is a random variable taking values in $\mathscr{L}$ and is called the *chordal* SLE_κ *in* $\mathbb{H}$ *from* 0 *to* ∞.

They are the unique processes having paths in $\mathscr{L}$ and satisfying the*scaling invariance* and the *domain Markov property. i.e.,* for $\lambda \in (0,\infty)$, the law is invariant under the scale

$$\mathscr{L} \to \mathscr{L}, \quad (K_t)_{t\geq 0} \mapsto (K_t^\lambda := \lambda K_{\lambda^{-2}t})_{t\geq 0}$$

and for all $s \in [0,\infty)$, set

$$K_t^{(s)} = K_{s,s+t} - \xi_s,$$

then the process $(K_t^{(s)})_{t\geq 0}$ has the same distribution as $(K_t)_{t\geq 0}$ and is independent of $\mathscr{F}_s = \sigma(\xi_r : r \geq s)$.

Theorem 5

For $\kappa < 4$, SLE_κ *is almost surely a simple curve* $(\gamma_t)_{t\geq 0}$ *starting at* 0. *Almost surely* $|\gamma_t| \to \infty$ *as* $t \to \infty$.

The chordal SLE_κ can also be defined in a simply connected domain D different from $\mathbb{R}^2$, with two distinguished points on the boundary by just taking the image of the conformal transformation from $\mathbb{H}$ to D, sending 0 and ∞ to these two points respectively. Since the only conformal transformation in $\mathbb{H}$ preserving 0 and ∞ are homotheties, the law of SLE traces is well defined thanks to the scaling invariance.

Different ordering on those two distinguished points does not change the distribution of the trace of SLE_κ for $\kappa \in (0,4]$ due to the highly non-trivial reversibility property of chordal SLEs. It was first conjectured by S. Rhode and O. Schramm in [SR05] and then proved by D. Zhan in 2008 [Zha08]. This property will be used repeatedly in this article.

Theorem 6 (*2008, [Zha08]*)

For $\kappa \in (0,4]$, the distribution of the trace SLE_κ in $\mathbb{H}$ coincides with its image under Rev : $z \to \frac{1}{\bar{z}}$.

The radial Loewner chain can be defined in a similar way in the unit disc $\mathbb{D}$.

Definition 2. The radial Loewner transform of $\xi \in C([0,\infty[,\mathbb{R})$ is the $(K_t)_{t\geq 0}$ family given by the radial Loewner equation:

$$\forall z \in \overline{\mathbb{D}}, \quad \begin{cases} \partial_t h_t(z) = -h_t(z)\frac{h_t(z)+e^{i\xi(t)}}{h_t(z)-e^{i\xi(t)}} \\ h_0(z) = z \end{cases}$$

where K_t is as in the chordal case. Taking $\xi(t) = \theta + \sqrt{\kappa}B_t$, the image of the radial Loewner transform is called radial SLE_κ starting from $e^{i\theta}$.

3 Minimizing curves

3.1 Large deviations on Brownian sample paths:

Let $T \in \mathbb{R}_+$ and $C_0([0,T])$ be the set of continuous function on $[0,T]$ for any f with $f(0) = 0$, endowed with the uniform norm.

Theorem 7 (*Schilder*)

(See eg. [DZ98])

The law of the sample path of $\sqrt{\kappa}B_t$ (the scaled Wiener measure W_κ) satisfies the large deviations principle with good rate function while κ approaches 0:

$$I_T(f) = \begin{cases} \frac{1}{2}\int_0^T \dot{f}(s)^2 \mathrm{d}s, & f \text{ absolutely continuous} \\ +\infty & \text{otherwise.} \end{cases}$$

where $\int_0^t \dot{f}(s)ds = f(t)$, $\forall t \geq 0$. i.e., For any closed set F and any open set O of $C_0([0,T])$,

$$\overline{\lim_{\kappa\to 0}}\, \kappa \ln W_\kappa(F) \leq -\inf_{f\in F} I_T(f),$$

$$\underline{\lim_{\kappa\to 0}}\, \kappa \ln W_\kappa(O) \geq -\inf_{f\in O} I_T(f);$$

the action functional *I_T is lower semi-continuous and $\{f \in C_0([0,T]) | \, I_T(f) \leq c\}$ is compact for every $c \geq 0$.*

We denote by H_T the set of all functions in $C_0([0,T])$ with finite I_T value, which is known as the Cameron-Martin space of Brownian motion on $[0,T]$. By elementary analysis, the following assertions are equivalent:

- $f \in H_T$;
- $f \in C_0([0,T])$ and $M_f := \sup_\Pi \sum_{t_{j-1},t_j \in \Pi} \frac{|f(t_j)-f(t_{j-1})|^2}{(t_j - t_{j-1})} < \infty$, where the supremum is taken over all finite partition $\Pi = (0 = t_0, t_1, t_2, \dots, t_k \leq T)$.

And in that case, $I_T(f) = M_f$. We define I_∞ in the same manner, even if the terminology has already been abused, we still call it the *energy* of f (or the Loewner chain associated), and the last equality holds also for $T = \infty$.

3.2 Existence of minimizing curves

Consider $D \subset C_0([0,+\infty[)$ which consists of continuous functions whose Loewner transform is a simple curve, goes to infinity when $t \to \infty$, and for $\kappa \leq 4$, the law of κB is supported on D. The general study of Loewner chain with finite energy is carried out in the section. 5.1. In particular it is shown that $H_\infty \subset D$. We write also $D_{z_\theta} \subset D$ of those passes to the right of z_θ (or through z_θ).

Lemma. 3 in [Sch01] gives:

Lemma 8. *For every $f \in D_{z_\theta}$ such that the associate Loewner chain does not pass through z_θ,*

$$\lim_{t\to\infty} w_t = -\infty,$$

where $w_t = \cot(arg(g_t(z_\theta) - f_t))$ and $(g_t)_{t\geq 0}$ is the Loewner flow.

A simple curve γ^* is called a *minimizing curve* in D_{z_θ} if it is the Loewner transform of a function $f^* \in C_0(\mathbb{R}_+)$ which minimizes the rate function $I_\infty(f) = \lim_{t\to\infty} I_t(f)$ among all functions in D_{z_θ}. It can be easily seen that if such function f^* exists, it should be constant for t sufficiently large.

Otherwise, one can choose $f(\cdot \wedge T)$ instead of f for some large T such that $w_T < 0$ by lemma 8. The Loewner chain remains on the right side of z_θ whereas the value of rate function decreases. We can also deduce that $\inf_{f\in D_{z_\theta}} I_\infty(f)$ is finite.

We first study properties of minimizing curves and show that there exists only one, and then it coincides with the optimal curve.

Proposition 9. *Let $h_\kappa(\theta) = W_\kappa(D_{z_\theta})$, depending only on the argument of z_θ. There is a function $l : \theta \to \mathbb{R}_+$, such that*

$$-l_\theta = \lim_{\kappa\to 0} \kappa \ln h_\kappa(\theta) = 8\ln(\sin(\theta)). \tag{2}$$

The quantity l_θ is called as the energy needed *by SLE_{0+} to pass to the right of a point of argument θ.*

The scaling invariance of SLE gives the non-dependence of the modulus of point z_θ. The existence and the value of that limit derive from explicit computation of $h_\kappa(\theta)$ with details given in Sec. 3.5. We explore first the fact that l_θ is the infimum of the rate function:

Let $t \in \mathbb{R}$ and the set $D^t_{z_\theta} \subset C_0([0,\infty[)$ denotes the collection of any function f such that $f(\cdot \wedge t) \in D_{z_\theta}$, or equivalently any driving function such that $w_t \leq 0$. The interior of $D^t_{z_\theta}$ consists of those functions such that $w_t < 0$.

Proposition 10. *There exists $\tau \in \mathbb{R}_+$ such that the function l_θ satisfies*

$$l_\theta = \inf_{f \in D_{z_\theta}} I_\infty(f) = \inf_{f \in D^t_{z_\theta}} I_t(f), \quad \forall t \in [\tau, \infty[. \tag{3}$$

Both infimum are realized, i.e., let $\Gamma_{z_\theta} \subset D_{z_\theta}$ denote the set of minimizing functions, then for any $f^* \in \Gamma_{z_\theta}$,

$$T_{f^*} = \inf\{t \geq 0, f^*(t) = l_\theta\} = \inf\{t \geq 0, w_t = 0\}.$$

Proof: For $R > |z_0|$, the stopping time $\tau_R = \inf\{t \geq 0 \,|\, |\gamma_t| = R\}$ is bounded. In fact, the trace $\gamma_{[0,R]} \subset \mathbb{D}_R \cap \mathbb{H}$, the half-plane capacity, is non-decreasing. Hence

$$\tau_R = a(\gamma_{[0,\tau_R]}) \leq a(\mathbb{D}_R \cap \mathbb{H}) = R^2.$$

We know that γ does not hit z_θ almost surely. Therefore, except from a zero measure set,

$$D_{z_\theta} \Delta D^{\tau_R}_{z_\theta} = \{\gamma \,|\, w_{\tau_R} > 0, \text{ but } w_t \xrightarrow[t\to\infty]{} -\infty\} \cup \{\gamma \,|\, w_{\tau_R} < 0 \text{ but } w_t \xrightarrow[t\to\infty]{} +\infty\}$$

One can see that $|w_{\tau_R}|$ is uniformly bounded from below by a number $\cot(\theta')$ almost surely, with arbitrarily small θ' unless we choose R sufficiently large. By now we fix $\theta' < \theta$ and a corresponding R.

