

Piecewise geodesic Jordan curves I: weldings, explicit computations, and Schwarzian derivatives

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Abstract

We consider Jordan curves $\gamma = \cup_{j=1}^n \gamma_j$ in the Riemann sphere for which each γ_j is a hyperbolic geodesic in the complement of the remaining arcs $\gamma \setminus \gamma_j$. These curves are characterized by the property that their conformal welding is piecewise Möbius. Among other things, we compute the Schwarzian derivatives of the Riemann maps of the two regions in $\widehat{\mathbb{C}} \setminus \gamma$, show that they form a rational function with second order poles at the endpoints of the γ_j , and show that the poles are simple if the curve has continuous tangents. Our key tool is the explicit computation of all *geodesic pairs*, namely pairs of chords $\gamma = \gamma_1 \cup \gamma_2$ in a simply connected domain D such that γ_j is a hyperbolic geodesic in $D \setminus \gamma_{3-j}$ for both $j = 1$ and $j = 2$.

1 Introduction

In this paper, we introduce a class of Jordan curves on the Riemann sphere and begin a systematic study of their geometric function theoretic aspects, including explicit constructions and computations. We call a Jordan curve $\gamma = \cup_{j=1}^n \gamma_j \subset \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ *piecewise geodesic* if each *edge* γ_j is a hyperbolic geodesic in the simply connected domain $\widehat{\mathbb{C}} \setminus (\gamma \setminus \gamma_j)$. Piecewise geodesic curves arise naturally as minimizers of a certain functional related to Loewner theory and Weil-Petersson geometry, and we explore these connections in our companion paper [MRW]. This type of geodesic property was motivated by a question of Wendelin Werner, and was also studied in [PW] for the configuration of several disjoint chords in a simply connected domain and in [BE] for the case of two disjoint arcs on $\widehat{\mathbb{C}}$.

It is natural to ask if piecewise geodesic Jordan curves consist of geodesics for the hyperbolic metric of a punctured sphere (with punctures at the endpoints of the γ_j , called *vertices*). In Lemma 4.2 we show that this is only the case if γ is a circle.

A crucial role in our investigation is played by *geodesic pairs*, defined in Section 2 as

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chords $\gamma = \gamma_1 \cup \gamma_2$ in a simply connected domain D such that γ_j is a hyperbolic geodesic in $D \setminus \gamma_{3-j}$ for both $j = 1$ and $j = 2$. If $\gamma = \cup_{j=1}^n \gamma_j$ is a piecewise geodesic Jordan curve, then two consecutive edges γ_j, γ_{j+1} form a geodesic pair in the complement of the other edges. Not all geodesic pairs are C^1 smooth (there can be logarithmic spirals at the vertex, see Example 2.7). Examples of smooth geodesic pairs first appeared in [Wa1] as the minimizer of a certain Loewner energy among all chords passing through a given point ζ . There the description was rather indirect (as limits of SLE_κ as $\kappa \rightarrow 0$ and by means of their Loewner driving function). We give an explicit construction when $D = \mathbb{D}$ by exhibiting a conformal map of the regions $\mathbb{D} \setminus (\gamma_j \cup \gamma_2)$ onto the upper and lower half planes. Among other things, this allows for an explicit computation of the Schwarzian derivative of the map.

In Section 3, we consider a notion of conformal welding that is tailored to the setting of a simply connected domain bisected by a chord. A key observation is Lemma 3.4 that characterizes hyperbolic geodesics as chords whose *welding map* is Möbius on its interval of definition. This is applied to describe the welding maps of geodesic pairs (Corollary 3.5), to prove uniqueness of C^1 geodesic pairs through a given point (Theorem 3.9), and again later in Section 4 to characterize piecewise geodesic Jordan curves by their welding.

Corollary 4.1 in Section 4 states that a Jordan curve γ is piecewise geodesic if and only if its welding homeomorphism is *piecewise Möbius* (see Section 4 for the definition). Of particular interest are C^1 piecewise geodesic Jordan curves. Theorem 4.3 gives an explicit computation of the Schwarzian derivatives of the two Riemann maps of the two sides of γ and shows that they form a rational function with simple poles at the vertices of the γ . Theorem 4.7 characterizes their welding homeomorphisms and can be viewed as a parametrization of the space of C^1 piecewise geodesic Jordan curves. We also discuss the case of non- C^1 piecewise geodesic Jordan curves and show that they form logarithmic spirals at each point where they are not C^1 . In this case, the Schwarzian derivative of the two Riemann maps has second order poles. In Theorem 4.13 we determine their coefficients in terms of the geometric *spiral rate* (see Definition 4.11). Finally, in Section 5 we investigate symmetry properties of piecewise geodesic curves with four edges and prove that in the C^1 case, the curve is invariant under all four automorphisms of the sphere punctured in the four vertices of γ .

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2 Geodesic pairs

A *Jordan curve* in \mathbb{C} is a continuous, injective image of the unit circle. A similar definition holds for curves in the extended plane, $\widehat{\mathbb{C}}$, where continuity refers to the spherical metric in the Riemann sphere. If $D \subsetneq \mathbb{C}$ is a simply connected region then by the Riemann

mapping theorem there is a conformal (i.e. one-to-one and analytic) map φ of the unit disk \mathbb{D} onto D . If ∂D is a Jordan curve, then φ extends to be a one-to-one and continuous map of $\overline{\mathbb{D}}$ onto \overline{D} by Carathéodory's theorem. If D is not bounded by a Jordan curve then we can view φ as a one-to-one map of $\partial\mathbb{D}$ onto the prime ends of $\partial D \subset \widehat{\mathbb{C}}$. Henceforth, $a \in \partial D$ shall mean that a is a prime end in ∂D .

Throughout this article, $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}z > 0\}$ is the upper halfplane, $\mathbb{L} = \{z \in \mathbb{C} : \text{Im}z < 0\}$ is the lower halfplane and $\mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}$ is the complement in the extended plane $\widehat{\mathbb{C}}$ of the closed unit disk $\overline{\mathbb{D}}$. We will denote the positive real axis by \mathbb{R}^+ , the positive imaginary axis by $i\mathbb{R}^+$, and the extended real line $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

If D is simply connected and if f is a conformal map of \mathbb{H} onto D then $\gamma = f(i\mathbb{R}^+)$ is called the **hyperbolic geodesic** from the prime end $a = f(0)$ to another prime end $b = f(\infty)$. Equivalently, the image of a diameter by a conformal map of \mathbb{D} onto D is a hyperbolic geodesic. Subarcs of a hyperbolic geodesic are called **hyperbolic geodesic arcs**. See Section I.4 in [GM] for an equivalent definition in terms of shortest curves in the hyperbolic metric on D . We note that a hyperbolic geodesic is an analytic curve.

If D is a region in $\widehat{\mathbb{C}}$ and if a, b are prime ends in ∂D , we say that γ is a **chord** in $(D; a, b)$ if γ has a continuous injective parameterization $\gamma : (0, T) \rightarrow D$ such that $\gamma(t) \rightarrow a$ as $t \rightarrow 0$ and $\gamma(t) \rightarrow b$ as $t \rightarrow T$. We allow $b = a$ in this definition.

Definition 2.1. Let $\zeta \in D$ and $a, b \in \partial D$. We define a **geodesic pair** in $(D; a, b, \zeta)$ to be a chord γ in $(D; a, b)$ such that $\gamma = \gamma_1 \cup \gamma_2$ and $\gamma_1 \cap \gamma_2 = \{\zeta\}$ where γ_1 is the (hyperbolic) geodesic in $(D \setminus \gamma_2; a, \zeta)$ and γ_2 is the geodesic in $(D \setminus \gamma_1; \zeta, b)$.

From the definition of hyperbolic geodesics, if φ is a conformal map defined on D , then γ is a geodesic pair in $(D; a, b, \zeta)$ if and only if $\varphi(\gamma)$ is a geodesic pair in $(\varphi(D); \varphi(a), \varphi(b), \varphi(\zeta))$.

For orientation of the reader, we note that if D is simply connected and if γ is a geodesic pair in $(D; a, b, \zeta)$ then γ cannot be the union of two geodesic arcs in D unless these arcs are subsets of the same geodesic in D . To see this we may suppose $D = \mathbb{D}$ and $\zeta = 0$. The hyperbolic geodesic arcs in \mathbb{D} from 0 to $\partial\mathbb{D}$ are the radial line segments. By explicit computation, the only radial geodesic in $\mathbb{D} \setminus [0, 1)$ is the interval $(-1, 0]$. By a rotation, if γ_1 is a radial line segment, then the only radial geodesic in $\mathbb{D} \setminus \gamma_1$ is $-\gamma_1$.

Similarly, a geodesic pair γ in $(D; a, b, \zeta)$ cannot be the union of two geodesic arcs in $D \setminus \{\zeta\}$ unless these arcs are subsets of the same geodesic in $D \setminus \{\zeta\}$. Again, we may suppose $D = \mathbb{D}$ and $\zeta = 0$. The hyperbolic geodesics in $\mathbb{D} \setminus \{0\}$ from 0 to $\partial\mathbb{D}$ are also the radial line segments because the hyperbolic geometry on $\mathbb{D} \setminus \{0\}$ is the hyperbolic geometry inherited from \mathbb{H} via the universal covering map of \mathbb{H} onto $\mathbb{D} \setminus \{0\}$, which is $\pi(z) = e^{iz}$. The hyperbolic geodesic in \mathbb{H} from ∞ to $x \in \mathbb{R}$ is the vertical half line ending at x , whose image by π is a radial line segment and we conclude the proof as above.

We begin with an explicit construction of a geodesic pair in $(\mathbb{D}; e^{i\theta}, -e^{-i\theta}, 0)$. We will see in Theorem 3.9 that it is the only C^1 geodesic pair in $(\mathbb{D}; e^{i\theta}, -e^{-i\theta}, 0)$. The C^1 regularity of the curve refers to the arclength parametrization. Equivalently, the unit tangent vector to the curve is continuous. Similarly, we say that a curve is C^α , if the arclength parametrization of the curve is C^k , where k is the largest integer such that $k \leq \alpha$, and its k -th derivative is $(\alpha - k)$ -Hölder continuous.

Define

$$(1) \quad G_\theta(z) = \frac{1}{2} \left(z + \frac{1}{z} \right) - ic \log z,$$

where $c = \sin \theta$, with $-\pi/2 \leq \theta \leq \pi/2$. Let

$$(2) \quad U_\theta = \{z : \operatorname{Re} z > -A \text{ and } \operatorname{Im} z > 0\} \cup \{z : \operatorname{Re} z > A \text{ and } \operatorname{Im} z < 0\} \cup (B, +\infty),$$

where $A = (\pi/2) \sin \theta$ and $B = \cos \theta + \theta \sin \theta$. Note $B \geq |A|$. See Figure 1. As an aside we remark that the functions G_θ can be used to compute the lift of an airfoil in classical two-dimensional aeronautics.

