Loewner energy via renormalization of Brownian loop measure

Yilin Wang (ETH Zürich)

MIT, November, 2018

[arXiv: 1810.04578]

[1802.01999]
[1710.04959] (Joint with Steffen Rohde)
[1601.05297]

Outline

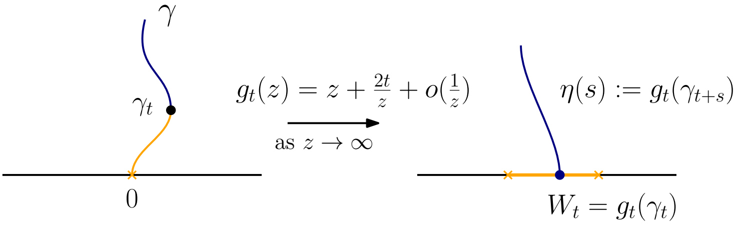
Introduction

- **Loewner energy** is a non-negative quantity associated to chords in (D, a, b) or Jordan curves (simple loops) in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, that measures the deviation of the chord from being the hyperbolic geodesic, or the loop from being round.
- It was introduced to describe the large deviation behavior of chordal SLE_κ as $\kappa \to 0$. SLEs are the random fractal non self-intersecting curves introduced by Oded Schramm in 1999, which successfully describe the scaling limit of interfaces in many statistical mechanics lattice models.
- This talk: we show an identity of the loop Loewner energy with a certain **Brownian loop measure**.

Outline

Chordal Loewner chains

Let γ be a simple chord in $\mathbb{H}$ from 0 to ∞. For $t \geq 0$, g_t is the conformal map $\mathbb{H} \setminus \gamma[0,t] \to \mathbb{H}$:



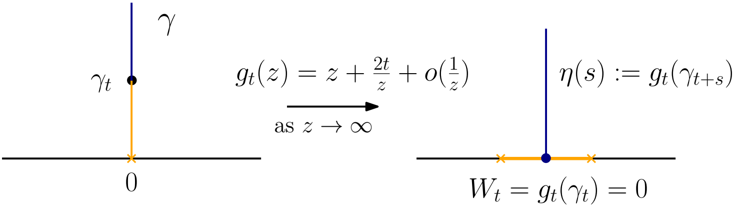
- γ is **capacity-parametrized** by $[0,\infty)$, *i.e. "size of* $\gamma[0,t]$ *seen from* ∞*"* $= t$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of γ.
- $W_0 = 0$.
- W is continuous.
- **Additivity:** the driving function of $\eta - W_t$ is $s \mapsto W(t+s) - W(t)$.
- **Scaling:** the driving function of $c\gamma$ is $t \mapsto cW(c^{-2}t)$.
- One can recover the curve γ from W using Loewner's differential equation.
- We say that γ is the **chordal Loewner chain** generated by W.

Chordal Loewner chain

- When the curve is driven by $W = \sqrt{\kappa} B$ where B is 1-d Brownian motion, the curve generated is the **Schramm-Loewner Evolution of parameter** κ (SLE_κ).
- In other domain (D, a, b), SLE is defined via a conformal transform $(\mathbb{H}, 0, \infty) \to (D, a, b)$.

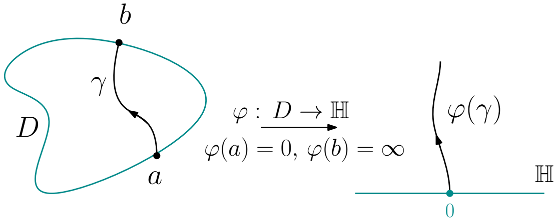
Chordal Loewner chain

- When the curve is driven by $W = \sqrt{\kappa} B$ where B is 1-d Brownian motion, the curve generated is the **Schramm-Loewner Evolution of parameter κ** (SLE$_\kappa$).
- In other domain (D, a, b), SLE is defined via a conformal transform $(\mathbb{H}, 0, \infty) \to (D, a, b)$.
- If $W \equiv 0$, then $\gamma = i\mathbb{R}_+$.