By the domain Markov property of SLE_κ, the probability of $\sqrt{k}B$ stays in the cross-difference of D_{z_θ} and $D^{\tau_R}_{z_\theta}$, which is bounded by $2F_\kappa(\theta')$. Using the upper-bound $\tau_R \leq R^2$ we get the cross-difference between D_{z_θ} and $D^{R^2}_{z_\theta}$:

$$\begin{aligned}
W_\kappa(D_{z_\theta} \Delta D^{R^2}_{z_\theta}) \leq & W_\kappa(D_{z_\theta} \Delta D^{\tau_R}_{z_\theta}) + W_\kappa(w_{\tau_R} \leq 0, w_{R^2} \geq 0, \lim_{t\to\infty} w_t = -\infty) \\
& + W_\kappa(w_{\tau_R} \geq 0, w_{R^2} \leq 0, \lim_{t\to\infty} w_t = \infty) \\
\leq & W_\kappa(D_{z_\theta} \Delta D^{\tau_R}_{z_\theta}) + W_\kappa(w_{\tau_R} \leq 0, w_{R^2} \geq 0, \lim_{t\to\infty} w_t = \infty) \\
& + W_\kappa(w_{\tau_R} \geq 0, w_{R^2} \leq 0, \lim_{t\to\infty} w_t = -\infty) \\
\leq & 2W_\kappa(D_{z_\theta} \Delta D^{\tau_R}_{z_\theta}) \\
\leq & 4F_\kappa(\theta').
\end{aligned}$$

And we set $\tau = R^2$.

Using the large deviation principle on Brownian sample paths on $[0,\tau]$ and that the set $D^\tau_{z_\theta}$ is closed, fix $\varepsilon>0$, then for κ sufficiently small,

$$\kappa \ln W_\kappa(D_{z_\theta}) \le \kappa \ln \left[W_\kappa(D^\tau_{z_\theta}) + 4F_\kappa(\theta')\right] \le -\inf_{f\in D^\tau_{z_\theta}} I_\tau(f) + \varepsilon \tag{4}$$

and

$$\kappa \ln W_\kappa(D_{z_\theta}) \ge \kappa \ln \left[W_\kappa(\overset{\circ}{D}{}^\tau_{z_\theta}) - 4F_\kappa(\theta')\right] \ge -\inf_{f\in \overset{\circ}{D}{}^\tau_{z_\theta}} I_\tau(f) - \varepsilon \tag{5}$$

Therefore

$$\inf_{f\in \overset{\circ}{D}{}^\tau_{z_\theta}} I_\tau(f) \le l_\theta \le \inf_{f\in D^\tau_{z_\theta}} I_\tau(f).$$

Both infimum are actually equal, for a function f which makes $w_\tau = 0$ by replacing the tail $f_{[\tau-\eta,\tau]}$ by a more "brutal" function, with η small and $w_{\tau-\eta}$ close enough to 0. One can get a function in its interior, and does not change too much of its energy. This again shows the existence of limit $\kappa \ln W_\kappa(D_{z_\theta})$.

All considerations above apply to $t>\tau$ and we get:

$$\inf_{f\in D_{z_\theta}} I_\infty(f) = \inf_{t\ge\tau} \inf_{f\in D^t_{z_\theta}} I_t(f) = l_\theta = \inf_{f\in D^t_{z_\theta}} I_t(f)$$

as previously announced. In consequence, it suffices to look at functions defined till τ to find a minimizer in D_{z_θ}. The set $D^\tau_{z_\theta}$ is closed and I_τ is a good rate function, and the infimum is realized for some $f\in D^\tau_{z_\theta}$. □

3.3 Minimizing curve v.s. optimal curve

We will see that there is a unique minimizing curve in 3.5. But before that, the following proposition shows that it's the good candidate for the optimal curve.

Proposition 11. *If γ^* be the unique minimizing curve to the right of z_θ, driven by a continuous function f^*. Let T denote the time from when f^* becomes stationary and γ^κ a SLE_κ in $\mathbb{H}$ driven by $X^\kappa = \sqrt{\kappa}B$. Then for every $n\in\mathbb{N}$, $\delta>0$,*

$$\tilde{W}_\kappa\{\|X^\kappa - f^*\|_{\infty,[0,n]} \ge \delta\} \xrightarrow[\kappa\to 0]{} 0.$$

Proof: We prove it for $n\ge\tau$ as defined in Prop. 10, and it is clear that $\tau\ge T$. For every $\varepsilon>0$, for κ small enough,

$$\begin{aligned}\kappa \ln \tilde{W}_\kappa\{\|X^\kappa - f^*\|_{\infty,[0,n]} \ge \delta\} =& \kappa \ln W_\kappa\left(\{\|X^\kappa - f^*\|_{\infty,[0,n]} \ge \delta\}\cap D_{z_\theta}\right) - \kappa\ln W_\kappa(D_{z_\theta})\\ \le& \kappa \ln \left[W_\kappa\left(\{\|X^\kappa - f^*\|_{\infty,[0,n]} \ge \delta\}\cap D^n_{z_\theta}\right) + W_\kappa(D_{z_\theta}\Delta D^n_{z_\theta})\right]\\ &+ l_\theta + \varepsilon\\ \le& -\min\left\{\inf_{f\in D^n_{z_\theta}, \|f-f^*\|_\infty\ge\delta} I_n(f), l_{\theta'}\right\} + l_\theta + 2\varepsilon\end{aligned}$$

Hence

$$\lim_{\kappa\to 0} \kappa \ln \tilde{W}_\kappa\{\|X^\kappa - f^*\|_{\infty,[0,n]} \geq \delta\} \leq -\min\left\{\inf_{f\in D^n_{z_\theta}, \|f-f^*\|_\infty\geq\delta} I_n(f), l_{\theta'}\right\} + l_\theta < 0,$$

where we have used

$$\inf_{f\in D^n_{z_\theta}, \|f-f^*\|_\infty\geq\delta} I_n(f) > l_\theta.$$

In fact, if, in the contrary, $\inf_{f\in D^n_{z_\theta}, \|f-f^*\|_\infty\geq\delta} I_n(f) = l_\theta$, then the intersection

$$\bigcap_{m\geq 1} \{f \in D^n_{z_\theta} \mid \|f - f^*\|_\infty \geq \delta, I_n(f) \leq l_\theta + \frac{1}{m}\}$$

is non-empty which contradicts to the uniqueness of the minimizing curve.

We conclude by simply taking the exponential. □

Now Theorem 1 is a corollary. Fix $\delta > 0$, there exists some sufficiently large $n \in \mathbb{N}$ s.t.

$$\sum_{i=n}^{\infty} \frac{1}{2^i} \leq \frac{\delta}{2}.$$

Therefore

$$\{d(f, f^*) > \delta\} \subset \{\|f - f^*\|_{\infty,[0,n]} \geq \frac{\delta}{2}\}.$$

Hence the unique minimizing curve is the optimal curve we look for,

$$\tilde{W}_\kappa(d(\gamma^\kappa, \gamma^*) \geq \delta) \leq \tilde{W}_\kappa\{\|f - f^*\|_{\infty,[0,n]} \geq \frac{\delta}{2}\} \xrightarrow[\kappa\to 0]{} 0.$$

3.4 Geometric properties of minimizing curves

In this section we will discuss and give an intuition of what a minimizing curve look like. We do not assume uniqueness in this case.

Let γ^* be a minimizing curve in Γ_{z_θ}, f^* and (g_t) be the associated driving function and Loewner flow. Let T_{f^*} be the first time f^* becomes stationary.

Proposition 12. *The trace of $\gamma^*|_{]0,T_{f^*}[}$ is strictly above the ray directed by $e^{i\theta}$ (i.e., whose argument $> \theta$) and $\gamma^*|_{[T_{f^*},\infty[}$ is above the ray and hits z_θ.*

Proof: Fix $0 < s < T_{f^*}$, $\theta' := \arg(\gamma^*_s)$, $\tilde{f}^*(t) := f^*(s \wedge t)$ induces a Loewner chain hitting γ^*_s, and thus

$$l_{\theta'} \leq \lim_{t\to+\infty} I_t(\tilde{f}^*) < l_\theta.$$

Since $\theta \mapsto l_\theta$ is decreasing, we get $\theta' > \theta$. The same argument works analogously for $s \geq T_{f^*}$, $\theta' := \arg(\gamma^*_s)$,

$$l_{\theta'} \leq \lim_{t\to+\infty} I_t(f^*) = l_\theta.$$

In consequence $\theta' \geq \theta$. We immediately have that γ^*_t hits z_θ. □

Define $T := \{t \geq 0 \,|\, \gamma_t = z_\theta\} \geq T_{f^*}$. It is not obvious that T coincides with T_{f^*}.

Proposition 13 (Regularity and asymptotic behavior of γ^*)**.** *A minimizing curve $\gamma^* \in \Gamma_{z_\theta}$ is a C^∞ parameterized curve in $]0, T]$ and $[T, +\infty[$, which has a vertical semi-tangent at 0 and a vertical asymptote when $t \to \infty$: $\{z \,|\, Re(z) = f^*(T)\}$.*

Proof: First we look at the trace of γ^* after T.

Recall that $g_T : \mathbb{H}\backslash\gamma^*([0,T]) \to \mathbb{H}$ is the conformal mapping-out function of $\gamma^*([0,T])$, which behaves like $g_T(z) - z = O(\frac{1}{z})$ for $z \to \infty$. Since $T \geq T_{f^*}$, f^* is stationary after T by definition, and hence

$$g_T(\gamma^*_{T+t}) = it + f^*(T), \quad t \leq 0$$

Thus, when $t \to \infty$,

$$\gamma^*_{T+t} = it + f^*(T) + O(\frac{1}{t}). \tag{6}$$

Furthermore, $t \to \gamma^*_t$ is smooth after time T and we get its asymptote at ∞.