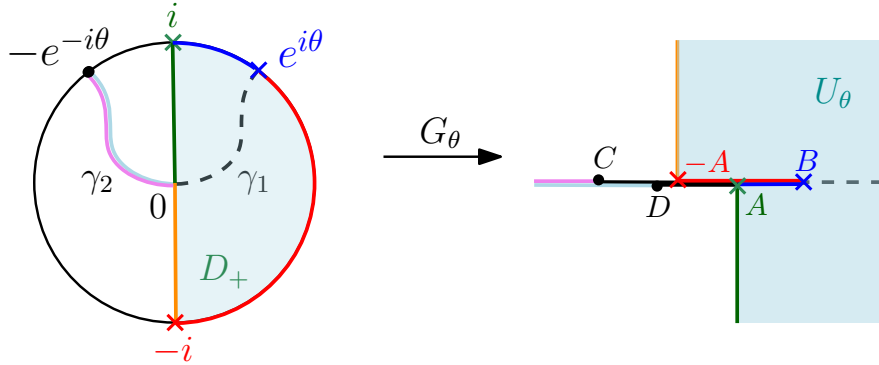


Figure 1: Construction of the geodesic pair $\gamma_1 \cup \gamma_2$ in $(\mathbb{D}; e^{i\theta}, -e^{-i\theta}, 0)$ using the conformal map $G_\theta: D_+ \rightarrow U_\theta$ sending 0 to ∞ . Here, $A = (\pi/2) \sin \theta$, $B = \cos \theta + \theta \sin \theta$, $C = -A - (B + A)$, $D = A - (B - A)$, $\gamma_1 = G_\theta^{-1}((B, \infty))$, and the analytic extension of G_θ^{-1} to $\mathbb{C} \setminus (-\infty, B]$ welds $(-\infty, C)$ to $(-\infty, D)$ to form γ_2 .

Lemma 2.2. *The function G_θ is a conformal map of the half disk $D_+ = \{z : \operatorname{Re} z > 0 \text{ and } |z| < 1\}$ onto U_θ , where the branch of $\log z$ is chosen so that $-\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2}$.*

Proof. Fix θ and write $G = G_\theta$ and $U = U_\theta$. Then

$$(3) \quad G'(z) = \frac{z^2 - 2icz - 1}{2z^2}.$$

Note that for $-1 \leq y \leq 1$, $y \neq 0$, $dG(iy)/dy > 0$, so as y decreases from 1 to 0, $\operatorname{Im} G$ decreases from 0 to $-\infty$ with $\operatorname{Re} G(iy) = G(i) = A$. Similarly as y decreases from

0 to -1 , $\text{Im}G$ decreases from $+\infty$ to 0 with $\text{Re}G(iy) = G(-i) = -A$. Furthermore $dG(e^{it})/dt = -\sin t + \sin \theta$ is positive on $(-\pi/2, \theta)$ and negative on $(\theta, \pi/2)$. Thus $G(e^{it})$ increases from $G(-i) = -A$ to $B = G(e^{i\theta}) = \cos \theta + \theta \sin \theta$, with $\text{Im}G = 0$, then decreases to $G(i) = A$, again with $\text{Im}G = 0$. We conclude that as z traces ∂D_+ , with positive orientation, $G(z)$ traces ∂U with positive orientation. Indeed, for ε sufficiently small, set $D_\varepsilon = \{z : |z| < \varepsilon\}$. As z traces the semi-circle $\{\varepsilon e^{it} : \pi/2 > t > -\pi/2\}$, then $G(\varepsilon e^{it}) \sim (2\varepsilon)^{-1} e^{-it}$ traces a curve from $G(\varepsilon i)$ to $G(-\varepsilon i)$. Thus if K is a compact subset of U , and if ε is sufficiently small then $G(\partial(D_+ \setminus D_\varepsilon))$ winds exactly once around each point of K . By the Argument Principle, G is a conformal map of D_+ onto U . \square

Set $\gamma_1 = G^{-1}[B, +\infty)$ and $\gamma_2 = -\bar{\gamma}_1 = \{-\bar{z} : z \in \gamma_1\}$. Then γ_1 is a curve in D_+ from $e^{i\theta}$ to 0, and $\gamma_2 \subset D_- = \mathbb{D} \setminus D_+$ is the reflection of γ_1 about the imaginary axis, which we parameterize as a curve from 0 to $-e^{-i\theta}$.

Corollary 2.3. *The curve $\gamma = \gamma_1 \cup \gamma_2$ is a C^1 geodesic pair in $(\mathbb{D}; e^{i\theta}, -e^{-i\theta}, 0)$ with a horizontal tangent at 0. When $-\pi/2 < \theta < \pi/2$, γ is perpendicular to $\partial\mathbb{D}$ at $e^{i\theta}$ and $-e^{-i\theta}$. When $\theta = \pi/2$, γ is a Jordan curve with both ends at i , forming three angles of $\pi/3$ with $\partial\mathbb{D}$. When $\theta = -\pi/2$, γ is the reflection of this latter curve about \mathbb{R} .*

Proof. Reflect across the vertical lines in $\partial U \setminus \{0\}$ and in ∂D_+ . By the Schwarz reflection principle, G extends to be a conformal map of $\mathbb{D} \setminus \gamma_2$ onto $\mathbb{C} \setminus (-\infty, B]$. Because $F(z) = B - z^2$ is a conformal map of \mathbb{H} onto $\mathbb{C} \setminus (-\infty, B]$ with $F(i\mathbb{R}^+) = (B, +\infty)$, the curve $\gamma_1 = G^{-1}(F(i\mathbb{R}^+))$ is a hyperbolic geodesic in $\mathbb{D} \setminus \gamma_2$. Moreover $H(z) = \overline{G(-\bar{z})}$ is a conformal map of $\mathbb{D} \setminus \gamma_1$ onto $\mathbb{C} \setminus (-\infty, B]$ with $H(\gamma_2) = (B, +\infty)$ and so γ_2 is a hyperbolic geodesic in $\mathbb{D} \setminus \gamma_1$. Thus $\gamma = \gamma_1 \cup \gamma_2$ is a geodesic pair in $(\mathbb{D}; e^{i\theta}, -e^{-i\theta}, 0)$.

We also note that

$$\arg(zG(z)) = \arg\left(\frac{1}{2}(z^2 + 1) - icz \log z\right) \xrightarrow{z \rightarrow 0} 0,$$

and $\arg G = 0$ along γ_1 , so that γ_1 has a horizontal tangent at 0. By reflection, so does γ_2 and hence γ is C^1 .

Finally note that by (1) and (3), G is analytic in a neighborhood of $e^{i\theta}$ and asymptotic to $d(z - e^{i\theta})^2$, with $d \neq 0$, near $e^{i\theta}$ such that the image of $\gamma_1 \cup \partial\mathbb{D}$ is contained in \mathbb{R} . Thus γ_1 must meet $\partial\mathbb{D}$ at right angles. A similar statement holds for γ_2 .

When $\theta = \pi/2$, the curve γ forms a Jordan curve beginning and ending at i . In this case $B = A = \pi/2$, so that U_θ has an angle of $3\pi/2$ at B and $[B, \infty)$ forms an angle of π with $[-A, A]$, or two thirds of $3\pi/2$. The map G is asymptotic to $K(z - i)^{3/2}$ near i . Thus the angle between γ_1 and $\partial\mathbb{D}$ at i is $\pi/3 = (2/3)(\pi/2)$. By reflection, the Corollary follows. \square

Remark 2.4. For $0 < \theta < \pi/2$, U_θ (see (2)) can be obtained from $U_{\pi/2}$ by removing the interval $J = [\pi/2, \cot \theta + \theta]$ then scaling by $\sin \theta$. If we remove the arc $\sigma =$

$G_{\pi/2}^{-1}([\pi/2, \cot \theta + \theta])$, together with its reflection $-\bar{\sigma}$ from \mathbb{D} and let f map $\mathbb{D} \setminus (\sigma \cup -\bar{\sigma})$ onto \mathbb{D} with $f(0) = 0$ and $f'(0) > 0$, then $f(\gamma_1 \setminus \sigma) \cup f(\gamma_2 \setminus -\bar{\sigma})$ is the geodesic pair in $(\mathbb{D}; e^{i\theta}, -e^{-i\theta}, 0)$ constructed above. Indeed, G_θ and $\sin \theta \cdot G_{\pi/2} \circ f^{-1}$ have the same image on D^+ and $i, 0, -i$ have the same image as boundary points so that these two maps are the same. A similar result holds for $-\pi/2 < \theta < 0$ using $U_{-\pi/2}$ and $G_{-\pi/2}$. In this sense, we can obtain all such geodesic pairs from the two Jordan curves beginning and ending at $\pm\pi/2$, though the formulae are more complicated because f is not given explicitly.

For later use, we record some elementary computations: We may write

$$(4) \quad G'(z) = \frac{z^2 - 2icz - 1}{2z^2} = \frac{(z - e^{i\theta})(z + e^{-i\theta})}{2z^2}$$

By logarithmic differentiation, the pre-Schwarzian of G is

$$(5) \quad P_G(z) = \frac{G''(z)}{G'(z)} = \frac{-2}{z} + \frac{1}{z - e^{i\theta}} + \frac{1}{z + e^{-i\theta}}$$

and Schwarzian derivative of G

$$(6) \quad S_G(z) = P'_G(z) - \frac{1}{2}P_G^2(z) = \frac{4i \sin \theta}{z} + R(z),$$

where R is rational with poles only at $e^{i\theta}$ and $-e^{-i\theta}$, and these poles have order 2.

Using the Riemann mapping theorem, we obtain easily a C^1 geodesic pair in other domains $(D; a, b)$ from the above construction. If $D \neq \mathbb{C}$ is simply connected, $a, b \in \partial D$ and $\zeta \in D$, then let $\gamma_{a,\zeta}$ and $\gamma_{\zeta,b}$ denote the hyperbolic geodesic arcs in D from a to ζ and from ζ to b , respectively.