The chordal Loewner energy

$D \subset \mathbb{C}$ a simply connected domain, a, b are two boundary points of D.



Definition: Loewner energy

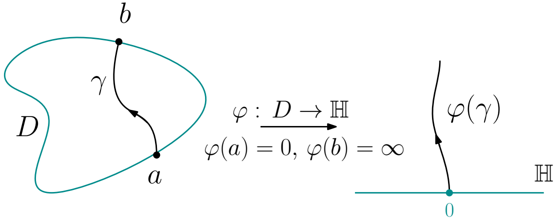
We define the **Loewner energy of a simple chord γ in** (D, a, b) to be

$$I_{D,a,b}(\gamma) := I_{\mathbb{H},0,\infty}(\varphi(\gamma)) := I(W) := \frac{1}{2}\int_0^\infty W'(t)^2 dt$$

where W is the driving function of $\varphi(\gamma)$ if W is absolutely continuous, otherwise the energy is defined to be ∞.

The chordal Loewner energy

$D \subset \mathbb{C}$ a simply connected domain, a, b are two boundary points of D.



- The Loewner energy is well-defined in (D, a, b) since for $c > 0$,

$$I_{\mathbb{H},0,\infty}(\gamma) = I_{\mathbb{H},0,\infty}(c\gamma).$$

- $I_{D,a,b}(\gamma) = 0$ *iff* γ is the hyperbolic geodesic connecting a and b.
- $I_{D,a,b}(\gamma) < \infty$, then γ is rectifiable *[Friz & Shekhar, 2015]*.

SLE_κ vs. Loewner energy

The Dirichlet energy $I(W)$ is the action functional of Brownian motion. Intuitively, the "Brownian path has the distribution on $C^0(\mathbb{R}_+, \mathbb{R})$"

$$\text{"}f(W) \propto \exp(-I(W))\mathcal{D}W.\text{"}$$

However, $I(B) = \infty$ *with probability* 1.

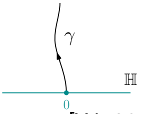
The Schilder's theorem states that $I(W)$ is also the **large deviation rate function** for Brownian motion $\sqrt{\kappa}B$ as $\kappa \to 0$. Loosely speaking,

$$\text{"}\mathrm{P}(\sqrt{\kappa}B \text{ stays close to } W) \approx \exp\left(-\frac{I(W)}{\kappa}\right).\text{"}$$

It should imply that the Loewner energy is the **large deviation rate function** of SLE_κ:

$$\text{"}\mathrm{P}(\text{SLE}_\kappa \text{ stays close to } \gamma) \approx \exp\left(-\frac{I(\gamma)}{\kappa}\right).\text{"} \quad (1)$$

The claim (1) *is made precise in [W. 2016].*



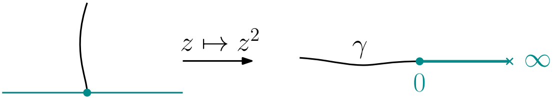
Using this interpretation we proved the **reversibility** of Loewner energy [W. 2016]:

$$I_{D,a,b}(\gamma) = I_{D,b,a}(\gamma).$$

Loewner loop energy

The loop energy generalizes the chordal energy:

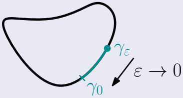
$$I_{\mathbb{C}\setminus\mathbb{R}_+,0,\infty}(\gamma) = I^L(\gamma \cup \mathbb{R}_+, \infty).$$



Definition (Rohde, W., 2017)

We define the **Loewner energy of a simple loop** $\gamma : [0,1] \mapsto \hat{\mathbb{C}}$ rooted at $\gamma_0 = \gamma_1$ to be

$$I^L(\gamma, \gamma_0) := \lim_{\varepsilon \to 0} I_{\hat{\mathbb{C}}\setminus\gamma[0,\varepsilon],\gamma_\varepsilon,\gamma_0}(\gamma[\varepsilon, 1]).$$



$I^L(\gamma, \gamma_0) = 0$ if and only if γ is a (round) circle.