For $\gamma^*|_{[0,T]}$, the regularity and tangency are deduced by the reversibility of SLE_κ for κ sufficiently small. Therefore Γ_{z_θ} is invariant under a multiple of Rev: $z \mapsto \frac{1}{\bar{z}}$ preserving z_θ. $\square$

Proposition 14. *If γ^* is differentiable at T, it is tangent to the line passing through 0 and directed by $e^{i\theta}$.*

Proof: Immediate deduction after Prop. 12. $\square$

3.5 Uniqueness of minimizing curve

Now we are ready to prove the uniqueness of the minimizing curve, and then deduce that it is the optimal curve of SLE_{0+} which passes to the right of z_θ, as we discussed in 3.3. We use asymptotic estimates of probability given in Prop. 15 to demonstrate uniqueness of the minimizing curve, which is not easy to be extended to a case of multiple-point restriction. Nevertheless, here we provide a rigorous proof for the one-point case.

Proposition 15. *For $\theta \leq \pi/2$, the value of l_θ defined in Prop. 10 is $-8\ln(\sin(\theta))$.*

Proof: We can compute this value by calculating $W_\kappa(D_{z_\theta})$. It was first calculated by Oded Schramm (see [Sch01]). Using the fact that $(W_\kappa(D_{z_\theta}))_{t\geq 0}$ is a martingale, the drift vanishes and hence the Itô formula gives:

Theorem 16

Let $w = \cot(\theta)$, the function $h_\kappa : w \mapsto W_\kappa(D_{z_\theta})$ satisfies the differential equation:

$$\begin{cases} \frac{\kappa}{2}h''_\kappa(w) + \frac{4w}{w^2+1}h'_\kappa(w) = 0 \\ h_\kappa(+\infty) = 0, \quad h_\kappa(-\infty) = 1. \end{cases} \tag{7}$$

Therefore $h'_\kappa(w) = Ce^{\int_0^w -\frac{8s}{k(s^2+1)}\mathrm{d}s} = C(w^2+1)^{-\frac{4}{\kappa}}$ for some constant C and

$$h_\kappa(w) = \int_w^\infty -C(s^2+1)^{-\frac{4}{\kappa}}\mathrm{d}s.$$

The following Lemma is crucial in further estimates, as it allows us to compute explicitly the energy l_θ.

Lemma 17. *Let $F(w) = \frac{8w}{w^2+1} \wedge 0$, and $\varepsilon_\kappa(w) = -\kappa\frac{h'_\kappa(w)}{h_\kappa(w)} - F(w)$, then*

$$\varepsilon_\kappa(w) = -\kappa\frac{h'_\kappa(w)}{h_\kappa(w)} - F(w) = \frac{\kappa(w^2+1)^{-\frac{4}{\kappa}}}{\int_w^\infty(s^2+1)^{-\frac{4}{\kappa}}\mathrm{d}s} - F(w) \xrightarrow[\kappa\to 0]{\textit{unif. in } w} 0.$$

Admit first the uniform convergence, then for $w \geq 0$,

$$\begin{aligned}\kappa\ln(h_\kappa(w)) &= \kappa\int_0^w \frac{h'_\kappa(s)}{h_\kappa(s)}\mathrm{d}s + \kappa\ln(h_\kappa(0))\\ &= \int_0^w \kappa\frac{h'_\kappa(s)}{h_\kappa(s)}\mathrm{d}s + \kappa\ln(\frac{1}{2})\\ &\xrightarrow{\kappa\to0} \int_0^w -\frac{8s}{s^2+1}\mathrm{d}s = -4\ln(w^2+1).\end{aligned}$$

Replacing w by $\cot(\theta)$, we obtain for $\theta \leq \pi/2$,

$$l_\theta = 4\ln(\cot(\theta)^2+1) = -8\ln(\sin(\theta))$$

as announced. □

Proof (of Lemma. 17) **: For** $w > 0$, we can write

$$\varepsilon_\kappa(w) = \frac{\kappa(w^2+1)^{-\frac{4}{\kappa}}}{\int_w^\infty(s^2+1)^{-\frac{4}{\kappa}}\mathrm{d}s} - \frac{8w}{w^2+1} = \frac{2}{(w^2+1)M_\kappa(w)} - \frac{8w}{w^2+1}$$

after a change of variable in the integral: $t^2 = \frac{(w^2+1)s^2}{s^2+1}$,

$$M_\kappa(w) = \frac{2}{\kappa}\int_w^{\sqrt{w^2+1}}(w^2+1-t^2)^{\frac{4}{\kappa}-\frac{3}{2}}\mathrm{d}t.$$

It can be bounded by

$$\begin{aligned}M_\kappa(w) &\leq \frac{1}{\kappa w}\int_w^{\sqrt{w^2+1}} 2t(w^2+1-t^2)^{\frac{4}{\kappa}-\frac{3}{2}}\mathrm{d}t\\ &= \frac{1}{\kappa w}\frac{2\kappa}{8-\kappa}[-(w^2+1-t^2)^{\frac{4}{\kappa}-\frac{1}{2}}]_w^{\sqrt{w^2+1}}\\ &= \frac{2}{w(8-\kappa)}\end{aligned}$$

For $1 > \varepsilon > 0$,

$$\begin{aligned} M_\kappa(w) &\geq \frac{1}{\kappa\sqrt{w^2+\varepsilon}}\int_w^{\sqrt{w^2+\varepsilon}} 2t(w^2+1-t^2)^{\frac{4}{\kappa}-\frac{3}{2}}\mathrm{d}t \\ &= \frac{1}{\kappa\sqrt{w^2+\varepsilon}}\frac{2\kappa}{8-\kappa}[-(w^2+1-t^2)^{\frac{4}{\kappa}-\frac{1}{2}}]_w^{\sqrt{w^2+\varepsilon}} \\ &= \frac{2}{(8-\kappa)\sqrt{w^2+\varepsilon}}(1-(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}}). \end{aligned}$$

Then we obtain bounds on ε_κ:

$$\varepsilon_\kappa(w) \geq \frac{-\kappa w}{w^2+1} \geq -\frac{\kappa}{2},$$

where we used $\frac{w}{w^2+1} \leq \frac{1}{2}$.

$$\begin{aligned} \varepsilon_\kappa(w) &\leq \frac{1}{w^2+1}\left(\frac{(8-\kappa)\sqrt{w^2+\varepsilon}}{1-(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}}} - 8w\right) \\ &\leq \frac{1}{w^2+1}\left|\frac{(8-\kappa)\sqrt{w^2+\varepsilon}-8w+8w(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}}}{1-(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}}}\right| \\ &\leq \frac{1}{w^2+1}\frac{(8-\kappa)(\sqrt{w^2+\varepsilon}-w)+\kappa w+8w(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}}}{1-(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}}} \end{aligned}$$

for every κ, we choose ε such that $(1-\varepsilon)^{\frac{4}{\kappa}-\frac{1}{2}} = \kappa$. One can check that $\varepsilon \sim \frac{\kappa\ln(\kappa)}{4}$, and notice that $\sqrt{w^2+\varepsilon} - w \leq \sqrt{\varepsilon}$. The upper-bound of $\varepsilon_\kappa(w)$ becomes

$$\varepsilon_\kappa(w) \leq \frac{1}{w^2+1}\frac{(8-\kappa)\sqrt{\varepsilon}+\kappa w+8w\kappa}{1-\kappa} \leq \frac{(8-\kappa)\sqrt{k\ln(k)}}{1-\kappa} + \frac{9\kappa}{1-\kappa}\frac{1}{2},$$

which does not depend on w and converges to 0 with rate $\sqrt{\kappa\ln(\kappa)}$.

For $w \leq 0$, $\varepsilon_\kappa(w)$ can be bounded brutally for any $a_\kappa > 1$:

$$0 \leq \varepsilon_\kappa(w) = \frac{\kappa(w^2+1)^{-\frac{4}{\kappa}}}{\int_w^\infty (s^2+1)^{-\frac{4}{\kappa}}\mathrm{d}s} \leq \frac{\kappa}{\int_0^\infty (s^2+1)^{-\frac{4}{\kappa}}\mathrm{d}s} = \frac{1}{\int_0^\infty e^{-4u}\frac{e^{u\kappa}}{2\sqrt{e^{uk}-1}}\mathrm{d}u}$$

$$\leq \frac{2}{\int_0^\infty \frac{e^{-4u}}{\sqrt{e^{uk}-1}}\mathrm{d}u} \leq \frac{2}{\int_0^{\ln(a_\kappa)/\kappa} \frac{e^{-4u}}{\sqrt{a_\kappa-1}}\mathrm{d}u} = \frac{4\cdot 2\sqrt{a_\kappa-1}}{1-e^{-4\ln(a_\kappa)/\kappa}}.$$

For example, choosing $a_\kappa = 1+\kappa$, we conclude that ε_κ converges uniformly to 0 with rate at least $\sqrt{\kappa}$ on $(-\infty, 0]$. □

Now we are going to prove the uniqueness of the minimizing curve.

To simplify the calculus we denote $L : [-\infty, +\infty] \to \mathbb{R}_+$ so that $L(\cot(\theta)) = l(\theta)$. Namely

$$L(w) = 4\ln(w^2+1), \quad \text{for } w > 0.$$

We consider the *centered Loewner flow* $\hat{g}_t := g_t - f_t^*$ for the convenience, write here and in the sequel $z_t = \hat{g}_t(z_\theta) = x_t + iy_t$, with $x_t, y_t \in \mathbb{R}$ and $w_t = \frac{x_t}{y_t} = \cot(\arg(z_t))$.

