BROWNIAN MOTION ON DIFF (S^1)

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Quantization days

QFT ORIGINS

M. J. Bowick, S. G.Rajeev, *String theory as the Kähler geometry of loop space*, Phys. Rev. Lett., 1987.

Kähler geometry of the loops on space-time (Minkowski space) describes bosonic string theory

Goal find a geometric and nonperturbative formulation similar to the Riemannian geometry of space-time in general relativity used to describe gravity

Geometry $\Longrightarrow$ a nonlinear equation of motion for the field similarly to Einstein's equations, vanishing of a generalization of the Ricci tensor

Virasoro group: physical string amplitudes are independent of parametrization, invariant under the group of diffeomorphisms of the circle $\mathbf{Diff}\,(S^1)$

S^1 pure rotations do not change the complex structure

$\mathcal{M}$ — the space of all complex structures of the loops in the Minkowski space transformed into each other by $\mathbf{Diff}\left(S^1\right)$ are identified with $\mathcal{M} = \mathbf{Diff}\left(S^1\right)/S^1$

This manifold thus plays a crucial role in string theory: the space $\mathcal{M}$ parameterizes vacuum states for Faddeev-Popov ghosts in the string field theory

B. Zumino, *The geometry of the Virasoro group for physicists*, the NATO Advanced Studies Institute Summer School on Particle Physics, the NATO, 1987

D. Freed, *The geometry of loop groups*, J. Differential geometry, 1988

CONNECTION TO THE UNIVERSAL TEICHMÜLLER SPACE

$\mathcal{M}$	as a dense complex submanifold of the universal Teichmüller space $T(1)$ of compact Riemann spaces of genus $g \geqslant 1$
Siegel disk	a holomorphic map of $T(1)$ into the infinite-dimensional Siegel disk (Nag-Sullivan)
$T(1)$	an infinite-dimensional complex Banach manifold
$\mathcal{M}$	an infinite-dimensional complex Fréchet manifold with a natural Kähler metric

The Ricci tensor for the space $\mathcal{M}$ is related to the problem of constructing a reparameterization-invariant vacuum for Faddeev-Popov ghosts.

D.K.Hong, and S.G. Rajeev, *Universal Teichmüller space and* $\mathbf{Diff}\left(S^1\right)/S^1$, Comm. Math. Phys., 1991

M. J. Bowick, and S.G. Rajeev, *The holomorphic geometry of closed bosonic string theory and* $\mathbf{Diff}\left(S^1\right)/S^1$, Nuclear Phys. B. 1987

A. A. Kirillov, 1987

A. A. Kirillov and D. V. Yur'ev, *Kähler geometry of the infinite-dimensional homogeneous space* $\mathcal{M} = \mathbf{Diff}_+(S^1)/\mathbf{Rot}(S^1)$, 1987

A. D. Popov, A. G. Sergeev, *Symplectic twistors and geometric quantization of strings*, 1994

RIEMANNIAN GEOMETRY OF DIFF $\left(S^1\right)/S^1$

$\mathrm{Diff}\left(S^1\right)$ smooth orientation-preserving diffeomorphisms of S^1

$\mathrm{diff}\left(S^1\right)$ $\varphi\left(\theta\right)\frac{d}{d\theta}$, where φ is a smooth periodic function

$\{f_k, g_k\}$ $f_k = \sin(k\theta)\frac{d}{dt}$, $g_k = \cos(k\theta)\frac{d}{d\theta}$
orthogonal basis of the Lie algebra $\mathrm{diff}\left(S^1\right)$

$\mathrm{diff}_0\left(S^1\right)$ $\left\{\varphi(\theta)\frac{d}{d\theta} \in \mathrm{diff}\left(S^1\right), \int\limits_0^{2\pi} \varphi\left(\theta\right) d\theta = 0\right\}$

J $Jf_k = f_k$, $Jg_k = -g_k$, an almost complex structure on $\mathrm{Diff}\left(S^1\right)/S^1$

$\omega_{c,h}(f,g)$ $\quad \int\limits_0^{2\pi} \left((2h - \frac{c}{12}) f'(\theta) - \frac{c}{12} f^{(3)}(\theta) \right) \frac{g(\theta) d\theta}{2\pi}$

a cocycle on $\operatorname{diff}\left(S^1\right)$

$B(f,g)$ $\quad \omega_{c,h}(f, Jg) = \omega_{c,h}(g, fg)$, an inner product on $\operatorname{diff}\left(S^1\right)$

∇ $\quad$ the covariant derivative determined by the inner product $B(f,g)$.

$$f = \sum_{k=1}^{\infty} (a_k f_k + b_k g_k) \in \operatorname{diff}_0\left(S^1\right)$$

$$\omega_{c,h}(f, Jf) = \sum_{k=1}^{\infty} \frac{1}{2} \left(hk + \frac{c}{12}\left(k^3 - k\right) \right) \left(a_k^2 + b_k^2\right)$$

$$\frac{f_k}{\alpha_k}, \frac{g_k}{\alpha_k} \quad k = 1, 2, \ldots$$

Theorem.