Alternatively, $I^L(\gamma, \gamma_0) = \int_{-\infty}^{\infty} W'(t)^2/2dt$.

Loewner loop energy

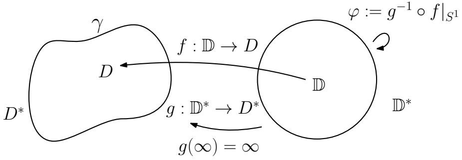
Theorem (Rohde, W. 2017)

The Loewner loop energy is **independent** of the parametrization of the loop. Moreover, $I^L(\gamma) < \infty$ implies that γ is quasicircle.

$\Longrightarrow$ I^L is invariant under Möbius transformations on the set of **free loops**.
$\Longrightarrow$ The loop setting is more natural than the chordal setting.

- The proof is based on the reversibility of the chordal energy.

Loewner Energy vs. Weil-Petersson Class



Theorem (W. 2018)

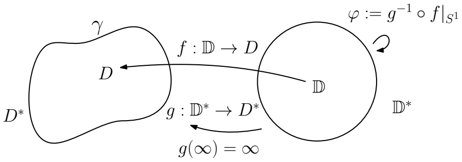
A bounded simple loop γ in $\hat{\mathbb{C}}$ has finite Loewner energy *if and only if* $\varphi \in \mathrm{WP}(S^1) \subset \mathrm{QS}(S^1)$. Moreover,

$$I^L(\gamma) = \frac{1}{\pi} \int_{\mathbb{D}} \left| \frac{f''}{f'}(z) \right|^2 \mathrm{d}z^2 + \frac{1}{\pi} \int_{\mathbb{D}^*} \left| \frac{g''}{g'}(z) \right|^2 \mathrm{d}z^2 + 4 \log \left| \frac{f'(0)}{g'(\infty)} \right|$$

is Kähler potential of the Weil-Petersson metric on the Weil-Petersson class.

- The root-invariance (and also the reversibility) of the loop energy follows immediately.

Loewner Energy vs. Weil-Petersson Class



Theorem (W. 2018)

A bounded simple loop γ in $\hat{\mathbb{C}}$ has finite Loewner energy *if and only if* $\varphi \in \mathrm{WP}(S^1) \subset \mathrm{QS}(S^1)$. Moreover,

$$I^L(\gamma) = \frac{1}{\pi}\int_{\mathbb{D}} \left|\frac{f''}{f'}(z)\right|^2 \mathrm{d}z^2 + \frac{1}{\pi}\int_{\mathbb{D}^*} \left|\frac{g''}{g'}(z)\right|^2 \mathrm{d}z^2 + 4\log\left|\frac{f'(0)}{g'(\infty)}\right|$$

is Kähler potential of the Weil-Petersson metric on the Weil-Petersson class.

- The root-invariance (and also the reversibility) of the loop energy follows immediately.
- Weil-Petersson metric is the unique homogeneous Kähler metric on $\mathrm{Diff}(S^1)/\mathrm{M\ddot{o}b}(S^1)$ and WP-class is its completion under the WP-metric [Cui].
- The Kähler potential on the right-hand side was derived by Takhtajan and Teo.