Proposition 18. *A minimizing driving function f^* must satisfy:*

$$L'(w_t)\partial_t(w_t) = -\frac{(\partial_t f_t^*)^2}{2} \tag{8}$$

In particular,

$$\begin{aligned}
\partial_t f_t^* &= \frac{8w_t}{y_t(1+w_t^2)} \\
\partial_t w_t &= \frac{-4w_t}{y_t^2(1+w_t^2)} \\
\partial_t y_t &= \frac{-2}{y_t(1+w_t^2)}
\end{aligned}$$

Proof: Let γ^* be a minimizing curve which hits z_θ at some time denoted T, by definition, it must be driven by a function f^* that satisfies:

$$\frac{1}{2}\int_0^T \dot{f}^*(s)^2 ds = L(w_0).$$

Let $t \in [0, T[$, the curve $\hat{g}_t(\gamma^*_{[t,\infty[})$ is the Loewner chain associated with $\tilde{f}_s^* = f^*(t+s) - f^*(t)$ so that $\partial_s \tilde{f}^*(s) = \partial_s f^*(t+s)$. Since $\tilde{f}^*$ is also a minimizer of I_T among functions whose Loewner chain passes to the right of z_t. We deduce that

$$\frac{1}{2}\int_t^T \partial_s f^*(s)^2 ds = L(w_t).$$

By differentiation w.r.t. t we get (8).

Now we compute $\partial_t(w_t)$.

$$\partial_t \hat{g}_t(z_\theta) = \frac{2}{\hat{g}_t(z_\theta)} - \dot{f}_t^* = \frac{2}{x_t + iy_t} - \partial_t f_t^*,$$

hence

$$\begin{aligned}
\partial_t x_t &= \frac{2x_t}{x_t^2 + y_t^2} - \partial_t f_t^* = \frac{2w_t}{y_t(1+w_t^2)} - \partial_t f_t^* \\
\partial_t y_t &= \frac{-2y_t}{x_t^2 + y_t^2} = \frac{-2y_t}{|z_t|^2} = \frac{-2}{y_t(1+w_t^2)}
\end{aligned}$$

$$\partial_t w_t = \partial_t(\frac{x_t}{y_t}) = \frac{\partial_t x_t}{y_t} - \frac{x_t \partial_t y_t}{y_t^2}$$
$$= -\frac{\partial_t f_t^*}{y_t} + \frac{4w_t}{y_t^2(1+w_t^2)}$$

Plug these relations into (9),

$$L'(w_t)(\frac{4w_t}{y_t^2(1+w_t^2)} - \frac{\partial_t f_t^*}{y_t}) = -\frac{(\partial_t f_t^*)^2}{2}.$$

We see that $\partial_t f_t^*$ is a root of a second order polynomial $X^2 - c_t X + d_t$, with

$$c_t = 2\frac{L'(w_t)}{y_t} \text{ et } d_t = 2\frac{4w_t L'(w_t)}{y_t^2(1+w_t^2)} = c_t^2 \frac{2w_t}{(1+w_t^2)L'(w_t)}.$$

Using Prop. 2 and Prop. 15:

$$L'(w_t) = \frac{8w_t}{(1+w_t^2)} \text{ and } c_t^2 - 4d_t = 0,$$

which gives the expression of $\partial_t f_t^*$. □

Remark. The function $\partial_t f^*$ depends not only on the argument of target point z_θ but also proportional to the inverse of y_t. This is not surprising, since the parameterization of f^* depends on the modulus of initial point. The dilatation of γ^* by $\lambda > 0$ is the Loewner chain associate with the minimizing driving function

$$\hat{f}_t^* = \lambda f^*_{\frac{t}{\lambda^2}}.$$

Then $\hat{T} = \lambda^2 T$ and $\partial_t \hat{f}_t^* = \frac{1}{\lambda}\partial_t f^*_{\frac{t}{\lambda^2}}$. The latter is also proportional to $1/|z_t|$.

From now on the uniqueness of minimizing curve is an immediate consequence.

Proposition 19 (existence and uniqueness of the optimal curve)**.** *For all* $z_\theta = (x_0, y_0)$ *with* $0 < \theta < \frac{\pi}{2}$*,*

1. *there exists a unique minimizing curve. The hitting time of* γ^* *at point* z_θ *is the first time from when the driving function become stationary, i.e.,* $T_{f^*} = T$ *and* $f^*(T) = \frac{4}{3}x_0$*;*
2. *the minimizing driving function* f^* *is increasing on* $[0, T]$*;*
3. *the map* $t \mapsto \theta_t$ *is increasing, converging to* $\frac{\pi}{2}$ *when* t *tends to* T*;*
4. *the optimal curve is tangent to the segment* $[0, e^{i\theta}]$*.*

Proof: 1. The proposition gives us a system of differential equation up to time T:

$$\partial_t f_t^* = \frac{8w_t}{y_t(1+w_t^2)} \tag{9}$$

$$\partial_t w_t = \frac{-4w_t}{y_t^2(1+w_t^2)} \tag{10}$$

$$\partial_t y_t = \frac{-2}{y_t(1+w_t^2)} \tag{11}$$

which could be resolved from the last two equations which provide a autonomous system, the solution is given explicitly:

$$\frac{\mathrm{dw}}{\mathrm{dy}} = \frac{2}{y}w \quad \Rightarrow \quad w_t = Cy_t^2 \quad \Rightarrow \quad x_t = Cy_t^3 + f^*(t).$$

where the constant $C = \frac{w_0}{y_0^2}$ is determined by initial conditions.

$$\frac{\mathrm{df}^*}{\mathrm{dy}} = -4w = -4Cy^2 \quad \Rightarrow \quad f^*(t) = -\frac{4}{3}Cy_t^3 + C',$$

for another C' determined using $f(0) = 0$.

$$f^*(0) = -\frac{4}{3}\frac{w_0}{y_0^2}y_0^3 + C' = -\frac{4}{3}x_0 + C',$$

so that $C' = \frac{4}{3}x_0$.

In particular, $T_{f^*} \stackrel{\text{def}}{=} \inf\{t \geq 0, w(t) = 0\} = \inf\{t \geq 0, y_t = 0\} \stackrel{\text{def}}{=} T$. The value $f^*(T)$ equals to C' (when $y = 0$), which is $\frac{4}{3}x_0$.

2. Immediate after the expression of $\partial_t f_t^*$.

3. Let $0 \leq r < t < T$,

$$L(w_t) - L(w_r) = \frac{1}{2}\int_r^t \partial_t f^*(s)^2 ds.$$

If $w_t = w_r$ then f^* is constant on $[r,t]$, $\hat{g}_t \circ \hat{g}_r^{-1}$ is the mapping-out function $\mathbb{H}\backslash[0, 2i\sqrt{t-r}] \to \mathbb{H}$, only points on imaginary axis have invariant argument, *i.e.,* $w_t = 0$, violates $t < T$.

When $t \to T$, $L(w_t) = \frac{1}{2}\int_t^T \partial_t f^*(s)^2 ds \to 0$ therefore $w_t \to 0$.

4. It's consequence of 3. We knew already that $\hat{g}_t^*(\gamma_{[t,\infty]})$ is contained in the cone delimited by the imaginary axis and $\{se^{i\theta_t}, s \in \mathbb{R}_+\}$ and that $\theta_t \to \pi/2$.

$\square$

3.6 Characterization of the minimizing curve in the unit disc

In this section we prove Thm. 3.

Since only the argument θ_t of z_t has an important place in determination of minimizing driving function, it urges us to consider the image of γ by the following transformation, which allows to get rid of the modulus of the initial point :

Let $\mathbb{D}$ denote the open unit disc, for every $t < T$, $\psi_t : \overline{\mathbb{H}} \cup \{\infty\} \to \overline{\mathbb{D}}$ the conformal transformation sending ∞ to $e^{-i\theta_t}$, z_t to 0. *i.e.,*

$$\psi_t : z \mapsto e^{-i\theta_t} \frac{z - z_t}{z - \overline{z_t}},$$

with inverse:

$$\varphi_t : h \mapsto \frac{e^{i\theta_t} z_t - e^{-i\theta_t}\overline{z_t} h}{1 - h}.$$

The image of the starting point of the Loewner chain at time t is

$$\psi_t(f_t) = e^{i\theta_t}$$

where $\theta_t = \arg(\hat{g}_t(z_0)) = \arg(z_t - f_t)$.

The image of γ is a curve from $e^{i\theta_t}$ to $e^{-i\theta_t}$ going through 0. The reversibility makes $\psi(\gamma^*)$ symmetric with respect to the real axis.

Write $h_t : \mathbb{D} \backslash \psi_t(\gamma_{[0,t]}) \to \mathbb{D}$,

$$h_t = \psi_t \circ g_t \circ \varphi_0,$$

We derive w.r.t. t, but as we noticed that the expression of f^* with variable t depends on the modulus of the target point, which is not convenient in the disc. The natural reparameterization is θ up to time $t = T$ or $\theta = \frac{\pi}{2}$, and the expression of $\partial_t \theta_t$ for the minimizing driving function $f = f^*$ is,

$$\partial_t \theta_t = \partial_t \arctan \frac{1}{w_t} = \frac{-1}{1 + w_t^2} \frac{-4w_t}{y_t^2(1 + w_t^2)} = \frac{4\cos(\theta_t)\sin(\theta_t)^3}{y_t^2}.$$

Hence the Loewner equation becomes :

$$\partial_\theta h_\theta(z) = h_\theta(z)\tan(\theta) \frac{e^{i\theta} + h_\theta(z)}{e^{i\theta} - h_\theta(z)}, \quad \theta_0 \le \theta < \frac{\pi}{2} \tag{12}$$

$$h_{\theta_0}(z) = z. \tag{13}$$

We see that it's the radial Loewner equation up to time change. Indeed, while defining for $\theta \in [\theta_0, \pi/2[$,

$$t(\theta) = \int_{\theta_0}^{\theta} tan(s) \mathrm{d}s, \quad \text{so that } \theta(t) \text{ satisfies} \begin{cases} \partial_t \theta(t) = \cot \theta(t) \\ \theta(0) = \theta_0 \end{cases} \tag{14}$$

hence,

$$\partial_t h_t(z) = -h_t(z)\frac{h_t(z)+e^{i\theta(t)}}{h_t(z)-e^{i\theta(t)}} \quad 0 \le t < \infty$$
$$h_0(z) = z.$$

We recognize the radial Loewner equation with driving function $\theta(t)$. In this way we show Theorem 3 characterizing the minimizing curve of SLE_{0+} from 0 to ∞ passing to the right of z_{θ_0} by its image in the unit disc.