- The torsion of the almost complex structure $\boldsymbol{J}$ vanishes on $\mathbf{diff}_0\left(\boldsymbol{S}^1\right)$;

- if $\boldsymbol{Q}$ be the tensor field of type $(\mathbf{1},\mathbf{2})$ defined by

$$4\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{\nabla}_{\boldsymbol{J}\boldsymbol{y}}\boldsymbol{J})\boldsymbol{x} + \boldsymbol{J}((\boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{J})\boldsymbol{x}) + 2\boldsymbol{J}((\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{J})\boldsymbol{y}),$$

then $\widetilde{\boldsymbol{\nabla}}_{\boldsymbol{x}}\boldsymbol{y} = \boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{y} - \boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y})$ can be extended to a bilinear torsion-free connection on $\mathfrak{m}_{\mathbb{C}} = \mathbf{diff}_0\left(\boldsymbol{S}^1\right)_{\mathbb{C}}$.

The curvature tensor for $\boldsymbol{x},\boldsymbol{y} \in \mathbf{diff}\left(\boldsymbol{S}^1\right)_{\mathbb{C}}$

$$\widetilde{\boldsymbol{R}}_{\boldsymbol{x}\boldsymbol{y}} = \widetilde{\boldsymbol{\nabla}}_{\boldsymbol{x}}\tilde{\boldsymbol{\nabla}}_{\boldsymbol{y}} - \widetilde{\boldsymbol{\nabla}}_{\boldsymbol{y}}\widetilde{\boldsymbol{\nabla}}_{\boldsymbol{x}} - \widetilde{\boldsymbol{\nabla}}_{[\boldsymbol{x},\boldsymbol{y}]_{\mathfrak{m}_{\mathbb{C}}}} - \mathbf{ad}([\boldsymbol{x},\boldsymbol{y}]_{\mathbb{C}\boldsymbol{f}_0});$$

The Ricci tensor $\mathrm{Ric}(\boldsymbol{x},\boldsymbol{y})$ is the trace of the map $\boldsymbol{z} \mapsto \widetilde{\boldsymbol{R}}_{\boldsymbol{z}\boldsymbol{x}}\boldsymbol{y}$.

An orthonormal basis of $\mathfrak{m}_{\mathbb{C}}$

$L_n = f_n + ig_n, n > 0$,

$L_n = f_{-n} - ig_{-n}, n < 0$.

Theorem. The only non-zero components of the Ricci tensor are

$$\mathbf{Ric}(\mathbf{L_n}, \mathbf{L_{-n}}) = -\frac{\mathbf{13n^3 - n}}{\mathbf{6}}, \ \mathbf{n} \in \mathbb{Z}, \mathbf{n} \neq \mathbf{0}.$$

- The parameters c and h do not appear in the Ricci curvature

- the Ricci tensor for the original covariant derivative ∇ diverges

- $\widetilde{\nabla}$ is the Levi-Civita covariant derivative, that is, it is metric compatible and torsion-free, but it is not a Hilbert-Schmidt operator

M. Gordina, P.Lescot 2006

HEAT KERNEL MEASURES ON DIFF $(S^1)/S^1$

(H.Airault, P. Malliavin, S. Fang, A. Thalmaier, M. Wu)

• Represent $\mathbf{Diff}\,(S^1)$ as a certain space of univalent functions, and then as the infinite-dimensional group Sp_{HS}

• construct the Brownian motion on Sp_{HS} as the solution to the stochastic differential equation

$$dG_t = \frac{1}{2}\sum_{j=1}^{\infty}\xi_j^2 G_t dt + dW_t G_t$$

with $Qe_{ij} = r^{i+j}e_{ij}$, $0 < r < 1$.

• this choice of renormalization forces the Brownian motion G_t to live in the group Sp_{HS}, but it also changes the geometry of the group H.Airault, P. Malliavin

• $H^{3/2}$ metric forces the Brownian motion to live in Hölderian homeomorphisms of S^1 P. Malliavin, S. Fang

$$\alpha_k = hk + \frac{c}{12}\left(k^3 - k\right) \sim k^3$$

• a stronger metric forces the Brownian motion to live in $\mathbf{Diff}\left(S^1\right)/S^1$
M. Wu 2011
rapidly decreasing scaling $\alpha(-k) = \alpha(k)$

$$\lim_{k \to \pm\infty} |k|^m \alpha_k = 0 \text{ for any } m \in \mathbb{N}.$$

HILBERT-SCHMIDT GROUPS

$B(H)$	bounded linear operators on a complex Hilbert space H.
$G = GL(H)$	invertible elements of $B(H)$.
Q	a bounded linear symmetric nonnegative operator on HS.
HS	Hilbert-Schmidt operators on H with the inner product $(A, B)_{HS} = TrB^{*}A$.
$\mathfrak{g} = \mathfrak{g}_{CM} \subseteq HS$	an infinite-dimensional Lie algebra with a Hermitian inner product $(\cdot, \cdot)$, $\lvert A\rvert_{\mathfrak{g}} = \lvert Q^{-1/2}A\rvert_{HS}$.