Characterizations of the WP-Class (an incomplete list)

[Nag, Verjovsky, Sullivan, Cui, Taktajan, Teo, Shen, etc.] The welding function φ is in Weil-Petersson class if one of the following equivalent conditions holds:

- $\int_{\mathbb{D}} |\nabla \operatorname{Re}(\log f'(z))|^2 \, \mathrm{d}z^2 = \int_{\mathbb{D}} |f''(z)/f'(z)|^2 \, \mathrm{d}z^2 < \infty$;
- $\int_{\mathbb{D}^*} |g''(z)/g'(z)|^2 \, \mathrm{d}z^2 < \infty$;
- $\int_{\mathbb{D}} |\mathcal{S}(f)|^2 \rho^{-1}(z) \, \mathrm{d}z^2 < \infty$;
- $\int_{\mathbb{D}^*} |\mathcal{S}(g)|^2 \rho^{-1}(z) \, \mathrm{d}z^2 < \infty$;
- φ has quasiconformal extension to $\mathbb{D}$, whose complex dilation $\mu = \partial_{\bar{z}}\varphi / \partial_z \varphi$ satisfies
$$\int_{\mathbb{D}} |\mu(z)|^2 \rho(z) \, \mathrm{d}z^2 < \infty;$$
- φ is absolutely continuous with respect to arc-length measure, such that $\log \varphi'$ belongs to the Sobolev space $H^{1/2}(S^1)$;
- Grunsky operator associated to f or g is Hilbert-Schmidt,

where $\rho(z) \, \mathrm{d}z^2 = 1/(1-|z|^2)^2 \, \mathrm{d}z^2$ is the hyperbolic metric on $\mathbb{D}$ or $\mathbb{D}^*$ and

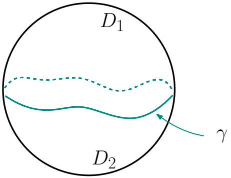
$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is the Schwarzian derivative of f.

Outline

The functional $\mathcal{H}$

- $g_0(z) = \frac{4}{(1+|z|^2)^2}\,\mathrm{d}z^2$ denote the spherical metric;
- $g = e^{2\varphi} g_0$ be a metric conformally equivalent to g_0;
- γ a C^∞ **smooth** simple loop in $\mathbb{C} \cup \{\infty\} \simeq S^2$;
- D_1 and D_2 two connected components $S^2 \backslash \gamma$;
- $\Delta_g(D_i)$ the positive Laplace-Beltrami operator with Dirichlet boundary condition on D_i.



Definition

Let $\det_\zeta$ be the ζ-regularized determinant, we introduce

$$\mathcal{H}(\gamma, g) := \log \det'_\zeta \Delta_g(S^2) - \log \mathrm{Area}_g(S^2) - \log \det_\zeta \Delta_g(D_1) - \log \det_\zeta \Delta_g(D_2).$$

Zeta-regularizated determinants

- $\Delta_g(S^2)$ is non-negative, essentially self-adjoint for the L^2 product.
- The spectrum is
$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \cdots$$
- Define the Zeta-function
$$\zeta_\Delta(s) := \sum_{i \ge 1} \lambda_i^{-s},$$
it can be analytically continued to a neighborhood of 0.
- Define (following Ray & Singer 1976)
$$\log \det{}'_\zeta(\Delta_g(S^2)) := -\zeta'_\Delta(0)$$
$$\text{“} = \sum_{i\ge 1} \log(\lambda_i)\lambda_i^{-s}|_{s=0} = \log(\prod_{i\ge 1} \lambda_i).\text{”}$$
- The Zeta-regularization of determinants has been used by physicists to perform Feynman path integrals, and is also important in Polyakov's quantum bosonic string theory.

Loewner Energy vs. Determinants

Recall $\mathcal{H}(\gamma, g) = \log \det_\zeta' \Delta_g(S^2) - \log \mathrm{Area}_g(S^2) - \log \det_\zeta \Delta_g(D_1) - \log \det_\zeta \Delta_g(D_2)$.