Remark. The implication (2. $\Rightarrow$ 3.) shows that one can determine a priori the minimizing curve without calculus knowing that the minimizing curve $\hat{g}_t(\gamma_{[t,\infty[})$ in $\mathbb{H}$ is conformally z_t-symmetric for all $t < T$.

Proof (Proof of Theorem. 3)**:** 1. $\Rightarrow$ 2. : These are properties of the minimizing curve we proved in sec. 3.5.

2. $\Rightarrow$ 3. : Let γ be a curve in $\mathbb{H}$ passing through z_θ at some time T, $z_t = \hat{g}_t(z_\theta) = g_t(z_\theta) - f_t$ and $\theta_t = \arg(z_t)$. Let $(\psi_t)_{t\le T}$ be the family of conformal transformations, such that $\psi_t(z_t) = 0$ and the vector carried by $ie^{i\theta_t}$ is sent to a vector proportional to 1, *i.e.*, $\psi_t'(z_t) \propto -ie^{-i\theta_t}$.

In the unit disc setting, define $\tilde{\gamma}_t = \psi_0(\gamma_t)$, and $h_t : \psi_t \circ g_t \circ \psi_0^{-1} : \mathbb{D}\backslash\tilde{\gamma}_{[0,t]} \to \mathbb{D}$. It suffices to show that $h_t(0) = 0$, $h_t'(0) \in \mathbb{R}_+$ and the image of $\tilde{\gamma}_{[t,\infty[}$ by h_t is conjugate-invariant. In fact, up to time reparameterization, these conditions ensures (h_t) to be the Loewner flow associated to $\tilde{\gamma}_{[0,T]}$ as radial Loewner chain with some driving function ξ_t. The symmetry w.r.t. real axis of image in $\mathbb{D}$ shows that $h_t(e^{-i\theta_0}) = h_t(\psi_0(\infty)) = \psi_t(\infty) = e^{-i\xi(t)}$ which is exactly the charaterization 3..

[**Conjugate-invariance**] Let $c_t \in \mathbb{R}_+$ such that $c_t\mathrm{Rev}(z_t) = z_t$. Since Rev is anti-holomorphic, preserving the unit circle, the vector $ie^{i\theta_t}$ in the tangent space over z_t is also preserved. Hence

$$\psi_t \circ (c_t\mathrm{Rev}) \circ (\psi_t^{-1}) : \mathbb{D} \to \mathbb{D}$$

is just the complex conjugate. By hypothesis 2., we know that $h_t(\tilde{\gamma}_{[t,\infty[}) = \psi_t(\hat{g}_t(\gamma_{[t,\infty[}))$ is conjugate-invariant for all $t < T$, thus the upper part corresponds to $\psi_t(\hat{g}_t(\gamma_{[t,T]}))$.

[**Real-derivative**] Since the curve is differentiable at z_θ, let v be the vector tangent to γ at z_θ directed towards γ_{T+} with argument φ.

$$\arg(\partial_s|_{s=0}\mathrm{Rev}\gamma_{T+s}) = \arg(-\frac{1}{\overline{\gamma_T}^2}\partial_s|_{s=0}\overline{\gamma_{T+s}}) = \pi + 2\theta - \varphi.$$

However the invariance gives also $\arg(\partial_s|_{s=0}\mathrm{Rev}\gamma_{T+s}) = \pi + \varphi$. Consequently the curve is directed by $e^{i\theta_t}$ at z_t for all t, then we deduce

$$g_t'(z_\theta) \propto e^{i(\theta_t - \theta)}.$$

Therefore $h_t'(0) \propto -ie^{-i\theta_t}e^{i(\theta_t-\theta_0)}ie^{i\theta_0} = 1$.

3. $\Rightarrow$ 1.: Let $\tilde{\gamma}$ denote the radial Loewner chain with driving function $\xi(t)$ satisfies the assumption 3. with mapping out functions $(\tilde{h}_t)$. Then

$$\partial_t \tilde{h}_t(e^{-i\theta}) = \partial_t e^{-i\xi(t)} = -ie^{-i\xi(t)}\partial_t \xi(t),$$

besides the radial Loewner equation requires:

$$\partial_t \tilde{h}_t(e^{-i\theta}) = \tilde{h}_t(e^{-i\theta_0})\frac{e^{i\xi(t)} + \tilde{h}_t(e^{-i\theta_0})}{e^{i\xi(t)} - \tilde{h}_t(e^{-i\theta})} = -ie^{-i\xi(t)}\cot\xi(t).$$

Hence $\partial_t\xi(t) = \cot\xi(t)$ and the initial condition $\xi(0) = \theta$, satisfies the same differential equation than $\theta(t)$. One concludes that $\forall t$, $\xi(t) = \theta(t)$ and $\tilde{h}_t = h_t$.

Readers familiar with $SLE_\kappa(\rho)$ processes, can recognize that the condition 3. gives exactly the $SLE_0(2)$ starting at $(e^{i\theta}, e^{-i\theta})$. If one assigns weight -8 to the center 0 which does not change the curve, one can make the coordinate change into half-plane chordal SLE since the weight sum up to -6. In other words, the minimizing curve stopped at z_θ is the $SLE_0(-8)$ curve starting from $(0, z_\theta)$ in the upper half-plane version. (See [SW05] for definition of $SLE_\kappa(\rho)$ and coordinates change.) $\square$

4 Another point of view by Girsanov theorem

4.1 Conditional law of driving function

Let us go back to the problem at the beginning. Let $(\Omega, \mathscr{F}, W)$ denotes a standard Wiener space, set B a Brownian motion, let $(\mathscr{F}_t)_{t\geq 0}$ the canonical completed filtration, write $X = \sqrt{\kappa}B$, with law W_κ. Fix z_θ, write as before $\tilde{W}_\kappa$ the probability conditioned by $\{\gamma$ passing to the right of $z_\theta\}$, $z_t = \hat{g}_t(z_\theta) = g_t(z_\theta) - X_t$ and $T = \inf\{t \,|\, z_t = 0\}$. Define the function $p_\kappa : \overline{\mathbb{H}} \to [0,1]$ by

$$p_\kappa(z) = W_\kappa(\gamma \text{ to the right of } z),$$

with convention $p_\kappa(a) = 1$ for $a \leq 0$ and 0 for $a > 0$.

The process $M_t := p_\kappa(z_t^T)_{t\geq 0}$ is a martingale converging almost surely to $\{0,1\}$. We weight $\mathbb{P}$ by $\frac{M_\infty}{\mathbf{E}(M_0)}$ to obtain the conditional probability $\tilde{\mathbb{P}}_\kappa$. And on the sub-$\sigma$-algebra $\mathscr{F}_t$:

$$\frac{\mathrm{d}\tilde{W}_\kappa}{\mathrm{d}W_\kappa}\Big|_{\mathscr{F}_t} = \mathbf{E}\left(\frac{M_\infty}{\mathbf{E}(M_0)}\,\Big|\,\mathscr{F}_t\right) = \frac{M_t}{p_\kappa(z_\theta)}.$$

By Girsanov Theorem, for $t < T$

$$X_t - [Y, X]_t = \sqrt{\kappa}\beta_t,$$

where β is a Brownian motion under $\tilde{W}$ and $\exp(Y_t - [Y]_t) = M_t$.

$$M_t dY_t = dM_t = \partial_x p_\kappa(z_t)(-\sqrt{\kappa}dB_t) = -\sqrt{\kappa}h'_\kappa(w_t)\frac{\partial w_t}{\partial x_t}dB_t = -\sqrt{\kappa}h'_\kappa(w_t)\frac{1}{y_t}dB_t$$

The function h_κ is defined in Thm. 16. Hence the driving function under $\tilde{W}_\kappa$ is solution of the stochastic differential equation system:

$$\begin{cases} dX_t = \sqrt{\kappa} d\beta_t - \frac{\kappa h'_\kappa(w_t)}{h_\kappa(w_t) y_t} dt \\ dz_t = \frac{2}{z_t} dt - dX_t \end{cases} \qquad \text{for } t < T$$

It can be expressed thanks to Lemma. 17

$$(E_\kappa): \quad \begin{cases} dX_t = \sqrt{\kappa} d\beta_t + \frac{F(w_t)}{y_t} dt + \frac{\varepsilon_\kappa(w_t)}{y_t} dt \\ dz_t = \frac{2}{z_t} dt - dX_t \end{cases} \qquad \text{for } t < T \tag{15}$$

where

$$F(w) = \frac{8w}{w^2+1} \wedge 0 \text{ and } \varepsilon_\kappa(w) \xrightarrow[\kappa \to 0]{\text{unif. in } w} 0.$$

Set $0 < a < y_0$ and $\tau_a = \inf\{t \geq 0 \,|\, y_t < a\}$, we see that the system 15 is Lipschitz up to time T_a, there exists a unique strong solution for given probability space, Brownian motion and initial condition. Here it means for any given $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbf{P})$-Brownian motion (β_t) and the initial condition $X_0 = 0, z_0 = z_\theta$, there exists a unique continuous adapted solution up to modification, moreover the solution is adapted to the filtration generated by β. Let $a \to 0$, we can then extend the solution to time T.