Theorem 2.5. *There is a geodesic pair $\gamma = \gamma_1 \cup \gamma_2$ in $(D; a, b, \zeta)$ with continuous tangent at ζ . The tangent to γ at ζ bisects the angle between $\gamma_{a,\zeta}$ and $\gamma_{\zeta,b}$ at ζ . If ∂D has a tangent at a , or b , with $a \neq b$, then γ meets ∂D at right angles at a , respectively b . If ∂D has a tangent at a and $b = a$, then γ forms three angles of $\pi/3$ with ∂D at a .*

Proof. If φ is a conformal map of D onto \mathbb{D} with $\varphi(\zeta) = 0$, then after rotating, we may suppose that $\varphi(a) = e^{i\theta}$ and $\varphi(b) = -e^{-i\theta}$ with $-\pi/2 \leq \theta \leq \pi/2$. Then $\varphi(\gamma_{a,\zeta})$ lies on the diameter of \mathbb{D} through $e^{i\theta}$, and $\varphi(\gamma_{b,\zeta})$ lies on the diameter of \mathbb{D} through $-e^{-i\theta}$. The geodesic pair $\sigma = \sigma_1 \cup \sigma_2$ constructed in Corollary 2.3 has a horizontal tangent at 0 which forms an angle θ with each of these diameters. Set $\gamma_j = \varphi^{-1}(\sigma_j)$, for $j = 1, 2$, and note that φ^{-1} preserves angles between curves through 0 by conformality. If ∂D has a tangent at a then φ^{-1} preserves angles between curves in $\overline{\mathbb{D}}$ meeting at $e^{i\theta}$, by Theorem II.4.2 in [GM]. Thus γ_1 meets ∂D at right angles at a . A similar statement holds at b . \square

Example 2.6. C^1 geodesic pairs on \mathbb{H} . For example, if $D = \mathbb{H}$ and $a = 0$ and $b = \infty$ and $\zeta = re^{it}$, $0 < t < \pi$, then

$$(7) \quad \tau(z) = -ie^{-it} \frac{z - \zeta}{z - \bar{\zeta}}$$

is a conformal map of \mathbb{H} onto \mathbb{D} with $\tau(\zeta) = 0$, $\tau(0) = -ie^{it}$, and $\tau(\infty) = -ie^{-it}$. Set $\theta = t - \pi/2$, $H = G_\theta \circ \tau$, $\sigma_1 = \tau^{-1}(\gamma_1)$ and $\sigma_2 = \tau^{-1}(\gamma_2)$, where $\gamma_1 \cup \gamma_2$ is the constructed geodesic pair in $(\mathbb{D}; e^{i\theta}, -e^{-i\theta}, 0)$. Then $\sigma = \sigma_1 \cup \sigma_2$ is a geodesic pair in $(\mathbb{H}; 0, \infty, \zeta)$ and H is a conformal map of $\mathbb{H} \setminus \sigma_1$ onto $\mathbb{C} \setminus (-\infty, B]$ where $B = \cos \theta + \theta \sin \theta$. The corresponding map of $\mathbb{H} \setminus \sigma_2$ onto the slit plane is $G(-\overline{\tau(z)})$. The hyperbolic geodesic in \mathbb{H} from 0 to $\zeta = re^{it}$ lies on the circle through 0 and ζ which is perpendicular to \mathbb{R} . The hyperbolic geodesic in \mathbb{H} from ζ to ∞ is the vertical half-line. Then by Theorem 2.5 the geodesic pair σ has a tangent direction that bisects the angle between these hyperbolic geodesics in \mathbb{H} . By elementary geometry, the tangent direction is e^{it} , so that at $\zeta = re^{it}$, σ is orthogonal to the circle of radius r centered at 0. This geodesic pair is also orthogonal to \mathbb{R} at 0 and has a vertical tangent at ∞ .

Not all geodesic pairs are C^1 . The following example is a geodesic pair which does not have a tangent at the intersection point ζ .

Example 2.7. A geodesic pair in $\widehat{\mathbb{C}} \setminus \{0\}$ with logarithmic spirals.

Let S be the strip $\{z : -\pi < \text{Im}z < \pi\}$. The image of \mathbb{R} , by the conformal map $ie^{z/2}$ of S onto \mathbb{H} , is the positive imaginary axis, so that \mathbb{R} is a hyperbolic geodesic in S . We consider another conformal image of S found by first applying the map $z \mapsto Az$ where A is chosen so that S is rotated by an angle θ then scaled so that the image by the map $z \mapsto Az$ of the two parallel lines in ∂S differ by $2\pi i$. More concretely, fix θ with $-\pi/2 < \theta < \pi/2$ and set $A = 1/(1 - i \tan \theta)$ then $A(\pi \tan \theta + \pi i) = \pi i$ and $A(-\pi \tan \theta - \pi i) = -\pi i$. The function e^{Az} maps the line $\{x + iy : y = \pi\} \subset \partial S$ onto a logarithmic spiral S_θ which spirals from 0 to ∞ as x increases. The line $\{x + iy : y = -\pi\} \subset \partial S$ is also mapped to exactly the same spiral. The image of \mathbb{R} is the spiral $-S_\theta$ from 0 to ∞ . Because e^{Az} is a conformal map on S , $-S_\theta$ is a hyperbolic geodesic in $\mathbb{C} \setminus S_\theta$. The map e^{Az} maps the region $S + \pi i$ onto $\mathbb{C} \setminus (-S_\theta)$. Similarly S_θ is a hyperbolic geodesic in $\mathbb{C} \setminus -S_\theta$. If we reparameterize $-S_\theta$ so that it is a curve from ∞ to 0, then $\gamma = S_\theta \cup (-S_\theta)$ is a geodesic pair in $(\widehat{\mathbb{C}} \setminus \{0\}; 0, 0, \infty)$.

3 Welding Maps

In this section we characterize piecewise geodesic curves using the welding maps.

Suppose Ω is a simply connected region contained in the extended plane $\widehat{\mathbb{C}}$ and suppose γ is a chord in Ω from a to b , with $a, b \in \partial\Omega$. By the Jordan curve theorem, $\Omega \setminus \gamma$ consists of two simply connected regions Ω_+ and Ω_- . We choose Ω_+ to be the region which lies to the left of the oriented curve γ , and choose Ω_- to be the region that lies to the right of γ . Let f_+ and f_- be conformal maps of Ω_+ and Ω_- onto the upper half

plane \mathbb{H} and the lower half plane \mathbb{L} , respectively. By Caratheodory's theorem, f_+ and f_- extend to be homeomorphisms of $\Omega_+ \cup \gamma$ and $\Omega_- \cup \gamma$ onto $\mathbb{H} \cup f_+(\gamma)$ and $\mathbb{L} \cup f_-(\gamma)$. The function $\omega = f_- \circ f_+^{-1}$ is called a **welding map** associated with the curve γ with respect to the region Ω . For $\zeta \in \gamma$, $x = f_+(\zeta)$ if and only if $\omega(x) = f_-(\zeta)$. So we can view f_+^{-1} and f_-^{-1} as mapping \mathbb{H} and \mathbb{L} onto Ω_+ and Ω_- pasting together, or welding, the intervals $f_+(\gamma) \subset \partial\mathbb{H}$ and $f_-(\gamma) \subset \partial\mathbb{L}$, according to the prescription ω .

The function ω is an increasing homeomorphism of $f_+(\gamma)$ onto $f_-(\gamma)$. Here we extend the notion of an increasing homeomorphism on an interval in \mathbb{R} to functions u defined on an interval $J^* \in \widehat{\mathbb{R}}$, the extended real line, where we allow $\infty \in J^*$ and $\infty \in f(J^*)$. An increasing homeomorphism on J^* is a continuous, injective function u defined on J^* which is increasing on each subinterval of J^* which contains neither ∞ nor $u^{-1}(\infty)$. Our choice for Ω_+ and Ω_- means that f_+ and f_- are also increasing functions of the parameterization of γ .

Example 3.1. Welding maps for geodesic pairs in Corollary 2.3.

Suppose $\gamma_1, \gamma_2 \subset \mathbb{D}$ are defined as in Corollary 2.3. Then G_θ provides a map of the two regions in $\mathbb{D} \setminus (\gamma_1 \cup \gamma_2)$ onto \mathbb{H} and \mathbb{L} . See Figure 1. The associated welding map ω is

$$(8) \quad \omega(x) = \begin{cases} x & \text{for } x > B = \cos \theta + \theta \sin \theta, \\ x + 2\pi \sin \theta & \text{for } x < C = -\cos \theta - (\theta + \pi) \sin \theta. \end{cases}$$

The expression for ω on (B, ∞) follows from the fact that G_θ is analytic across γ_1 . The welding for $x \in (-\infty, C)$ is given by first reflecting about the vertical line $x = -(\pi/2) \sin \theta$ then reflecting about the vertical line $x = (\pi/2) \sin \theta$. Two reflections form a shift (and this is why we considered regions of the form U_θ). Indeed, the first reflection is given by $z \mapsto -\bar{z} - \pi \sin \theta$, and the second reflection is given by $z \mapsto -\bar{z} + \pi \sin \theta$, so that $\omega(x) = x + 2\pi \sin \theta$, mapping the interval $(-\infty, C)$ onto $(-\infty, D)$.

Example 3.2. Welding maps for Example 2.7.

This case was explored in [KNS], with slightly different notation. For completeness, we give the construction here. Recall that the function $f(z) = e^{A \log z}$, with $-\pi < \arg z < \pi$ is a conformal map of \mathbb{H} and \mathbb{L} onto the two regions in $\mathbb{C} \setminus S_\theta \cup (-S_\theta)$, where $A = 1/(1 - i \tan \theta)$. This yields $f_+(z) = f^{-1}(z)$ for $z \in f(\mathbb{H})$ and $f_-(z) = f^{-1}(z)$ for $z \in f(\mathbb{L})$. The welding map $\omega = f_- \circ f_+^{-1}$ then satisfies

$$(9) \quad \omega(x) = \begin{cases} x & \text{for } x > 0, \\ ax & \text{for } x < 0, \end{cases}$$

where $a = \exp(-2\pi \tan \theta)$. Indeed, because f is analytic on $\mathbb{H} \cup \mathbb{L} \cup \mathbb{R}^+$, $\omega(x) = x$ for $x > 0$. If $x \in \partial\mathbb{H}$, $x < 0$ and $ax \in \partial\mathbb{L}$, then $\log x = \log |x| + i\pi$ and $\log ax = \log |ax| - i\pi$. An easy computation shows that $f(ax) = f(x)$, so that $\omega(x) = ax$ for $x < 0$.

Non-constant maps of the form $\sigma(z) = \frac{az+b}{cz+d}$, where a, b, c, d are complex numbers, are called **Möbius maps**. Note that σ is an increasing homeomorphism of $\widehat{\mathbb{R}}$ onto $\widehat{\mathbb{R}}$ if and only if $\sigma(\mathbb{H}) = \mathbb{H}$, and this occurs if and only if $\sigma(\mathbb{L}) = \mathbb{L}$.

We can identify the Möbius maps that preserve \mathbb{H} with the group

$$(10) \quad \text{PSL}(2, \mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\} /_{A \sim -A}$$

which acts on \mathbb{H} (and \mathbb{L}) by $A : z \mapsto \frac{az+b}{cz+d}$. The set of Möbius maps of $\widehat{\mathbb{C}}$ is similarly identified with the group $\text{PSL}(2, \mathbb{C})$.

If ω is an increasing homeomorphism defined on an interval $J \subset \widehat{\mathbb{R}}$ and if σ_1, σ_2 are Möbius maps with $\sigma_j(\mathbb{H}) = \mathbb{H}$, $j = 1, 2$, then $\omega_1 = \sigma_1 \circ \omega \circ \sigma_2$ is also an increasing homeomorphism, defined on $\sigma_2^{-1}(J)$.

Definition 3.3. We say that two increasing homeomorphisms ω_1, ω_2 defined on intervals in $\widehat{\mathbb{R}}$ are **equivalent** if $\omega_1 = \sigma_1 \circ \omega_2 \circ \sigma_2$, for some Möbius maps with $\sigma_j(\mathbb{H}) = \mathbb{H}$ (and hence $\sigma_j(\mathbb{L}) = \mathbb{L}$), $j = 1, 2$.