Theorem (W., 2018)

If $g = e^{2\varphi} g_0$ is a metric conformally equivalent to the spherical metric g_0 on S^2, then:

1. $\mathcal{H}(\cdot, g) = \mathcal{H}(\cdot, g_0)$
2. Circles minimize $\mathcal{H}(\cdot, g)$ among all C^∞ smooth Jordan curves.
3. Let γ be a smooth Jordan curve on S^2. We have the identity

$$\begin{aligned} I^L(\gamma, \gamma(0)) &= 12\mathcal{H}(\gamma, g) - 12\mathcal{H}(S^1, g) \\ &= 12 \log \frac{\det_\zeta(\Delta_g(\mathbb{D}_1))\det_\zeta(\Delta_g(\mathbb{D}_2))}{\det_\zeta(\Delta_g(D_1))\det_\zeta(\Delta_g(D_2))}, \end{aligned}$$

   where $\mathbb{D}_1$ and $\mathbb{D}_2$ are two connected components of the complement of S^1.

The above identity gives again the parametrization independence of the Loewner loop energy for smooth loops.

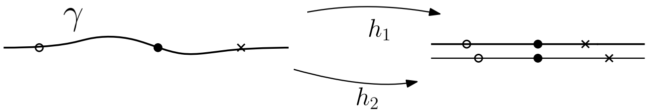
Proof of the identity (sketch)

$$I^L(\gamma, \gamma(0)) = 12 \log \frac{\det_\zeta(\Delta_{\mathbb{D}_1,g})\det_\zeta(\Delta_{\mathbb{D}_2,g})}{\det_\zeta(\Delta_{D_1,g})\det_\zeta(\Delta_{D_2,g})}$$

- When γ passes through ∞, we show

$$I^L(\gamma, \infty) = \frac{1}{\pi}\left(\int_{\mathbb{C}\setminus\gamma} \left|\nabla(\mathrm{Re}\log h'(z))\right|^2 \mathrm{d}z^2\right) = \frac{1}{\pi}\left(\int_{\mathbb{C}\setminus\gamma} \left|\frac{h''}{h'}\right|^2 \mathrm{d}z^2\right),$$

  where h maps conformally the complement of γ to two half-planes and fixes ∞.



  It is the large deviation analog of the quantum zipper by S. Sheffield. The proof is purely deterministic.

- Use the **Polyakov-Alvarez conformal anomaly formula** to compare determinants of Laplacians. □

Outline

Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let $x \in M$, $t > 0$, consider the sub-probability measure $\mathbb{W}_x^t$ on the path of Brownian motion (diffusion generated by $-\Delta_M$) on M started from x on the time interval $[0, t]$, killed if it hits the boundary of M.

Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let $x \in M$, $t > 0$, consider the sub-probability measure $\mathbb{W}_x^t$ on the path of Brownian motion (diffusion generated by $-\Delta_M$) on M started from x on the time interval $[0, t]$, killed if it hits the boundary of M.
The measures $\mathbb{W}_{x \to y}^t$ on paths from x to y are obtained from the disintegration of $\mathbb{W}_x^t$ according to its endpoint y:

$$\mathbb{W}_x^t = \int_M \mathbb{W}_{x \to y}^t \,\mathrm{dvol}(y).$$

Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let $x \in M$, $t > 0$, consider the sub-probability measure $\mathbb{W}_x^t$ on the path of Brownian motion (diffusion generated by $-\Delta_M$) on M started from x on the time interval $[0, t]$, killed if it hits the boundary of M.
The measures $\mathbb{W}_{x \to y}^t$ on paths from x to y are obtained from the disintegration of $\mathbb{W}_x^t$ according to its endpoint y:

$$\mathbb{W}_x^t = \int_M \mathbb{W}_{x \to y}^t \, \mathrm{dvol}(y).$$

Define the **Brownian loop measure** on M:

$$\mu_M^{loop} := \int_0^\infty \frac{\mathrm{d}t}{t} \int_M \mathbb{W}_{x \to x}^t \, \mathrm{dvol}(x).$$

In particular, the total mass of $\mathbb{W}_{x \to x}^t$ is $p_t(x, x)$, the heat kernel with Dirichlet boundary condition evaluated at (x, x).

Brownian loop measure

Introduced by Greg Lawler and Wendelin Werner.