This point of view offers an explicit way to couple the conditional law of SLE_κ for different κ.

4.2 Convergence of solution

The SDE (E_κ) will be seen as a perturbation of the deterministic differential equation system:

$$(E_0): \quad \begin{cases} dX_t = \frac{F(w_t)}{y_t} dt \\ dz_t = \frac{2}{z_t} dt - dX_t \\ X_0 = 0, \quad z_0 = z_\theta \end{cases} \qquad \text{for } t < T = T^* \tag{16}$$

where T^* denotes the hitting time of the deterministic solution. It is easy to check that it is exactly the $SLE_0(8)$, with driving function f^*, the same as the minimizer in the large deviation approach. We write z_t^* as the flow starting from z_θ in (E_0).

Theorem 20

If the initial condition (X_0^κ, z_0^κ) of (E_κ) converges in probability to $(0, z_\theta)$, then, the solution $(X^\kappa, z^\kappa)|_{[0,T^*-\eta]}$ restricted to the interval $[0, T^* - \eta]$ for arbitrarily small $\eta > 0$ satisfies also,

$$\mathbf{P}\left\{\|(X^\kappa, z^\kappa) - (f^*, z^*)\|_{\infty,[0,T^*-\eta]} > \delta\right\} \xrightarrow[\kappa \to 0]{} 0, \quad \forall \delta > 0. \tag{17}$$

This is the same theorem as Thm. 1 but only up to time $[0,T[$, since the optimal curve is not defined as solution of differential equation at time T. Even though (E_κ) starting at $(0,z_\theta)$ is almost surely defined on $[0,\infty[$ (the SLE_κ doesn't hit the target point z_θ), and one could also extend the flow of z^*_θ by the flow of $\hat{g}_{t-T}(0^-)$ after time T, the convergence might still be doable. But here we only consider what happens before T, it is purely an ODE perturbation problem.

Before the proof we recall the classical Gronwall lemma which is used repeatedly:

Lemma 21 (Gronwall inequality)**.** *Let I denote an interval of $\mathbb{R}$ which might be infinite. Let α,β and u be real-valued functions defined on I. Assume that β and u are continuous.*

If the function β is non-negative, α is non-decreasing and if u satisfies the integral inequality

$$u(t) \le \alpha(t) + \int_a^t \beta(s)u(s)\,\mathrm{d}s, \qquad \forall t\in I,$$

then

$$u(t) \le \alpha(t)\exp\Big(\int_a^t \beta(s)\,\mathrm{d}s\Big), \qquad \forall t\in I.$$

Proof (of the theorem. 20)**: Step 1:** Show the convergence on an positive interval $[0,t']$.

Fix $a>0$ small, for every Loewner flow starting at a point $g_t(x_0,y_0)=(x_t,y_t)$, set $\tau_a = \inf\{t\ge 0\,|\, y_t\le a\}$ which is obviously smaller than $T=\inf\{t\ge 0\,|\, g_t(x_0,y_0) \text{ is not defined}\}$. The stopping time τ_a can be bounded from below:

$$\partial_t y_t = -\frac{2y_t}{x_t^2+y_t^2} \ge -\frac{2}{y_t}.$$

we deduce that $\tau_a \ge \frac{y_0^2-a^2}{4}$.

In the setting of the theorem, define $t' = \frac{(\frac{y_0^*}{2})^2-a^2}{4}$, then τ_a^κ is uniformly bounded from below by t' on the event $\mathscr{E}_\kappa := \{y_0^\kappa > \frac{y_0}{2}\}$, and $\mathbf{P}(\mathscr{E}_\kappa)\to 1$.

Now we control the difference of the solution restricted to $[0,t']$ of the perturbed equation on the event $\mathscr{E}_\kappa$.

The process $z_t := z_t^k$ satisfies by definition

$$z_t = z_0 + \int_0^t \frac{2}{z_s} - \frac{F(w_s)}{y_s} - \frac{\varepsilon_\kappa(w_s)}{y_s}\mathrm{d}s - \sqrt{\kappa}\beta_t.$$

Since y is larger than a, and Lemma. 17 ensures that

$$\forall\varepsilon>0, \exists\kappa_\varepsilon>0, \text{ s.t. } \forall\kappa<\kappa_\varepsilon, \forall t\le\tau_a, \quad \left|\frac{\varepsilon_\kappa(w_s)}{y_s}\right| \le \varepsilon.$$

The map $z\mapsto \frac{2}{z}-\frac{F(w)}{y}$ is b-Lipschitz on $\{y\ge a\}$ for some $b>0$. The difference between z_t and z_t^* can be controlled almost surely:

$$\begin{aligned}|z_t - z_t^*| &= \left|z_0 - z_0^* + \int_0^t \frac{2}{z_s} - \frac{2}{z_s^*} - \frac{F(w_s)}{y_s} + \frac{F(w_s^*)}{y_s^*}\mathrm{d}s - \int_0^t \frac{\varepsilon_\kappa(w_s)}{y_s}\mathrm{d}s - \sqrt{\kappa}\beta_t\right| \\ &\le \int_0^t b\,|z_s - z_s^*|\,\mathrm{d}s + \varepsilon t + \sqrt{\kappa}S_t + |z_0 - z_0^*|\,.\end{aligned}$$

Where $S_t = \sup_{0\le s\le t} \beta_s$. Using Gronwall inequality we have for all $t \le t'$:

$$|z_t - z_t^*| \le (\varepsilon t + \sqrt{\kappa} S_t + |z_0 - z_0^*|) \exp(bt)$$

hence

$$\|z_t^\kappa - z_t^*\|_{\infty,[0,t']} \le (\varepsilon t' + \sqrt{\kappa} S_{t'} + |z_0^\kappa - z_0^*|) \exp(bt').$$

The random variable $S_{t'}$ has the same law than $|B_{t'}|$, we conclude that for any $\delta > 0$, $\kappa < \kappa_\varepsilon$,

$$\mathbf{P}\left\{\|z_t^\kappa - z_t^*\|_{\infty,[0,t']} > \delta\right\} \le \mathbf{P}\left\{(\varepsilon t' + \sqrt{\kappa} S_{t'} + |z_0^\kappa - z_0^*|) \exp(bt') > \delta\right\} + \mathbf{P}(\mathscr{E}_\kappa^c) \xrightarrow[\varepsilon,\kappa\to 0]{} 0.$$

Similarly for the driving function:

$$X_t^\kappa = X_0^\kappa + \sqrt{\kappa}\beta_t + \int_0^t \frac{F(w_s)}{y_s} + \frac{\varepsilon_\kappa(w_s)}{y_s} \mathrm{d}s.$$

on the event $\left\{\|z_t^\kappa - z_t^*\|_{\infty,[0,t']} \le \delta\right\}$, we have for $\kappa < \kappa_\varepsilon$,

$$\|X_t^\kappa - f_t^*\|_{\infty,[0,t']} \le |X_0^\kappa| + \sqrt{\kappa} S_{t'} + t' b\delta + t'\varepsilon,$$

We obtain therefore the convergence in probability of the couple $(X_t^\kappa, z_t^\kappa)|_{[0,t']}$.

Step 2: We will extend the result to $[0, T^* - \eta]$ for any $\eta > 0$. Let

$$c = \inf\{\eta > 0 \,|\, 17 \text{ holds.}\}.$$

Step 1 shows already that the set is not empty, and it suffices to prove $c = 0$. Assume the contrary, at time $s = T^* - c$, $y_s^* > 0$, for a time $s' = s - \varepsilon$ slightly earlier than time s, $\|z_t^\kappa - z_t^*\|_{\infty,[0,s']}$ converges in probability to 0, therefore so is $|z_{s'}^\kappa - z_{s'}^*|$. We apply again step one together with the Markov property, hence we get the uniform convergence in probability with $t' = \frac{(\frac{y_{s'}^*}{2})^2 - a^2}{4}$:

$$\mathbf{P}(\|g_{s'}(z_{s'+t}^\kappa) - g_{s'}^*(z_{s'+t}^*)\|_{\infty, t\in[0,t']} > \delta) \xrightarrow[\kappa\to 0]{} 0.$$

It results in the control of $z_{s'+t}^\kappa$:

$$\begin{aligned} |z_{s'+t}^\kappa - z_{s'+t}^*| &\le \left|(g_{s'})^{-1}(g_{s'}(z_{s'+t}^\kappa)) - (g_{s'}^*)^{-1}(g_{s'}(z_{s'+t}^\kappa))\right| \\ &\quad + \left|(g_{s'}^*)^{-1}(g_{s'}(z_{s'+t}^\kappa)) - (g_{s'}^*)^{-1}(g_{s'}^*(z_{s'+t}^*))\right| \\ &\quad \xrightarrow[\kappa\to 0]{\text{in prob., unif in } t\in[0,t']} 0, \end{aligned}$$

in the last convergence we have used that $\|g_{s'}^{-1} - (g_{s'}^*)^{-1}\|_\infty$ converges in probability to 0 by the Loewner-Kufarev theorem 4, and the uniform continuity of $(g_{s'}^*)^{-1}$. In consequence

$$\mathbf{P}(\|z^\kappa - z^*\|_{\infty,[0,s'+t']} > \delta) \to 0$$

and similarly for X^κ. Since $y_{s'}^* \searrow_{s'\to s} y_s^*$, it is easy to choose convenient a and s' that make $t' + s' > s$ which leads to a contradiction if $s \ne T^*$. $\square$

5 A more geometric point of view & Generalization

The optimal curve γ of SLE_{0+} in disc from $e^{i\theta}$ to $e^{-i\theta}$ passing to the left of 0 can be considered as a embedded graph in $\mathbb{D}$ with three vertices: $e^{-i\theta}, 0, e^{i\theta}$ and two edges e_1 (with vertices 0 and $e^{-i\theta}$) and e_2 ($e^{i\theta}$ and 0). The graph satisfies that e_2 is geodesic (w.r.t. the hyperbolic metric) within the simply connected domain $\mathbb{D}\backslash e_1$ and vice versa. We say that $(\mathbb{D}, \gamma)$ satisfies the property (P). More precisely, for a Riemann surface $\mathscr{O}$, and G connected, we say that the embedded graph $(\mathscr{O}, G)$ satisfies the property (P) if only vertices can be on the boundary of $\mathscr{O}$ and:

(P) For each edge e of G, the connected component $\mathscr{O}_e$ containing e in $\mathscr{O}\backslash \bigcup_{e' \neq e} e'$ is simply connected. The conformal transform $\psi_e : \mathscr{O}_e \to \mathbb{H}$ (up to homotheties) sending ends of e to 0 and ∞, the image of e by ψ_e is the imaginary axis..