If $\omega = f_- \circ f_+^{-1}$ is a welding map associated with a curve γ with respect to a region Ω then every increasing homeomorphism equivalent to ω is also associated with the same curve γ in the region Ω . Indeed, if $g_+ = \sigma_2^{-1} \circ f_+$ and $g_- = \sigma_1 \circ f_-$, then $g_- \circ g_+^{-1} = \sigma_1 \circ \omega \circ \sigma_2$. By the Riemann mapping theorem, every welding associated with the curve γ is equivalent to ω . We remark that this notion of equivalence depends on the orientation of the curve γ . If ω is a welding map associated to $\gamma \subset \Omega$, then $-\omega^{-1}(-x) = g_- \circ g_+^{-1}$, where $g_+ = -f_-$ and $g_- = -f_+$, is also an increasing homeomorphism, not equivalent to ω , but associated with γ traced in the opposite direction.

Lemma 3.4. *Suppose $\Omega \subset \widehat{\mathbb{C}}$ is a simply connected region and suppose $\gamma \subset \Omega$ is a chord in Ω . Let $\omega = f_- \circ f_+^{-1}$ be a welding map associated with γ with respect to Ω . Then γ is a hyperbolic geodesic in Ω if and only if ω is a Möbius map on its interval of definition.*

Proof. If γ is a hyperbolic geodesic from a to b with $a, b \in \partial\Omega$, then let g be a conformal map of Ω onto \mathbb{H} such that $g(\gamma) = i\mathbb{R}^+$, so that $g(a) = +\infty$ and $g(b) = 0$. Then $g^2|_{\Omega_+}$ and $g^2|_{\Omega_-}$ are conformal maps of the two regions Ω_+ and Ω_- , with $\Omega \setminus \gamma = \Omega_+ \cup \Omega_-$, onto \mathbb{H} and \mathbb{L} respectively. The welding map ω_0 on $(-\infty, 0)$, in this case, is the identity map. But $\omega = \sigma_1 \circ \omega_0 \circ \sigma_2$ for some Möbius maps σ_1, σ_2 by the Riemann mapping theorem. Thus $\omega = \sigma_1 \circ \sigma_2$, a Möbius map, on $\sigma_2^{-1}(-\infty, 0)$.

Conversely suppose ω is a Möbius map on its interval of definition I . Let σ_1 be a Möbius map with $\sigma_1(\omega(I)) = (-\infty, 0)$ and let σ_2 be a Möbius map with $\sigma_2(I) = (-\infty, 0)$. We can also arrange that $\sigma_j(\mathbb{H}) = \mathbb{H}$ and $\sigma_j(\mathbb{L}) = \mathbb{L}$, $j = 1, 2$. Then if c is a positive constant,

$\omega_1 = c\sigma_1 \circ \omega \circ \sigma_2^{-1}$ fixes 0 and ∞ , so we can choose c so that ω_1 is the identity map on $(-\infty, 0)$. Set $g_+ = \sigma_2 \circ f_+$ and $g_- = c\sigma_1 \circ f_-$. Then $\omega_1 = g_- \circ g_+^{-1}$ is a welding map associated with γ . Moreover $g_+^{-1} = g_-^{-1}$ on $(-\infty, 0)$ so that the function h defined by $h = g_+^{-1}$ on \mathbb{H} and $h = g_-^{-1}$ on \mathbb{L} extends to be conformal on $\mathbb{H} \cup \mathbb{L} \cup (-\infty, 0)$. Then $k(z) = h(z^2)$ is a conformal map of \mathbb{H} onto Ω with $k(i\mathbb{R}^+) = \gamma$. \square

Corollary 3.5. *A welding map for a geodesic pair in $(D; a, b, \zeta)$ is equivalent to a welding map given by*

$$(11) \quad \omega(x) = \begin{cases} a_1x + b_1, & \text{for } x > c_1 \\ a_2x + b_2, & \text{for } x < c_2 \end{cases}$$

where $a_1 > 0$, $a_2 > 0$, $b_1, b_2, c_1, c_2 \in \mathbb{R}$, and $c_2 \leq c_1$, and $a_2c_2 + b_2 \leq a_1c_1 + b_1$.

Proof. By Lemma 3.4, the welding map ω for a geodesic pair in $(D; a, b, \zeta)$ is given by Möbius maps on two adjacent intervals. By pre and post composition with increasing Möbius maps, we may suppose that the common point of these two intervals in $\widehat{\mathbb{R}}$ is ∞ , and $\omega(\infty) = \infty$. A Möbius map which fixes ∞ and is increasing as a map \mathbb{R} to \mathbb{R} must be of the form $ax + b$ with $a > 0$ and $b \in \mathbb{R}$. The lemma follows. \square

If an increasing homeomorphism is of the form (11), we say it is piecewise Möbius near ∞ . If it also satisfies $a_1 = a_2$ then we say that ω has a continuous derivative equal to a_1 at ∞ .

Theorem 3.6. *If ω is a piecewise Möbius map near ∞ which has a continuous derivative at ∞ then ω is a welding map associated to exactly one of the geodesic pairs constructed in Corollary 2.3, unless ω is a linear map on all of \mathbb{R} .*

Proof. Suppose $\omega(x) = ax + b_1$, for $x > c_1$, and $\omega(x) = ax + b_2$ for $x < c_2$, where $c_2 \leq c_1$ and $a > 0$.

If $b_1 \neq b_2$ then set

$$\varphi(z) = \left(\frac{|b_1 - b_2|}{a} \right) z + c_1$$

and

$$\tau(z) = \frac{z - (ac_1 + b_1)}{|b_1 - b_2|}.$$

It is elementary to check the following: If $b_2 > b_1$ then

$$(12) \quad \omega_1(x) \equiv \tau \circ \omega \circ \varphi(x) = \begin{cases} x, & \text{for } x > 0 \\ x + 1 & \text{for } x < \frac{(c_2 - c_1)a}{b_2 - b_1} \leq -1. \end{cases}$$

The last inequality in (12) holds because $\omega(c_2) \leq \omega(c_1)$.

If $b_1 > b_2$ then

$$(13) \quad \omega_1(x) \equiv \tau \circ \omega \circ \varphi(x) = \begin{cases} x, & \text{for } x > 0 \\ x - 1 & \text{for } x < \frac{(c_2 - c_1)a}{b_1 - b_2} \leq 0. \end{cases}$$

If $b_1 = b_2$ and $c_2 < c_1$ then set

$$\varphi(z) = \frac{c_1 - c_2}{2}x + \frac{c_1 + c_2}{2}$$

and

$$\tau(z) = \frac{2x}{a(c_1 - c_2)} - \frac{c_1 + c_2 + 2b_1/a}{c_1 - c_2}.$$

Then

$$(14) \quad \omega_1(x) \equiv \tau \circ \omega \circ \varphi(x) = x \text{ for } x > 1 \text{ and } x < -1.$$

If $b_1 = b_2$ and $c_1 = c_2$ and if ω is a welding for $\mathbb{D} \setminus \gamma = D_+ \cup D_-$ then $f_+^{-1}(x) = f_-^{-1}(ax + b_1)$ for $x \neq c_1$. This implies that the function F given by f_+^{-1} on \mathbb{H} and $f_-^{-1}(ax + b_1)$ on \mathbb{L} extends to be an entire function mapping onto the disk. But then F is constant, which is a contradiction. We remark that these linear maps on all of \mathbb{R} are welding maps for circles and lines in $\widehat{\mathbb{C}}$, but not for geodesic pairs.

Apply the above normalization to the welding maps in Example 3.1. So $a = 1$, $b_1 = 0$, $b_2 = 2\pi \sin \theta$, $c_1 = \cos \theta + \theta \sin \theta$ and $c_2 = -\cos \theta - (\theta + \pi) \sin \theta$.

If $\sin \theta > 0$ then (12) applies with $\frac{(c_2 - c_1)a}{b_2 - b_1} = -\left(\frac{\cot \theta + \theta}{\pi}\right) - \frac{1}{2}$.

If $\sin \theta < 0$ then (13) applies with $\frac{(c_2 - c_1)a}{b_1 - b_2} = \left(\frac{\cot \theta + \theta}{\pi}\right) + \frac{1}{2}$.

If $\sin \theta = 0$, then $\theta = 0$ and (14) applies.

Note that $-(\cot \theta + \theta)/\pi - 1/2$ has positive derivative and increases from $-\infty$ to -1 as θ increases from 0 to $\pi/2$. Similarly $(\cot \theta + \theta)/\pi + 1/2$ decreases from 0 to $-\infty$ as θ increases from $-\pi/2$ to 0. Thus the examples in Corollary 2.3 have a welding map in each equivalence class of piecewise Möbiusmaps with continuous derivative at ∞ .

Suppose $\gamma_\alpha = \gamma_{\alpha,1} \cup \gamma_{\alpha,2}$ and $\gamma_\beta = \gamma_{\beta,1} \cup \gamma_{\beta,2}$ are geodesic pairs created in Corollary 2.3 for $\theta = \alpha$ and $\theta = \beta$. If their welding maps are equivalent, then there is a conformal map φ of \mathbb{D} onto \mathbb{D} with $\varphi(\gamma_\alpha) = \gamma_\beta$, by the comment after Definition 3.3. But then φ must be a Möbius map with $\varphi(0) = 0$, and hence φ is a rotation. But if γ_β is a rotation of γ_α , then $\alpha = \beta$. This proves that there is exactly one welding map from

the examples in Corollary 2.3 in each equivalence class of piecewise Möbiusmaps with continuous derivative at ∞ . This latter fact can also be proved by showing that no two maps of the form (12), (13), or (14) are equivalent. \square

If φ is a conformal map of Ω onto another simply connected region $\varphi(\Omega)$ and if ω is a welding map associated with $\gamma \subset \Omega$, then ω is also a welding map associated with $\varphi(\gamma) \subset \varphi(\Omega)$. The next lemma, which is well-known, says that a piecewise analytic Jordan curve is determined by its welding map, up to conformal equivalence. This lemma and the subsequent corollary are also true under the less restrictive assumption that γ_1 is a quasi-arc. Because the audience for this paper may come from other fields, we decided to keep the proofs as simple as possible.

Lemma 3.7. *Suppose ω is a welding map for chords $\gamma_1 \subset \Omega_1$ and $\gamma_2 \subset \Omega_2$. If γ_1 is piecewise analytic, then there is a conformal map F of Ω_1 onto Ω_2 such that $F(\gamma_1) = \gamma_2$.*

Proof. Let f_+, f_- be the conformal maps onto \mathbb{H} and \mathbb{L} of the two regions determined by γ_1 , and let g_+, g_- be the conformal maps onto \mathbb{H} and \mathbb{L} of the two regions determined by γ_2 . The functions $F_1 = g_+^{-1} \circ f_+$ and $F_2 = g_-^{-1} \circ f_-$ are conformal maps of the regions determined by γ_1 onto the regions determined by γ_2 . By assumption $f_- \circ f_+^{-1} = \omega = g_- \circ g_+^{-1}$ on $f_+(\gamma_1)$, so that $F_1 = F_2$ on γ_1 . Because γ_1 is piecewise analytic, the function F_1 extends to be analytic on Ω_1 except for the isolated points where γ_1 is not analytic. Isolated points are removable for continuous analytic functions, so the function F_1 extends to be a conformal map of Ω_1 onto Ω_2 . \square

It is not true that a welding ω for a curve γ with respect to a region Ω is also a welding with respect to a subregion $\Omega_1 \subset \Omega$ for $\gamma \cap \Omega_1$, but we can cut Ω along γ to give the following local result.