[Following J. Dubédat] Let $x \in M$, $t > 0$, consider the sub-probability measure $\mathbb{W}_x^t$ on the path of Brownian motion (diffusion generated by $-\Delta_M$) on M started from x on the time interval $[0, t]$, killed if it hits the boundary of M.
The measures $\mathbb{W}_{x \to y}^t$ on paths from x to y are obtained from the disintegration of $\mathbb{W}_x^t$ according to its endpoint y:

$$\mathbb{W}_x^t = \int_M \mathbb{W}_{x \to y}^t \, \mathrm{dvol}(y).$$

Define the **Brownian loop measure** on M:

$$\mu_M^{loop} := \int_0^\infty \frac{\mathrm{d}t}{t} \int_M \mathbb{W}_{x \to x}^t \, \mathrm{dvol}(x).$$

In particular, the total mass of $\mathbb{W}_{x \to x}^t$ is $p_t(x, x)$, the heat kernel with Dirichlet boundary condition evaluated at (x, x).
We consider μ_M^{loop} as measure on unrooted Brownian loops by forgetting the starting point.

Property of Brownian loop measure

The Brownian loop measure satisfies the following two remarkable properties

- *(Restriction property)* If $M' \subset M$, then $d\mu_{M'}^{loop}(\delta) = 1_{\delta \in M'} d\mu_M^{loop}(\delta)$.
- *(Conformal invariance)* On the surfaces $M_1 = (M, g)$ and $M_2 = (M, e^{2\sigma} g)$ be two conformally equivalent Riemann surface, where $\sigma \in C^\infty(M, \mathbb{R})$, then

$$\mu_{M_1}^{loop} = \mu_{M_2}^{loop}.$$

Loop measure vs. determinant of Laplacian

If we compute formally, the total mass of μ_M^{loop} is given by

$$\text{“}\left|\mu_M^{loop}\right| = \int_0^\infty \frac{\mathrm{d}t}{t}\int_M p_t(x,x)\,\mathrm{dvol}(x) = \int_0^\infty t^{-1}\mathrm{Tr}\left(e^{-t\Delta}\right)\,\mathrm{d}t.\text{”}$$

On the other hand, $1/\Gamma(s)$ is analytic and has the expansion near 0 as

$$1/\Gamma(s) = s + O(s^2).$$

Therefore for any analytic function f in a neighborhood of 0,

$$\left(\frac{f(s)}{\Gamma(s)}\right)'\Bigg|_{s=0} = f(0).$$

Take formally $f(s) = \int_0^\infty t^{s-1}\mathrm{Tr}(e^{-t\Delta})\,\mathrm{d}t$, we have

$$\text{“}-\log\det{}_\zeta(\Delta) = \zeta_\Delta'(0) = \int_0^\infty t^{-1}\mathrm{Tr}(e^{-t\Delta})\,\mathrm{d}t = \left|\mu_M^{loop}\right|\text{”}. \qquad (2)$$

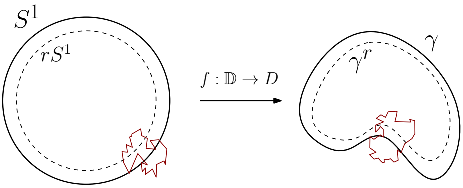
Loop measure vs. Loewner energy (heuristic)

The determinant expression of Loewner energy suggests that we have formally

$$
\begin{aligned}
\text{``}\frac{1}{12} I^L(\gamma) &= \log \frac{\det_\zeta(\Delta_{\mathbb{D}_1,g})\det_\zeta(\Delta_{\mathbb{D}_2,g})}{\det_\zeta(\Delta_{D_1,g})\det_\zeta(\Delta_{D_2,g})} \\
&= \left|\mu^{loop}_{D_1}\right| + \left|\mu^{loop}_{D_2}\right| - \left|\mu^{loop}_{\mathbb{D}_1}\right| - \left|\mu^{loop}_{\mathbb{D}_2}\right| + \left|\mu^{loop}_{S^2}\right| - \left|\mu^{loop}_{S^2}\right| \\
&= \mu^{loop}_{S^2}(\{\delta; \delta \cap S^1 \neq \emptyset\}) - \mu^{loop}_{S^2}(\{\delta; \delta \cap \gamma \neq \emptyset\}).\text{''}
\end{aligned}
$$

However, both terms diverge due to the small and large Brownian loops (from the conformal invariance).