The property (P) interested us as it provides a canonical embedding of a certain class of combinatorial graph (a finite set of vertices, a finite collection undirected edges (multiple edges between two vertices allowed, but no loop) and a cyclical order on edges adjacent to each vertex) into Riemann surfaces with prescribed image for vertices.

Remark. We can prove that for $\mathscr{O} = \hat{\mathbb{C}}$ and $|V| = 2$, assume that two vertices are 0 and ∞, and there are n edges between them, the only embedding satisfying (P) are

$$\{\exp(i\theta + i\frac{k}{2\pi} + \lambda), \lambda \in \mathbb{R}, k \in \{0, 1, \cdots, n-1\}\},$$

for some $\theta \in \mathbb{R}$, which is unique up to rotation.

It reveals that a local minimizer of points-constrained curve and the preimage of $\mathbb{R}$ by a holomorphic covering $S \to \hat{\mathbb{C}}$ ramified only at real points provides examples for (P)-embedded graphs (Prop.24 and Prop. 25).

5.1 Curves with finite energy

Somehow the title of this section should be clarified, why a Loewner chain generated by a function with finite energy is called a *curve*? The purpose of this section is summarized in the following proposition:

Proposition 22. *For all f in H_∞, there exists $K > 0$, s.t. the Loewner transform $(\gamma_t)_{t\in[0,T]}$ of f is a K-quasiarc on any interval $[0, T]$, from 0 to ∞, and never touches the real line for $t > 0$, with vertical tangent at 0.*

The Loewner chain generated by a continuous function satisfies only the local growth property, but by no means it is a simple curve, take for example the SLE_κ for $\kappa > 4$. In the deterministic setting, Rhode, Marshall investigates first when is the radial Loewner chain generated a quasislit halfplane in [MR05]. By K-quasislit halfplane we mean the image of $\mathbb{H}\backslash[0, i]$ by a K-quasiconformal mapping fixing $\mathbb{H}$ and ∞, its complement in $\mathbb{H}$ is a quasiarc which is not tangent to the real line. Piece-wise smooth curve without 0-angle cusp are quasiarcs, Then Lind gives a sharp condition in [Lin05] for generating quasislit halfplane.

Recall $Lip(\frac{1}{2})$ is the space of Hölder continuous functions with exponent $\frac{1}{2}$, which consists of functions $\lambda(t)$ satisfying

$$|\lambda(s) - \lambda(t)| \leq c\,|s-t|^{1/2}$$

for some $c < \infty$, equipped with the norm $\|\lambda\|_{\frac{1}{2}}$ being the smallest such c.

Theorem 23 (*[Lin05]*)

If $f \in Lip(\frac{1}{2})$ with $\|f\|_{\frac{1}{2}} < 4$, then the domains $\mathbb{H}\backslash\gamma_{[0,t]}$ generated by f are K-quasislit halfplane for all t, for some $K = K(\|f\|_{\frac{1}{2}})$.

It is a sharp condition because $t \mapsto 4\sqrt{1-t}$ generates a infinite spiral when t approaches 1 where the local Hölder norm is 4.

We remark that H_∞ injects into $Lip(\frac{1}{2})$ since

$$|f(t_1) - f(t_2)| \leq \int_{t_1}^{t_2} \left|\dot{f}\right| \mathrm{d}t \leq (t_2 - t_1)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \left|\dot{f}\right|^2 (t)\right)^{1/2} \leq (t_2 - t_1)^{\frac{1}{2}} I_\infty(f)^{\frac{1}{2}}.$$

Proof (of Prop. 22)**:** Even though the $\frac{1}{2}$-norm of $f \in H_\infty$ can be large than 4, it is not a local effect. More precisely, we can choose a finite sequence $0 = t_0, t_1, t_2, \cdots, t_n = \infty$ such that

$$\sqrt{\int_{t_i}^{t_{i+1}} \dot{f}^2(t)\mathrm{d}t} \leq 3.$$

Since concatenation of quasislits are still quasislits (see eg. [LMR+10], proof of Thm 4.1), we have that $\mathbb{H}\backslash\gamma_{[0,T]}$ are K-quasislit halfplane for all T, with some $K > 0$ since only finite number of concatenations have been done. From now we can say that H_∞ generates curves as Loewner chain.

For a curve to touch the real line or itself, it has first to touch a point of argument θ with θ or $\pi - \theta$ arbitrarily small, which implies infinite energy of f.

Since the energy functional is scaling invariant, it suffices to show that it is absurd if γ touches infinite times the circle $C_1 := \{|z| = 1\}$. Assume that is the case, for $R > 0$ large, when γ touches C_R at some time T_R, the image $g_{T_R}(C_1) - f_{T_R}$ is included in two cones $\{z \in \mathbb{H} \,|\, arg(z) < \theta' \text{ or } \pi - arg(z) < \theta'\}$ with θ' arbitrarily small unless we take R large. Since the trace of γ is not bounded, T_R is finite, and the energy of γ after time T_R is arbitrarily large, which contradicts the finiteness of energy.

For any small $\varepsilon > 0$, there is $t_\varepsilon > 0$ such that $I_{t_\varepsilon}(f) < \varepsilon$, therefore the beginning of the curve $\gamma_{[0,t_\varepsilon]}$ is contained in the cone $\{z \in \mathbb{H} \,|\, \theta_\varepsilon \leq arg(z) \leq \pi - \theta_\varepsilon\}$, where $\theta_\varepsilon = l^{-1}(\varepsilon)$, which goes to $\pi/2$ when $\varepsilon \to 0$. □

5.2 Minimizing curve conditioned by several points

Fix n distinct and unordered points $\overline{z} = \{z_1, \cdots, z_n\}$ in $\mathbb{H}$, and assign left $(+)$ or right $(-)$ to each point. A curve from 0 to ∞ is said to be *compatible* with $\overline{z}$ if it passes to the left or the right according to assignment. A minimizing curve γ of SLE_{0+} with constraints $\overline{z}$ in $\mathbb{H}$ is a Loewner chain driven by a function f who minimizes the energy functional among all curves compatible with $\overline{z}$. There is no reason that the minimizing curve is unique, take two different points symmetric w.r.t. the imaginary axis, the left one assigned $(+)$ and the right one assigned $(-)$. There should be at least two minimizing curves symmetric to each other.

We write $D_{\overline{z}}$ for the set of all curves compatible with $\overline{z}$ and $D^T_{\overline{z}}$ for the set that $f(\cdot \wedge T) \in D_{\overline{z}}$. Let $Rev : \mathbb{H} \to \mathbb{H}$ be the map $z \mapsto \frac{1}{\overline{z}}$. Which acts also on $\overline{z}$ by taking the image one by one and keep the same assignment on points. Recall that $C_0([0,\infty[)$ is equipped with the metric which induces uniform convergence on compacts.

Proposition 24. *For every isolated local minimizing curve γ^* of constraints $\overline{z}$ in $\mathbb{H}$, we write $\overline{z}' = \overline{z} \cap \gamma^*$ with the same assignment. Then γ^**

1. *is local minimizer of constraints $\overline{z}'$;*
2. *is an embedded graph with vertices $\overline{z}'$, 0 and ∞ which has property (P);*
3. *is smooth everywhere;*
4. *satisfies: if γ^* is globally minimal, then $\exists \tau > 0$, such that*

$$I_\infty(f^*) = \lim_{\kappa \to 0} \kappa \ln W_\kappa(D_{\overline{z}}) = \inf_{f \in D_{\overline{z}}} I_\infty(f) = \inf_{f \in D^t_{\overline{z}}} I_t(f), \quad \forall t \geq \tau$$

   where f^* is its driving function. If γ^* is just local minimizer, the equality holds when we replace $D_{\overline{z}}$ by $D_{\overline{z}} \cap B_\varepsilon(f^*)$ for some ε small enough, where $B_\varepsilon(f^*)$ is the closed ball of radius ε centered at f^*,

5. *has its reversed curve $\tilde{\gamma}^* := Rev(\gamma^*)$, driven by a function $\tilde{f}^*$ which is also a local minimizer of constraints $\tilde{z} := Rev(\overline{z})$. Moreover,*

$$I_\infty(f^*) = I_\infty(\tilde{f}^*).$$

Proof: Property 4 is an adaptation of the Prop. 10.