Suppose ω_1, ω_2 are weldings for piecewise analytic chords γ_1, γ_2 with respect to regions Ω_1, Ω_2 , respectively, and suppose f_+, f_- and g_+, g_- are conformal maps of the corresponding regions $(\Omega_j)_+$ and $(\Omega_j)_-$ onto \mathbb{H} and \mathbb{L} . If $\omega_1|_I = \omega_2|_I$ for some interval $I \subset f_+(\gamma_1) \cap g_+(\gamma_2)$, then $\omega_1|_I$ is a welding map associated with $(\Omega_1 \setminus \gamma_1) \cup f_+^{-1}(I)$. Similarly, $\omega_2|_I$ is a welding map associated with $(\Omega_2 \setminus \gamma_2) \cup g_+^{-1}(I)$. Lemma 3.7 implies the following result.

Corollary 3.8. *If $\omega_1|_I = \omega_2|_I$ for some interval $I \subset f_+(\gamma_1) \cap g_+(\gamma_2)$, then there is a conformal map from $(\Omega_1 \setminus \gamma_1) \cup f_+^{-1}(I)$ onto $(\Omega_2 \setminus \gamma_2) \cup g_+^{-1}(I)$ mapping $f_+^{-1}(I) \subset \gamma_1$ onto $g_+^{-1}(I) \subset \gamma_2$.*

In other words, the local behavior of γ_1 is, up to local conformal equivalence, determined by the local behavior of ω_1 , as was observed in [KNS].

A Jordan curve $\gamma \subset \widehat{\mathbb{C}}$ can be viewed as a chord in $\Omega = \widehat{\mathbb{C}} \setminus \{\zeta_0\}$ with endpoints tending to $\partial\Omega$, where $\zeta_0 \in \gamma$. In particular, if γ_1 and γ_2 are Jordan curves associated with the same

welding map ω , and if γ_1 is piecewise analytic, then by Corollary 3.8 there is a Möbius map τ such that $\gamma_2 = \tau(\gamma_1)$. Thus increasing piecewise analytic homeomorphisms of $\widehat{\mathbb{R}}$, modulo pre and post composition with Möbius maps, correspond to piecewise analytic Jordan curves in $\widehat{\mathbb{C}}$ modulo post composition with Möbius maps.

Corollary 3.8 also yields the following uniqueness result.

Theorem 3.9. *If D is simply connected, if $a, b \in \partial D$, and if $\zeta \in D$, then there is a unique C^1 geodesic pair γ in $(D; a, b, \zeta)$. There is a conformal map φ of D onto \mathbb{D} so that $\varphi(\gamma)$ is one of the geodesic pairs constructed in Corollary 2.3.*

Proof. By applying a conformal map of D onto \mathbb{D} , we may suppose $D = \mathbb{D}$ and $\zeta = 0$. Let $\gamma = \gamma_1 \cup \gamma_2$ be a geodesic pair in $(\mathbb{D}; a, b, 0)$. If ω is a welding map associated with γ in \mathbb{D} , we may suppose that ω is defined on an interval $J \subset \widehat{\mathbb{R}}$ containing ∞ with $\omega(\infty) = \infty$. Then

$$(15) \quad \omega(x) = \begin{cases} a_1x + b_1 & \text{for } x > c_1 \\ a_2x + b_2 & \text{for } x < c_2 \end{cases}$$

where $a_1, a_2 > 0$, $b_1, b_2, c_1, c_2 \in \mathbb{R}$, with $c_2 \leq c_1$. If $a_1 \neq a_2$, set $\varphi(x) = x/a_1 + (b_2 - b_1)/(a_1 - a_2)$ and $\tau(x) = x + (a_2b_1 - a_1b_2)/(a_1 - a_2)$. Then

$$(16) \quad \omega_1(x) \equiv \tau \circ \omega \circ \varphi(x) = \begin{cases} x & \text{for } x > \varphi^{-1}(c_1) \\ \frac{a_2}{a_1}x & \text{for } x < \varphi^{-1}(c_2) \end{cases}$$

Choose θ so that $a_2/a_1 = \exp(-2\pi \tan \theta)$ with $-\pi/2 < \theta < \pi/2$. By Example 2.7, for x sufficiently large, the welding map $\omega_1(x)$ agrees with a welding map associated with a logarithmic spiral S_θ . By Corollary 3.8 there is a conformal map F defined in a neighborhood U of 0 with $F(S_\theta \cap U) = \gamma \cap F(U)$. In particular γ does not have a continuous (unit) tangent at 0. So if γ has a continuous (unit) tangent at 0, then it has a welding map with continuous derivative at ∞ . By Theorem 3.6, the welding map ω for γ in \mathbb{D} is associated with exactly one of the geodesic pairs constructed in Corollary 2.3. By Lemma 3.7 there is a conformal map H of \mathbb{D} onto \mathbb{D} so that $H(\gamma)$ is one of these constructed geodesic pairs. Because H is a Möbius map fixing the origin, H must be a rotation. There is exactly one rotation of \mathbb{D} and one θ , $-\pi/2 \leq \theta \leq \pi/2$, so that a, b are rotated to $e^{i\theta}$ and $-e^{-i\theta}$. \square

The proof of Theorem 3.9 also gives the following information about geodesic pairs which do not have a continuous tangent.

Corollary 3.10. *If D is simply connected and if $\gamma \in (D; a, b, \zeta)$ is a geodesic pair which does not have a continuous unit tangent at ζ , then γ is asymptotically a logarithmic spiral near ζ .*

The *Schwarzian derivative* of an analytic function f is given by

$$(17) \quad S_f(z) = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2.$$

The composition rule for Schwarzian derivative is

$$(18) \quad S_{g \circ h}(z) = S_g(h(z)) (h'(z))^2 + S_h(z).$$

The Schwarzian derivative of a Möbius map is zero. If $S_f = S_g$, then there is a Möbius map σ so that $f = \sigma \circ g$.

Our construction also gives some information about the Schwarzian derivative of the conformal map f_{\pm} in the next Corollary.

Corollary 3.11. *If $D \neq \mathbb{C}$ is simply connected and if $\gamma = \gamma_1 \cup \gamma_2$ is a C^1 geodesic pair in $(D; a, b, \zeta)$, then the Schwarzian derivative S_f of the conformal map $f_{\pm} : D \setminus \gamma_1 \cup \gamma_2 \rightarrow \mathbb{H} \cup \mathbb{L}$ extends to be analytic on D , except for a simple pole at ζ . The residue of the pole of S_f at ζ is $(4i \sin \theta)h'(\zeta)$, where h is the conformal map of D onto \mathbb{D} with $h(\zeta) = 0$ and $h(a) = e^{i\theta}$ and $h(b) = -e^{-i\theta}$.*

Proof. Conformal maps have non-vanishing derivative, so that S_g is analytic whenever g is conformal. The composition rule for Schwarzian derivatives (18) applied to a Möbius map g and f_{\pm} shows that the choice of the maps onto \mathbb{H} and \mathbb{L} does not affect S_f . If h is a conformal map of D onto \mathbb{D} with $h(\zeta) = 0$ and $h(a) = e^{i\theta}$ and $h(b) = -e^{-i\theta}$, and $g = G_{\theta}$ as in (1), then by (18), Theorem 3.9 and (6) the corollary follows. \square

If $\gamma_{a,\zeta}$ and $\gamma_{\zeta,b}$ are the hyperbolic geodesics in D from a to ζ and from ζ to b with tangent directions e^{it} and e^{is} at ζ , then by Theorem 2.5, the tangent direction of the geodesic pair in $(D; a, b, \zeta)$ is e^{iu} at ζ , where $u = (t + s)/2$, and the corresponding θ is given by $\theta = (t - s)/2$. The density of the hyperbolic metric $\rho(z)|dz|$ in D at z is given by $\rho(z) = |h'(z)|$, where h is a conformal map of D onto \mathbb{D} . Thus we can write the residue of S_f at ζ purely in terms of the hyperbolic geometry on D as

$$4i \sin \left(\frac{t - s}{2} \right) \rho(\zeta) e^{-i(t+s)/2}.$$

4 Piecewise geodesic curves

4.1 General facts

If $\gamma = \gamma_1 \cup \dots \cup \gamma_n$ is a Jordan curve in $\widehat{\mathbb{C}}$, then a welding map ω for γ in $\widehat{\mathbb{C}}$ is an increasing homeomorphism of $\widehat{\mathbb{R}} = J_1 \cup \dots \cup J_n$ into $\widehat{\mathbb{R}}$. Note that $\omega|_{J_j}$ is a welding map for γ_j in $\widehat{\mathbb{C}} \setminus (\gamma \setminus \gamma_j)$.

An increasing homeomorphism h of \mathbb{R} onto \mathbb{R} will be called **piecewise Möbius** if there are $\{x_j\}$ with $-\infty = x_0 < x_1 < \dots < x_n = +\infty$ so that h is given by a Möbius map L_j on (x_{j-1}, x_j) , for each $j = 1, \dots, n$. A homeomorphism h of \mathbb{R} onto \mathbb{R} is called **quasisymmetric** if there is a constant $M < \infty$ such that

$$(19) \quad \frac{1}{M} \leq \frac{|h(x+t) - h(x)|}{|h(x) - h(x-t)|} \leq M,$$

for all $x, t \in \mathbb{R}$.

A Jordan curve $\gamma \subset \widehat{\mathbb{C}}$ will be called **piecewise geodesic** if $\gamma = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_n$ where γ_j is a hyperbolic geodesic in $\widehat{\mathbb{C}} \setminus (\gamma \setminus \gamma_j)$, for $j = 1, \dots, n$. The points $\zeta_j = \gamma_j \cap \gamma_{j+1}$ will be called the *vertices* of γ , and the γ_j the *edges*.

Corollary 4.1. *A piecewise Möbius increasing homeomorphism of $\widehat{\mathbb{R}}$ onto $\widehat{\mathbb{R}}$ is a welding map associated with a piecewise geodesic Jordan curve. Conversely if γ is a piecewise geodesic Jordan curve, then the welding maps associated with γ are piecewise Möbius homeomorphisms.*

Proof. Suppose ω is an increasing homeomorphism of $\widehat{\mathbb{R}} = J_1 \cup \dots \cup J_n$ onto $\widehat{\mathbb{R}}$ such that $\omega|_{J_j}$ is Möbius. By composing ω with a single increasing Möbius map, we may suppose that $\omega(\infty) = \infty$. Then there is a constant $c < \infty$ such that $1/c < \omega'(x) < c$ for all $x \in \mathbb{R}$. By the mean-value theorem applied to the numerator and denominator of (19), ω is quasisymmetric. By [LV, page 92] or [A], ω is a welding map for a Jordan curve $\gamma = \gamma_1 \cup \dots \cup \gamma_n$. By Lemma 3.4, because $\omega|_{J_j}$ is a welding map for γ_j in $\widehat{\mathbb{C}} \setminus (\gamma \setminus \gamma_j)$, γ_j is a hyperbolic geodesic in $\widehat{\mathbb{C}} \setminus (\gamma \setminus \gamma_j)$. The converse also follows immediately from Lemma 3.4. \square

A similar result holds for piecewise geodesic Jordan chords in a simply connected region $D \neq \mathbb{C}$. In this case the welding maps are piecewise Möbius homeomorphisms of a proper sub-interval in $\widehat{\mathbb{R}}$.