Loop measure vs. Loewner energy

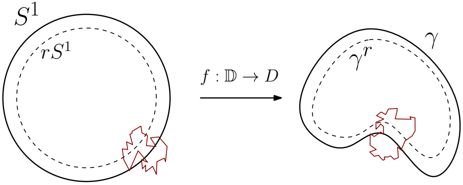


For a Brownian loop $\delta \subset D$, where $D \subset \mathbb{D}$ is simply connected, we denote δ^{out} its outer boundary (therefore of $\text{SLE}_{8/3}$ type).
Let $A, B \subset \mathbb{C}$ be disjoint compact sets,

$$\mathcal{W}(A, B; D) := \left|\mu^{loop}\{\delta \subset D; \delta^{out} \text{ intersects both } A \text{ and } B\}\right| < \infty.$$

Introduced by W. Werner.

Loop measure vs. Loewner energy



For a Brownian loop $\delta \subset D$, where $D \subset \mathbb{D}$ is simply connected, we denote δ^{out} its outer boundary (therefore of $\mathrm{SLE}_{8/3}$ type).
Let $A, B \subset \mathbb{C}$ be disjoint compact sets,

$$\mathcal{W}(A, B; D) := \left|\mu^{loop}\{\delta \subset D; \delta^{out} \text{ intersects both } A \text{ and } B\}\right| < \infty.$$

Introduced by W. Werner.

Theorem (W., 2018)

For all Jordan curve γ (no regularity assumption),

$$\frac{1}{12} I^L(\gamma) = \lim_{r \to 1} \mathcal{W}(S^1, rS^1; \mathbb{C}) - \mathcal{W}(\gamma, \gamma^r; \mathbb{C}).$$

Proof: Chordal Conformal restriction

Lemma 1: Chordal Conformal restriction

Let (D, a, b) and (D', a, b) be two simply connected domains in $\mathbb{C}$ coinciding in a neighborhood of a and b, and γ a simple curve in both (D, a, b) and (D', a, b). Then we have

$$\begin{aligned} I_{D',a,b}(\gamma) - I_{D,a,b}(\gamma) =& I_{D,a,b}(\psi(\gamma)) - I_{D,a,b}(\gamma) \\ =& 3\log\left|\psi'(a)\psi'(b)\right| + 12\mathcal{W}(\gamma, D\backslash D'; D) - 12\mathcal{W}(\gamma, D'\backslash D; D'), \end{aligned}$$

where $\psi : D' \to D$ is a conformal map fixing a and b.

The SLE partition function

$$\mathcal{Z}^{\mathrm{SLE}_\kappa}_{(D,a,b)} = H_D(a,b)^\beta \det{}_\zeta(\Delta)^{-c/2}$$

where as $\kappa \to 0$,

$$b = \frac{6-\kappa}{2\kappa} \sim \frac{3}{\kappa}, \quad c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa} \sim -\frac{24}{\kappa}.$$

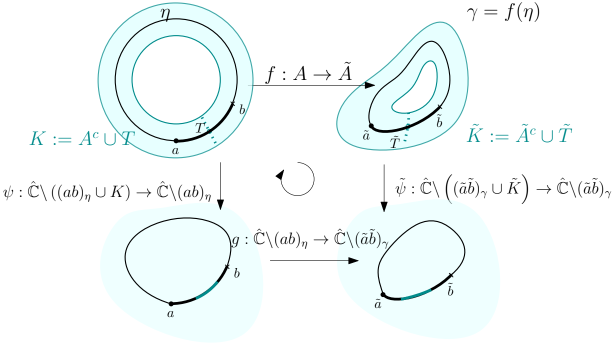
The Energy $=$ "$-\kappa \log(\cdot)$"