Property 1 is true since small perturbations (of topology of uniform convergence on compacts) of γ^* will not affect the final quadrant in which points of $\overline{z}$ out side of γ^* are located or must have large energy (the beginning is very close to γ^* but comes back late to change quadrants for other points needs big energy). In the former case if γ^* is not local minimizer with constraints $\overline{z}'$ then it is neither for $\overline{z}$, in the latter the energy is much too large than γ^*'s.

For property 5, if γ^* is the global minimizer then $\tilde{\gamma}^*$ is also the global minimizer of $\tilde{z}$. In fact, the conditional law by $\overline{z}$ of κB w.r.t. the $\|\cdot\|_\infty$ converges to f^*, which implies the convergence of γ_κ to γ^* w.r.t. the Hausdorff metric since γ^* is a simple curve. Therefore the convergence for the Hausdorff metric takes place also in the reversed SLE_κ conditioned by $\tilde{z}$ to $\tilde{\gamma}^*$.

by the reversibility of SLE_κ,

$$W_\kappa(D_{\overline{z}}) = W_\kappa(D_{\tilde{z}}),$$

therefore

$$I_\infty(\tilde{f}^*) \geq \inf_{\tilde{f}\in D_{\tilde{z}}} I_\infty(\tilde{f}) = \lim_{\kappa\to 0} \kappa \ln W_\kappa(D_{\tilde{z}}) = \lim_{\kappa\to 0} \kappa \ln W_\kappa(D_{\overline{z}}) = I_\infty(f^*).$$

If γ^* is just local minimizer, we could add some constraint points to both side of the curve to make it global (we shall use again the compactness of $\{f \in D^n_{\overline{z}} \,|\, I_n(f) \leq I_n(\gamma^*)\}$ for some n large). Its reversed curve is also global minimizer therefore local minimizer when we get rid of the additional points.

Now we come to property 2. If the set of n points has already their sign of real-part as the assignment, then the unique local minimizing curve (also the global one) of $\overline{z}'$ is the imaginary axis. It is clear that (P) is satisfied.

Otherwise, the last point in $\overline{z}$, say z_1 for example, which arrives in the presumed quadrant following its flow must stay on the imaginary axis, and the driving function becomes stationary. Therefore γ passes through z_1. By 5, $\tilde{\gamma}^*$ is also minimizer and passes through $Rev(z_1)$ but not any other point before $Rev(z_1)$. Denote t_1 the half-plane capacity of the part of reversed curve from 0 to $Rev(z_1)$. Let $\tilde{g}_{t_1}$ be the mapping-out function $\mathbb{H}\backslash\tilde{\gamma}^*_{[0,t_1]} \to \mathbb{H}$ sending the tip to 0.

The image $\tilde{g}_{t_1}(\tilde{\gamma}^*_{[t_1,\infty[})$ is local minimizer of $Rev(\overline{z}\backslash z_1)$. By induction, the curve $\tilde{g}_{t_1}(\tilde{\gamma}^*_{[t_1,\infty[})$ with points $\tilde{z}' \subset \tilde{z}$ as vertices has property (P). The property (P) with $\overline{z}' = Rev(\tilde{z}') \cup z_1$ for the initial curve is clearly satisfied.

The proof of Property 3 is similar with Property 2. The fact that the minimizing curve of a 1-point constraint is smooth at the point, gives the smoothness of γ^* when it passes through points of $\overline{z}'$. $\square$

If γ^* is global minimizer of $\overline{z}$, it is not always global minimizer of $\overline{z}'$. We have to keep points in $\overline{z}$ even if γ^* doesn't pass through them, they allows to rule out other smaller energy curves which are not in the neighborhood.

One can think about the example where the constraints are symmetric w.r.t. the imaginary axis. Any local minimizing curve must has a symmetric one with the same energy, but visits points in different order. We can play with this idea to find a counterexample.

Some questions raise naturally:

1. Are those local minimizers isolated?
2. If a simple curve with marked points on it has property (P), is it local minimizer of those points with some assignment?
3. If we fix $\overline{z}'$ and the order of visited points, is there a unique local minimizer?
4. The energy of a simple curve in $\mathbb{H}$ from 0 to ∞ is firstly defined through the large deviation principle on its driving function. It is shown to be reverse-invariant for

local minimizers as shown in Prop. 24 which is not obvious through the definition of Loewner chain. Is the energy reverse-invariant on all curves?

The answers are not obvious to me and the work is ongoing. Let's just mention an observation about Q.4.

Let γ be a curve from 0 to ∞ in $\mathbb{H}$ with energy L. If every $\varepsilon > 0$, there is a choice of constraint points put on both side of the curve (finitely many and with assignment compatible with γ), such that the minimizing curve compatible with it has energy larger than $L-\varepsilon$. Then $I_\infty(\gamma) = I_\infty(Rev(\gamma))$. This suggests that the energy is reverse-invariant on regular enough curves or it gives a natural notion of the regularity of a Loewner curve.

5.3 Dessin d'enfant

Let S be a compact Riemann surface and $\varphi : S \to \hat{\mathbb{C}}$ a holomorphic covering ramified only on real points.

A well known example is Belyi pair (S, φ), namely a holomorphic function $\varphi : S \to \hat{\mathbb{C}}$ with critical values $\{0, 1, \infty\}$. The Belyi theorem [Bel80] states that a complex smooth projective curve S is defined over a number field, if and only if there exists a non-constant morphism : $S \to \hat{\mathbb{C}}$ with at most 3 critical values. This urges Grothendieck to develop his theory of Dessin d'enfant, the preimage of $[0, 1]$ together with 0 and 1. And its combinatorial type, characterize the function φ up to biholomorphism.

Here we consider the preimage of the real line and consider it as a embedded graph with preimages of critical values as vertices. For a Belyi pair, we get a triangulation on S.

Proposition 25. *The graph $G_\varphi = \varphi^{-1}(\mathbb{R})$ embedded in S has the property (P).*

For the simplicity of notation, we prove the proposition for a Belyi pair, but one can see that the general case (not only three ramified points) is also true. This is not surprising since the property (P) is by no means of algebraic nature.

Proof: We distinguish vertices and edges of different image, and denote the graph $G_\varphi = (V_0, V_1, V_\infty, E_{0,1}, E_{1,\infty}, E_{-\infty,0})$ in the obvious way, 'V' for a set of vertices, 'E' for set of edges and the subscript indicates its image by φ.

The graph G_φ is a triangulation of S. Each triangle has exactly one vertex in V_0, V_1 and V_∞. If a triangle's vertices $0, 1, \infty$ are in anti-clockwise order, it is sent by φ onto $\mathbb{H}$ the upper half-plane, triangles whose vertices are in clockwise order, are sent to the lower half-plane.

Let e be an edge in $E_{0,1}$. The open set $\mathscr{O}_e$ is reunion of two open triangles and e. Therefore $\varphi(\mathscr{O}_e) = \mathbb{C}\backslash]-\infty, 0] \cup [1, \infty[$, we see that $\varphi(e) =]0, 1[$ is a geodesic in $\varphi(\mathscr{O}_e)$. The same is true for $E_{1,\infty}$ and $E_{-\infty,0}$. Hence the property (P). $\square$

5.4 Geometric property of minimizers with finite constraints

The minimizing curve γ of SLE_{0+} in disc from $e^{i\theta}$ to $e^{-i\theta}$ passing to the left of 0 can be considered as an embedded graph in $\mathbb{D}$ with three vertices: $e^{-i\theta}, 0, e^{i\theta}$ and two edges

e_1 (with vertices 0 and $e^{-i\theta}$) and e_2 ($e^{i\theta}$ and 0). The graph satisfies that e_2 is the conformal geodesic (w.r.t. the hyperbolic metric) in the simply connected domain $\mathbb{D}\backslash e_1$ and vice versa. More generally, for a Riemann surface $\mathscr{O}$, and G connected, we say that the embedded graph $(\mathscr{O}, G)$ has the property (P) if:

(P) For each edge e of G, the connected component $\mathscr{O}_e$ containing e in $\mathscr{O}\backslash \bigcup_{e'\neq e} e'$ is simply connected, and e is the conformal geodesic in $\mathscr{O}_e$ between its two ends.

Remark. We can prove that for $\mathscr{O} = \hat{\mathbb{C}}$ and $|V| = 2$, assuming that two vertices are 0 and ∞ and there are n edges between them, the embedding satisfying (P) is under the form

$$\{\exp(i\theta + ik/2\pi + \lambda), \lambda \in \mathbb{R}, k \in \{0, 1, \cdots, n-1\}\},$$

for some $\theta \in \mathbb{R}$, hence unique up to rotation.

Property (P) provides a canonical embedding of a certain class of combinatorial graph (a finite set of vertices, a finite collection undirected edges, where multiple edges between two vertices are allowed, but no loop, and a cyclical order on edges adjacent to each vertex) into Riemann surfaces with prescribed image for vertices. In [**?**], we study such embedding of Eulerian graphs through minimizing curves, which is inspired by the following observation.

Proposition 26. *For every minimizing curve γ constraint to pass through successively $(z_1, z_2, \cdots, z_n)$ in $\mathbb{H}$, γ is an embedded graph satisfying (P) with vertices (z_i), 0 and ∞.*

Proof: If $n = 0$, γ is the imaginary axis thus satisfies (P).

For $n \geq 1$, assuming that γ hits z_n at time T, the driving function is stationary after time T. Thus (P) for the edge $[z_n, \infty]$ is checked. By Theorem **??**, γ is also a minimizer among curves passing through $(z_n, z_{n-1}, \cdots, z_1)$ in the domain $(\mathbb{H}, \infty, 0)$. Hence, it is a minimizer in the domain $(\mathbb{H} \setminus \gamma[T, \infty), z_n, 0)$ through the remaining $n-1$ points. The property (P) follows by induction. $\square$

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