It is natural to ask if the edges of a piecewise geodesic Jordan curve are hyperbolic geodesics of the punctured sphere, with punctures at the vertices. This is generally not the case:

Lemma 4.2. *If $\gamma = \cup_{i=1}^n \gamma_i$ is a piecewise geodesic Jordan curve and if each edge γ_i is a hyperbolic geodesic in the punctured sphere $\Sigma := \widehat{\mathbb{C}} \setminus \{\zeta_1, \dots, \zeta_n\}$ between ζ_i and ζ_{i+1} , then γ is a circle.*

Proof. For each $i \in 1, \dots, n$, γ_i is the hyperbolic geodesic in the simply connected domain $\Omega_+ \cup \Omega_- \cup \gamma_i$. Hence, there is an antiholomorphic involution $R_i : \Omega_+ \cup \Omega_- \cup \gamma_i \rightarrow \Omega_+ \cup \Omega_- \cup \gamma_i$ where γ_i is the set of fixed points. Both maps $R_i|_{\Omega_+}$ and $R_i|_{\Omega_-}$ extend to the boundary and induce two homeomorphisms $\gamma \rightarrow \gamma$.

We claim that $R_i|_{\Omega_+}$ and $R_i|_{\Omega_-}$ also fix all ζ_k , for $k \in 1, \dots, n$. In fact, let $p : \mathbb{D} \rightarrow \Sigma$ be the universal cover map. Consider a connected component D of $p^{-1}(\Omega_+ \cup \Omega_- \cup \gamma_i)$. Its boundary is formed by circular arcs which project to $\gamma \setminus \gamma_i$ (since γ_k are assumed to be hyperbolic geodesics in Σ), and $p^{-1}(\gamma_i)$ is both a conformal geodesic in D and a circular arc orthogonal to the boundary. Without loss of generality, we assume $p^{-1}(\gamma_i) = (-1, 1)$. Then the involution R_i lifts to an antiholomorphic map \tilde{R}_i on D which equals the identity on $(-1, 1)$ and therefore is the map $z \mapsto \bar{z}$ on D . It implies that \tilde{R}_i maps cusps to cusps. Therefore R_i fixes all ζ_k .

Now if $i \neq j$, then $R_i \circ R_j|_{\Omega_+}$ is a conformal map $\Omega_+ \rightarrow \Omega_+$ whose extension to γ fixes all ζ_k , so $R_i \circ R_j|_{\Omega_+}$ is the identity map. This implies that $R_i = R_j$. We obtain that R_i extends to a antiholomorphic involution $\widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with set of fixed points γ . Therefore γ is a circle. \square

See also the closely related comment after Definition 2.1.

4.2 C^1 piecewise geodesic Jordan curves

Let us first comment on the existence of C^1 piecewise geodesic Jordan curves. In [RW, Prop. 2.13] two of the authors introduced the notion of *Loewner energy* for Jordan curves and showed the existence of minimizers among all curves passing through a given set of vertices. The union of two adjacent edges in those minimizers are geodesic pairs in the complement of the rest of the curve, and do not have logarithmic spirals. By Corollary 3.10, they are C^1 piecewise geodesic Jordan curves¹.

One of the consequences of the explicit formula for C^1 geodesic pairs is the following. As before if $\gamma = \gamma_1 \cup \dots \cup \gamma_n$ is a piecewise geodesic curve, let $\zeta_j = \gamma_j \cap \gamma_{j+1}$, $j = 1, \dots, n$ where $\gamma_{n+1} \equiv \gamma_1$.

Theorem 4.3. *Suppose $\gamma \subset \mathbb{C}$ is a C^1 piecewise geodesic Jordan curve. If Ω_+, Ω_- are the two regions given by $\widehat{\mathbb{C}} \setminus \gamma$, and if f_+, f_- are conformal maps onto \mathbb{H} and \mathbb{L} respectively, set $f = f_+$ on Ω_+ and $f = f_-$ on Ω_- . Then the Schwarzian derivative, S_f , of f extends to be meromorphic in $\widehat{\mathbb{C}}$ and is given by*

$$(20) \quad S_f(z) = \sum_{j=1}^n \frac{c_j}{z - \zeta_j},$$

for some constants c_j satisfying

$$(21) \quad \sum_{j=1}^n c_j = \sum_{j=1}^n c_j \zeta_j = \sum_{j=1}^n c_j \zeta_j^2 = 0.$$

¹The proof of the regularity stated in [RW, Prop. 2.13] was sketched out and referred to the present work for details. However, the proof of the existence of the minimizer was self-contained.

Proof. By Corollary 3.11 applied $\gamma_j \cup \gamma_{j+1}$ in $D = \widehat{\mathbb{C}} \setminus (\gamma \setminus (\gamma_j \cup \gamma_{j+1}))$, $j = 1, \dots, n$, S_f is analytic in $\widehat{\mathbb{C}}$ except for simple poles at each ζ_j . To examine the behavior of S_f near ∞ , we can use (18) to obtain $S_f(z) = S_{f \circ g}(1/z)z^{-4}$, where $g(z) = 1/z$. Because $f \circ g$ is analytic at 0, $S_f(z) = O(z^{-4})$ near ∞ . Thus we can write S_f in the form (20), and expanding this form in powers of $1/z$ we obtain (21). \square

The residues c_j are reminiscent to the *accessory parameters* on the n -punctured sphere. Similar to [TZ1, TZ2], we relate c_j to the variation of the Loewner energy in the companion paper [MRW].

If $\gamma \subset \mathbb{C}$ is a C^1 piecewise geodesic curve, then it follows from Theorem 4.3 that $S_f(z) = q(z) \prod_1^n (z - \zeta_j)^{-1}$, where q is a polynomial of degree $\leq n - 4$.

Remark 4.4. One consequence is that $S_f(z) \equiv 0$ if $n \leq 3$, which implies that γ is a circle or a line through the points $\{\zeta_j\}$. This implies the curve γ is unique when $n \leq 3$.

If $n = 4$ we must have

$$(22) \quad S_f = \frac{C}{(z - \zeta_1)(z - \zeta_2)(z - \zeta_3)(z - \zeta_4)}.$$

A version of Theorem 4.3 holds for a piecewise geodesic curve $\gamma \subset \widehat{\mathbb{C}}$ with continuous tangent. If ∞ is not one of the ζ_j , we can use (18) with $g = f$ and Möbius h to obtain the same statement as in Theorem 4.3. If ∞ is one of the ζ_j , then again by using (18) with $g = f$ and Möbius h we obtain (20) with $n - 1$ terms, one for each finite ζ_j . In this case $S_f = O(z^{-3})$ near ∞ so that (21) is replaced by

$$(23) \quad \sum_{j=1}^n c_j = \sum_{j=1}^n c_j \zeta_j = 0.$$

We were unable to explicitly solve the following.

Problem 4.5. *Given four distinct points $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{C}$. Find the explicit expression of a C^1 piecewise geodesic Jordan curve γ with vertices $\zeta_1, \zeta_2, \zeta_3, \zeta_4$.*

Problem 4.6. *For which values of C does there exist a piecewise geodesic Jordan curve γ with vertices $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ such that the Schwarzian derivative of the associated conformal maps of $\widehat{\mathbb{C}} \setminus \gamma$ onto \mathbb{H}, \mathbb{L} satisfy (22)?*

There are several ways to normalize the welding maps. For the next theorem, the following normalization is convenient. If ω is a piecewise Möbiuswelding map of $\widehat{\mathbb{R}}$ onto $\widehat{\mathbb{R}}$ with continuous derivative, then we can replace ω with $\omega \circ \sigma_1$, where σ_1 is a Möbius map so that ω is analytic on $\widehat{\mathbb{R}} \setminus \{x_j\}_0^n$ with $x_0 = \infty$, $0 = x_1 < x_2 < \dots < x_n = 1$.

Moreover, post composing ω with another Möbius map, we may suppose that $w(x) = x$ for $x < 0$. Because ω has continuous derivative at ∞ , we then have $\omega(x) = x + b$ for some $b \in \mathbb{R}$, when $x > 1$. If $n \geq 2$, then there is exactly one welding map with this normalization in each equivalence class of piecewise Möbius welding maps modulo pre and post composition by Möbius maps.

Theorem 4.7. *Suppose $0 = x_1 < x_2 < \dots < x_n = 1$ and $0 = y_1 < y_2 < \dots < y_n$. Then there is a normalized piecewise Möbius welding map ω of $\widehat{\mathbb{R}}$, analytic on $\mathbb{R} \setminus \{x_1, \dots, x_n\}$, with continuous derivative such that*

$$\omega(\infty) = \infty, \omega'(\infty) = 1, \text{ and } \omega(x_j) = y_j, \quad j = 1, \dots, n$$

if and only if

$$(24) \quad 1 = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} \cdot \frac{x_{n-1} - x_{n-2}}{y_{n-1} - y_{n-2}} \cdot \frac{y_{n-2} - y_{n-3}}{x_{n-2} - x_{n-3}} \dots$$

where the last term in (24) is $\frac{y_2 - y_1}{x_2 - x_1}$ when n is even and $\frac{x_2 - x_1}{y_2 - y_1}$ when n is odd. Moreover such a piecewise Möbius welding map is unique.

Remark 4.8. By Corollary 4.1, this theorem gives a parameterization of the family of C^1 piecewise geodesic Jordan curves in terms of the real coordinates $(x_2, \dots, x_{n-1}, y_2, \dots, y_n)$. Together with the constraint (24), this family has real dimension $2n - 4$ which coincides with the dimension of the Teichmüller space of the $(n + 1)$ -punctured sphere.

Proof. The proof is based on the following elementary identity which holds for all Möbius maps T . If $a \neq b$, then

$$(25) \quad T'(a)T'(b) = \left(\frac{T(b) - T(a)}{b - a} \right)^2.$$

Given $a, b, \alpha, \beta, \delta$ then there exist a unique Möbius map T such that $T(a) = \alpha$, $T(b) = \beta$ and $T'(a) = \delta$. Indeed, if

$$S_1(z) = \frac{\delta(a - b)}{\alpha - \beta} \frac{z - a}{z - b} \text{ and } S_2(z) = \frac{z - \alpha}{z - \beta},$$

Then $T = S_2^{-1} \circ S_1$ works. Moreover if T_1 is another such map then $S = S_2 \circ T_1 \circ S_1^{-1}$ satisfies $S(0) = 0$, $S(\infty) = \infty$ and $S'(0) = 1$, so $S(z) = z$ and $T_1 = T$. The identity (25) allows us to determine $T'(b)$ from $T(b) - T(a)$, $b - a$ and $T'(a)$.