Lemma 2: Loop conformal restriction

If η is a Jordan curve with finite energy and $\gamma = f(\eta)$, where $f : A \to \tilde{A}$ is conformal on a neighborhood A of η, then

$$I^L(\gamma) - I^L(\eta) = 12\mathcal{W}(\eta, A^c; \mathbb{C}) - 12\mathcal{W}(\gamma, \tilde{A}^c; \mathbb{C}).$$

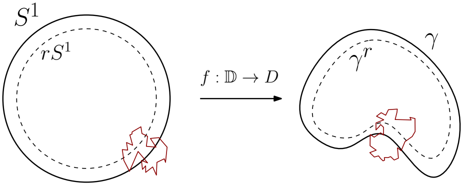
Proof of Lemma 2:



Let $b \to a$

Proof: Equipotentials

When $\eta = rS^1$, $\gamma^r = f(rS^1)$ is the equipotential, and $A = \mathbb{D}$.



We deduce

$$I^L(\gamma^r) = 12\mathcal{W}(rS^1, S^1; \mathbb{C}) - 12\mathcal{W}(\gamma^r, \gamma; \mathbb{C}).$$

Lemma 3

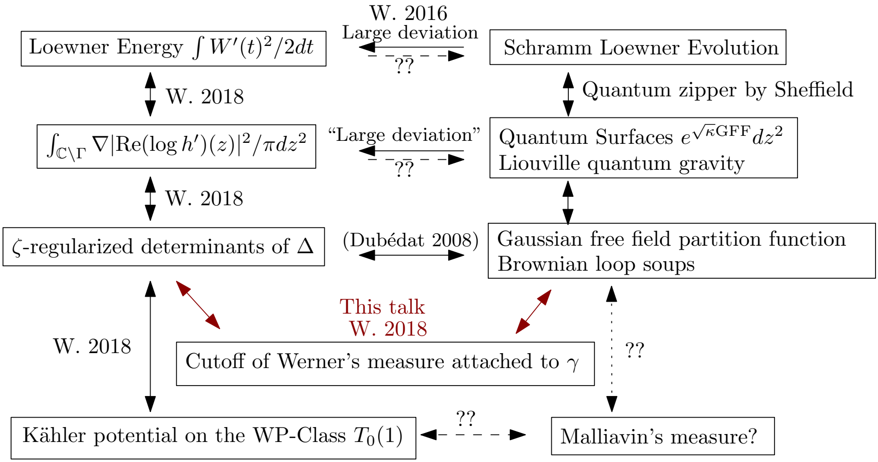
We have: $I^L(\gamma^r) \xrightarrow{r \to 1} I^L(\gamma)$.

□

In fact, $r \mapsto I^L(\gamma^r)$ is increasing if $I^L(\gamma) > 0$, namely when γ is not a circle.
[Viklund, W. 2018+]

Outline

Action functionals vs. Random objects



What's next?

- What is the random model naturally associated to the WP-class?
  $\implies$ An intrinsic description of SLE loop ($\kappa \leq 4$)? $\implies$ Reversibility?
- In which space does the random welding belong to?
- How is the Kähler structure on the WP-class encoded in the Loewner's driving function.
- Topological group structure on the WP-class $\implies$ meaning in the Loewner setting?
- Isometric welding of two finite energy domains still has finite Loewner energy. Coupling identity between field Dirichlet energies and the Loewner energy (*work in progress* with Fredrik Viklund).
- The (stochastic) gradient flow of the Loewner energy on loops?
- Generalization to other Riemann surfaces?

Thank you for your attention!