Suppose ω exists. Because ω is linear on $(-\infty, 0)$ and on $(1, +\infty)$ and $\omega'(\infty) = 1$, we have $\omega'(0) = 1 = \omega'(1)$. By (25)

$$(26) \quad \omega'(x_{j+1}) = \left(\frac{\omega(x_{j+1}) - \omega(x_j)}{x_{j+1} - x_j} \right)^2 \frac{1}{\omega'(x_j)}.$$

So we successively find a formula for $\omega'(x_{j+1})$, $j = 1, \dots, n-1$, starting with $\omega'(x_1) = 1$. Because $\omega'(1) = 1$, by induction (24) must hold. Here we note that individual ratios in (24) are all positive, so that the square is not needed in the statement. The map ω restricted to each interval (x_j, x_{j+1}) , $j = 1, \dots, n-1$, is uniquely determined from $\{x_j\}$, $\{y_j\}$ once we know $\omega'(x_j)$. Note that we must have $\omega(x) = x$, on $(-\infty, 0)$ and $\omega(x) = x - 1 + y_n$ on $(1, +\infty)$, because $\omega(0) = 0$ and $\omega(1) = y_n$.

Conversely, assume (24) holds. Define $\omega(x) = x$, for $x < 0$. Then $\omega'(0) = 1$, so we can uniquely define ω on each subsequent interval because of (25). Because (24) holds, we obtain $\omega'(1) = 1$. Then set $\omega(x) = x - 1 + y_n$ for $x > 1$. \square

If we do not assume that $\omega'(\infty) = 1$ then the statement of the corresponding result is somewhat more complicated. We must split the result into two cases, when n is even and when n is odd. When n is even, there is only one possible derivative at ∞ , and it is given in terms of the product. When n is odd, any choice of the derivative at ∞ will produce a welding map for these intervals, if and only if (24) holds. In the latter case, once the derivative at ∞ is chosen, the welding map for the set of intervals is unique. We are interested, though, in C^1 piecewise geodesic curves. So if ω is a welding map for a Jordan curve γ with respect to a region Ω , then so is the normalization of ω . So characterizing the normalized maps gives a characterization of the equivalence classes of all piecewise Möbius welding maps with continuous derivatives, modulo pre and post composition by Möbius maps.

We obtain another proof of Remark 4.4:

Corollary 4.9. *A C^1 piecewise geodesic Jordan curve with three edges is a circle.*

Proof. By Corollary 4.1, the welding homeomorphism ω of a C^1 piecewise geodesic Jordan curve with three edges can be chosen to be piecewise Möbius as in Theorem 4.7 with $n = 2$, $0 = x_1 < x_2 = 1$, $0 = y_1 < y_2$, and $(y_2 - y_1)/(x_2 - x_1) = 1$. It implies that $y_2 = x_2 = 1$. Therefore ω is the identity map on $\widehat{\mathbb{R}}$ which implies that the welded curve is a circle. \square

Our explicit construction of geodesic pairs also gives a little more information about the behavior of a piecewise geodesic curve $\gamma = \gamma_1 \cup \dots \cup \gamma_n$, $n \geq 3$, near a point $\zeta_j = \gamma_j \cap \gamma_{j+1}$. Applying a conformal map of $\widehat{\mathbb{C}} \setminus (\gamma \setminus (\gamma_j \cup \gamma_{j+1}))$ onto \mathbb{D} , it is enough to investigate the behavior of geodesic pairs in $(\mathbb{D}; e^{i\theta}, -e^{-i\theta}, 0)$ near 0. If γ is a Jordan curve with continuous tangent at ζ then the conformal maps f_{\pm} of the two regions in $\widehat{\mathbb{C}} \setminus \gamma$ onto \mathbb{H} and \mathbb{L} respectively are $C^{1-\varepsilon}$ for $\varepsilon > 0$, but not necessarily in C^1 . See [GM], Section II.4.

Corollary 4.10. *A C^1 piecewise geodesic Jordan curve is piecewise analytic and in $C^{2-\varepsilon}$ for all $\varepsilon > 0$.*

Proof. Piecewise geodesic curves are piecewise analytic by definition. To show the regularity near vertices, it is enough to prove the corollary for geodesic pairs in $(D; e^{i\theta}, -e^{-i\theta}, 0)$. Fix θ and set $G = G_\theta$ as in (1) and let $\gamma = \gamma_1 \cup \gamma_2$ be the corresponding geodesic pair in \mathbb{D} . Set $H(z) = 1/G(z)$. Then H maps γ onto the real axis with $H(0) = 0$. Then

$$(27) \quad H'(z) = \frac{2(1 + 2icz - z^2)}{(1 - 2icz \log z + z^2)^2} = 2 + 8icz \log z + O(z).$$

Thus $(H'(z) - H'(0))/z^{1-\varepsilon}$ is continuous at 0, for $\varepsilon > 0$. By Lemma II.4.4 in [GM], γ is in $C^{2-\varepsilon}$, for $\varepsilon > 0$.

This result can also be deduced using a result in [Wo] on Loewner slits. \square

We remark that Corollary 4.10 implies that a piecewise Möbius welding map ω in C^1 is in $C^{2-\varepsilon}$, a fact that can more easily be deduced directly from its definition.

4.3 Piecewise geodesic Jordan curves with spirals

Now we turn to general piecewise geodesic Jordan curves γ . From Corollaries 3.8 and 3.10, γ has a logarithmic spiral at the vertices where γ does not have continuous (unit) tangent.

Definition 4.11. If $\gamma(t) = \zeta_0 + e^{r(t)+i\varphi(t)}$ is a spiral with “eye” at ζ_0 , then the **spiral rate** of γ at ζ_0 is given by

$$(28) \quad R(\gamma, \zeta_0) = \lim_{r \rightarrow 0} \frac{\partial \varphi}{\partial r}$$

So γ has a large spiral rate at ζ_0 if $|\arg \gamma(t)|$ increases rapidly as $\gamma(t) \rightarrow \zeta_0$, and it has a small spiral rate if $|\arg \gamma(t)|$ increases slowly as $\gamma(t) \rightarrow \zeta_0$. If $R(\gamma, \zeta_0)$ is positive, then γ spirals clockwise as $\gamma(t) \rightarrow \zeta_0$. If $R(\gamma, \zeta_0)$ is negative, then γ spirals counter-clockwise as $\gamma(t) \rightarrow \zeta_0$.

If S_θ is the spiral given in Example 2.7, then we can write $S_\theta(t) = e^{r(t)+i\varphi(t)}$ where

$$r(t) = \cos^2 \theta (t - \pi \tan \theta)$$

and

$$\varphi(t) = \cos^2 \theta (\pi + t \tan \theta),$$

so that the spiral rate at 0 is given by

$$(29) \quad R(S_\theta, 0) = \tan \theta.$$

Recall that an associated welding map is given by (9) with $a = \exp(-2\pi \tan \theta)$. Thus

$$(30) \quad R(S_\theta, 0) = \frac{1}{2\pi} \log \frac{\omega'(0^+)}{\omega'(0^-)}.$$

This allows us to compute the jump in the derivative of the welding map for a piecewise geodesic curve in terms of the geometry of the curve, as follows.

Lemma 4.12. *Suppose $\gamma = \gamma_1 \cup \dots \cup \gamma_n$ is a piecewise geodesic Jordan curve, with piecewise Möbius welding map ω . Set $\zeta_j = \gamma_j \cap \gamma_{j+1}$. If $x_j \in \partial\mathbb{H}$ corresponds to $\zeta_j \in \mathbb{R}$, then $\omega'(x)$ has one-sided limits $\omega'(x_j^+)$ and $\omega'_j(x_j^-)$ at x_j . The “jump” in the derivative at x_j given by*

$$\frac{\omega'(x_j^+)}{\omega'_j(x_j^-)} = e^{2\pi R(\gamma, \zeta_j)}.$$

Proof. Set $1/a = \frac{\omega'(x_j^+)}{\omega'_j(x_j^-)}$. By the chain rule, a does not change if we apply a Möbius map of \mathbb{H} onto \mathbb{H} , so that a does not depend on our choice of the welding map for γ . If F is conformal near ζ_j then the spiral rate of $F(\gamma)$ at $F(\zeta_j)$ is the same as the spiral rate of γ at ζ_j . If $a \neq 1$ then by Corollary 3.8 and Example 3.2, γ is the conformal image of S_θ near ζ_j where $\log a = -2\pi \tan \theta$. The lemma then follows from (29) and (30). \square

If $x_j = \infty$, then the jump is $\lim_{x \rightarrow +\infty} \omega(x) / \lim_{x \rightarrow -\infty} \omega(x)$, as can be seen by replacing ω with $\varphi \circ \omega \circ \varphi^{-1}$ where $\varphi(z) = 1/(1-z)$, then computing the jump at $0 = \varphi(\infty)$.

We remark that for a piecewise geodesic curve γ , because $\frac{\partial \varphi}{\partial r}$ is constant for the curve S_θ , the spiral rate is also equal to

$$(31) \quad R(\gamma, \zeta_j) = \lim \frac{2\pi}{\log \frac{|\alpha - \zeta_j|}{|\beta - \zeta_j|}},$$

where the limit is taken over $\alpha, \beta \ni \gamma \rightarrow \zeta_j$ such that $\arg(\alpha - \zeta_j) = 2\pi(k+1)$ and $\arg(\beta - \zeta_j) = 2\pi k$. This gives the spiral rate in terms of the asymptotic decay rate of $|\gamma - \zeta_j|$ as γ loops once around ζ_j .

A similar discussion of the relation of the jump in the welding map to a spiral rate can be found in [KNS], though with a different definition of the spiral rate.

Theorem 4.13. *Suppose $\gamma \subset \widehat{\mathbb{C}}$ is a piecewise geodesic Jordan curve, analytic except at $\zeta_1, \dots, \zeta_n \in \mathbb{C}$. If Ω_+, Ω_- are the two regions given by $\widehat{\mathbb{C}} \setminus \gamma$, and if f_+, f_- are conformal maps onto \mathbb{H} and \mathbb{L} respectively, set $f = f_+$ on Ω_+ and $f = f_-$ on Ω_- . Let R_j be the geometric spiral rate of γ at ζ_j given in (28) and (31). Then the Schwarzian derivative, S_f , of f extends to be meromorphic in $\widehat{\mathbb{C}}$ and is given by*

$$S_f(z) = \sum_{j=1}^n \frac{\frac{1}{2}R_j^2 + iR_j}{(z - \zeta_j)^2} + \frac{c_j}{z - \zeta_j},$$

for some constants c_j . Near ∞ , $S_f = O(z^{-4})$.

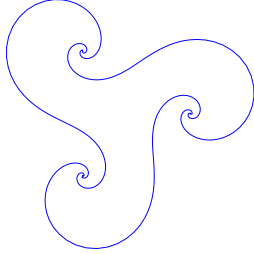


Figure 2: A piecewise geodesic curve consisting of three spirals with spiral rate 2, reminiscent of the Triskele, one of the oldest Irish Celtic symbols.

Proof. By Corollary 3.11, if γ has a continuous tangent at ζ_j then S_f has a simple pole at ζ_j . If γ does not have a continuous tangent at ζ_j then to examine the local behavior of S_f near ζ_j , by Lemma 3.8, (14), and Example 2.7 we can write $f = g \circ \varphi$ where φ is analytic and one-to-one in a neighborhood of ζ_j with $\varphi(\zeta_j) = 0$ and $g(z) = e^{(1-i \tan \theta) \log z}$, for the appropriate θ . Then $S_g(z) = \lambda z^{-2}$, where $\lambda = i \tan \theta + \frac{1}{2}(\tan \theta)^2$. By the identity (18), $S_{g \circ \varphi}(z) = \frac{\lambda}{(z-\zeta_j)^2} + \dots$ has a double pole at ζ_j with the same coefficient λ . By (29), this proves the expansion of the Theorem at points where the curve is not continuously differentiable. \square

Corollary 4.10 and Theorem 4.13 yield the dichotomy that a piecewise geodesic curve is either $C^{2-\varepsilon}$ for every $\varepsilon > 0$ at a vertex or a logarithmic spiral at a vertex.

A welding map of the form (11) with $a_1 \neq a_2$ can be conjugated using linear maps to an equivalent welding of the form

$$(32) \quad \omega(x) = \begin{cases} x & \text{for } x > c_1, \\ ax & \text{for } x < c_2, \end{cases}$$

with $c_2 \leq c_1$. If $c_1 > 0$ and $c_2 < 0$, we can then use the examples created earlier for welding maps h of the form $h(x) = ax$, $x > 0$ and $h(x) = x$, when $x < 0$ to create geodesic pair (for each $a > 0$) in \mathbb{C} which meet at 0 and ∞ and form logarithmic spirals at 0 and ∞ . We reduce the size of the intervals $x > 0$ and $x < 0$ by removing portions of the two spirals from \mathbb{C} to obtain a simply connected region and a geodesic pair with the desired welding map. See Corollary 3.8. (The explicit form of a conformal map from the upper half plane to the complement in \mathbb{C} of a portion of S_θ is given in [LMR], proposition 3.3). In this way we obtain geodesic pairs for all welding maps of the form $h(x) = x$, for $x < c_1$, and $h(x) = ax$ for $x > c_2$, provided $c_2 \leq 0 \leq c_1$, and of course any welding map conjugate to these.

The maps for which we do not have an explicit construction are when $0 < c_1 \leq c_2$ or $c_1 \leq c_2 < 0$. The point is that Example 2.7 creates an infinite spiral at points corresponding to both 0 and ∞ , the two fixed points of the maps $h(x) = ax$ and $h(x) = x$. These fixed points cannot be interior to the intervals where the welding map is the identity or where it is given by ax because the associated maps f_+^{-1} and f_-^{-1} must be analytic where the welding is Möbius.

Problem 4.14. *Analogous to (1), if $a < 1$, find an expression for a conformal map from $\mathbb{D} \setminus \gamma$ onto $\mathbb{H} \cup \mathbb{L}$, where $\gamma = \gamma_1 \cup \gamma_2$ is a geodesic pair in \mathbb{D} , such that the welding map is given by $h(x) = ax$ for $x < 1$ and $h(x) = x$ for $x > 2$ onto \mathbb{H} .*

One might also ask for geodesic pairs in the simply connected region \mathbb{C} . In this case, because the welding map must map $\widehat{\mathbb{R}}$ onto $\widehat{\mathbb{R}}$, the welding is conjugate to $h_a(x) = x$ for $x < 0$ and $h_a(x) = ax$ for $x > 0$, $a > 0$. This of course gives the logarithmic spirals discussed earlier, if $a \neq 1$, and a straight line or circle if $a = 1$.

5 C^1 piecewise geodesic curves with four edges

As we noted in Remark 4.4 and Corollary 4.9 that a C^1 piecewise geodesic curve with three edges has to be a circle, we now consider C^1 piecewise geodesic curves with four edges.

The conformal modulus of a Jordan domain marked by four consecutive boundary points can be defined as the aspect ratio of a rectangle that arises as the image of the domain under a conformal map sending the marked points to the corners. Our description of piecewise Möbius weldings Theorem 4.7 easily gives the following symmetry:

Lemma 5.1. *If $\gamma = \gamma_1 \cup \dots \cup \gamma_4$ is a C^1 piecewise geodesic curve, then the conformal moduli of the two complementary domains Ω_+ and Ω_- , marked by the vertices ζ_1, \dots, ζ_4 of γ , coincide.*

Proof. By Corollary 4.1, the welding maps associated with γ are piecewise Möbius homeomorphisms. The conformal maps $f_+ : \Omega_+ \rightarrow \mathbb{H}$ and $f_- : \Omega_- \rightarrow \mathbb{L}$ can be chosen so that $f_+(\zeta_4) = f_-(\zeta_4) = \infty$ and such that the welding $\omega = f_- \circ f_+^{-1}$ satisfies the normalization of Theorem 4.7 with $x_j = f_+(\zeta_j)$, $y_j = f_-(\zeta_j)$, $j = 1, 2, 3$. Then (24) gives

$$1 = \frac{y_3 - y_2}{1 - x_2} \cdot \frac{x_2 - x_1}{y_2 - y_1}$$

and it follows that $x_2 = y_2/y_3$ so that the upper half-plane, marked by $\infty, 0, x_2, 1$ is anti-conformally equivalent to the lower half plane marked by $\infty, 0, y_2, y_3$. \square

A special feature of four edge piecewise geodesic curves is that they have more symmetries, namely the invariance under the four conformal automorphisms of $\widehat{\mathbb{C}} \setminus \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ that permute the vertices:

Theorem 5.2. *If γ is a C^1 piecewise geodesic curve and if T is a conformal automorphism of $\widehat{\mathbb{C}} \setminus \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, then $T(\gamma) = \gamma$. Moreover, if T is the identity or if $T(\zeta_1) = \zeta_3$ and $T(\zeta_2) = \zeta_4$, then $T(\Omega_+) = \Omega_+$ and $T(\Omega_-) = \Omega_-$, whereas the other two automorphisms interchange Ω_+ and Ω_- .*

Proof. We have described earlier (see the discussion after Definition 3.3) that equivalent homeomorphisms of the real line arise as welding homeomorphisms of the same curve. However, if the homeomorphism and the curve are suitably normalized, then the welding uniquely determines the curve and the (suitably normalized) conformal maps representing it. Our proof is based on this uniqueness, together with our characterization of C^1 piecewise Möbiushomeomorphisms.

We may assume that the four points are $0, \zeta, 1, \infty$ and that the two conformal maps f_+ and f_- are normalized to fix $0, 1, \infty$. Then the welding homeomorphism $\omega = f_- \circ f_+^{-1}$ fixes $0, 1, \infty$ and by Lemma 5.1 also fixes $x = f_+(\zeta) \in (0, 1)$. Note that $d = \omega'(\infty) > 0$ but does not necessarily equal 1. By Theorem 4.7 (and the discussion preceding its statement), ω is the unique C^1 piecewise Möbius homeomorphism fixing $0, x, 1, \infty$ with derivative d at ∞ .

If T is the automorphism interchanging $0, \infty$ and $\zeta, 1$, namely $T(z) = \zeta/z$, let $S(z) = x/z$ and notice that S induces the same permutation on $\{0, x, 1, \infty\}$ as T on $\{0, \zeta, 1, \infty\}$, and that S interchanges \mathbb{H} and \mathbb{L} . The conformal maps $g_+ = T \circ f_-^{-1} \circ S$ of \mathbb{H} and $g_- = T \circ f_+^{-1} \circ S$ of \mathbb{L} solve the same normalized welding problem as f_+^{-1} and f_-^{-1} : Indeed, the homeomorphism

$$g_-^{-1} \circ g_+ = S^{-1} \circ f_+ \circ f_-^{-1} \circ S = S \circ \omega^{-1} \circ S$$

of $\widehat{\mathbb{R}}$ is piecewise Möbius with continuous derivative, fixes $0, x, 1, \infty$, and has the same derivative as ω at ∞ by a simple computation (using $\omega'(0) = d$ since ω is linear on the interval $(-\infty, 0)$). It follows that $g_+ = f_+^{-1}$ so that $g_+(\mathbb{H}) = T f_-^{-1}(\mathbb{L})$ and therefore $T(\Omega_+) = \Omega_-$ and $T(\gamma) = \gamma$. The same argument applies when T interchanges $0, \zeta$ and $1, \infty$.

If T interchanges $0, 1$ and ζ, ∞ , the corresponding Möbius transformation S that interchanges $0, 1$ and x, ∞ fixes \mathbb{H} and \mathbb{L} , and the above reasoning applied to $g_+ = T \circ f_+^{-1} \circ S$ on \mathbb{H} and $g_- = T \circ f_-^{-1} \circ S$ on \mathbb{L} again yields $g_+ = f_+^{-1}$, hence $T(\Omega_+) = \Omega_+$. \square

An alternative proof, again assuming without loss of generality that the four points are $0, \zeta, 1$, and ∞ , goes as follows: The maps $\tau_1(z) = \zeta/z$, $\tau_2(z) = (z - \zeta)/(z - 1)$, and $\tau_3(z) = \zeta(z - 1)/(z - \zeta) = \tau_1 \circ \tau_2(z)$ each preserve the set $P = \{0, \zeta, 1, \infty\}$, so that if γ is a piecewise geodesic Jordan curve which is analytic except at these points, then $\tau_j^{-1}(\gamma)$ is also a piecewise geodesic Jordan curve which is analytic except at these points. Using the notation of Theorem 4.3, if f maps the two regions determined by γ onto $\mathbb{H} \cup \mathbb{L}$, then $f \circ \tau_j$ maps the two regions determined by $\tau_j^{-1}(\gamma)$ onto $\mathbb{H} \cup \mathbb{L}$. By Theorem 4.3,

(18), and (22)

$$(33) \quad S_{f \circ \tau_j}(z) = S_f(\tau_j(z))(\tau_j'(z))^2 = S_f(z).$$

Thus there are Möbius maps σ_j so that

$$(34) \quad f \circ \tau_j = \sigma_j \circ f.$$

But $\text{Im} f \neq 0$ if and only if $z \notin \gamma$, and similarly $\text{Im} f \circ \tau_j \neq 0$ if and only if $z \notin \tau_j^{-1}(\gamma)$. Because $\tau_j(P) = P$, there must be at least 4 distinct points in $\sigma_j(\widehat{\mathbb{R}}) \cap \widehat{\mathbb{R}}$. Because $\sigma(\widehat{\mathbb{R}})$ is either a circle or a line, σ_j must map $\widehat{\mathbb{R}}$ onto $\widehat{\mathbb{R}}$, and hence $\tau_j^{-1}(\gamma) = \gamma$. This equality is an equality of sets. The orientation can be reversed. Moreover $\tau_j(\Omega_+ \cup \Omega_-) = \Omega_+ \cup \Omega_-$. The last statement in the theorem follows because we can compute the direction of $\tau_j^{-1}(\gamma)$ by the order of the image points $\tau_j^{-1}(0), \tau_j^{-1}(\zeta), \tau_j^{-1}(1), \tau_j^{-1}(\infty)$.

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