$G_{CM} \subseteq GL(H)$ Cameron-Martin group $\{x \in GL(H), d(x, I) < \infty\}$

$d(x, y)$ the Riemannian distance induced by $|\cdot|$

$$d(x,y) = \inf_{\substack{g(0)=x \\ g(1)=y}} \int_0^1 |g(s)^{-1} g'(s)|_{\mathfrak{g}} ds$$

HS as infinite matrices

HS= matrices $\{a_{ij}\}$ such that $\sum_{i,j} |a_{ij}|^2 < \infty$.

$$e_{ij} = \begin{array}{cc} & j \\ i & \begin{pmatrix} \cdots\cdots\cdots \\ \ldots\ 1\ \ldots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{pmatrix} \end{array}, \quad Qe_{ij} = \lambda_{ij} e_{ij},\ \lambda_{ij} \geqslant 0.$$

$$\xi_{ij} = \sqrt{\lambda_{ij}}\, e_{ij}.$$

Q is a trace class operator $\Longleftrightarrow \sum_{i,j} \lambda_{ij} < \infty$.

For example, $\lambda_{ij} = r^{i+j}$, $0<r<1$.

(i) The Hilbert-Schmidt general group

$\boldsymbol{GL_{HS}{=}GL(H)\bigcap(I+HS)}$,

Lie algebra $\mathfrak{gl}_{HS}{=}\boldsymbol{HS}$, $\mathfrak{g}_{CM}=\boldsymbol{Q^{1/2}HS}$.

(ii) The Hilbert-Schmidt orthogonal group $\boldsymbol{SO_{HS}}$ is the connected component of

$\{\boldsymbol{B:B-I\in HS,\ \ B^TB{=}BB^T{=}I}\}$.

Lie algebra $\mathfrak{so}_{HS}{=}\{\boldsymbol{A:A\in HS,\ \ A^T{=}-A}\}$,

$\mathfrak{g}_{CM}=\boldsymbol{Q^{1/2}}\mathfrak{so}_{HS}$.

(iii) The Hilbert-Schmidt symplectic group

$Sp_{HS} = \{X : X - I \in HS, \;\; X^T J X = J\}$, where

$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

Lie algebra $\mathfrak{sp}_{HS} = \{X : X \in HS, \;\; X^T J + J X = 0\}$,

$\mathfrak{g}_{CM} = Q^{1/2} \mathfrak{sp}_{HS}$.

BROWNIAN MOTION:
MANG WU'S CONSTRUCTION

$$\frac{\sin(k\theta)}{\alpha_k}, \frac{\cos(k\theta)}{\alpha_k} \quad k = 1, 2, \ldots$$

$\alpha_k = 1$ orthonormal basis for $L^2(S^1)$

α_k^2 $\frac{1}{2}\left(hk + \frac{c}{12}\left(k^3 - k\right)\right)$ orthonormal basis on $\mathrm{diff}\left(S^1\right)$ with respect to the inner product $\omega_{c,h}(f, Jg) = \omega_{c,h}(g, fg)$

$\mathcal{H}_\alpha$ Hilbert space with the orthonormal basis $\frac{\sin(k\theta)}{\alpha_k}, \frac{\cos(k\theta)}{\alpha_k}$

α determines the covariance of the Brownian motion W_t

$\mathbf{diff}\,(S^1)$ Fréchet space

$$\mathbf{diff}\left(S^1\right) = \bigcup_{\alpha} \mathcal{H}_{\alpha} = \bigcap_{n} W^{n,2}\left(S^1\right) = \bigcap_{n} W^{n,\infty}\left(S^1\right)$$

Stochastic differential equation on $\mathbf{Diff}\,(S^1)$ to define a Brownian motion in the group $\mathbf{Diff}\,(S^1)$

Composition is the group operation on $\mathbf{Diff}\,(S^1)$ $\Longrightarrow$ two actions of the group $\mathbf{Diff}\,(S^1)$ on the Lie algebra $\mathbf{diff}\,(S^1)$

$$\sigma(g)\,f = f \circ g \quad \text{or} \quad \sigma(g)\,f = g' \cdot f$$

$$f \in \mathbf{diff}\left(S^1\right), g \in \mathbf{Diff}\left(S^1\right)$$

$\mathbf{Diff}\left(S^1\right) \subseteq \mathbf{id} + \mathbf{diff}\left(S^1\right)$ affine space

$$\boldsymbol{dX_t = \sigma\left(\mathrm{id} + X_t\right) dW_t}$$

$\boldsymbol{\sigma}$ is locally Lipschitz

Theorem. (Mang Wu 2011) There exists a unique solution $\boldsymbol{X_t}$ with a.s. continuous sample paths. Furthermore, the solution is non-explosive and lives in $\mathbf{Diff}\left(S^1\right)$.

KÄHLERIAN STRUCTURE ON LOOP GROUPS

• Loop group $LG = \{g : [0,1] \to G, g(0) = g(1) = e\}$
$H_0 = \{h : [0,1] \to \mathfrak{g}, h(0) = h(1) = 0\}$ with $\|h'\|_{L^2} < \infty$

• the Riemannian structure on LG can be described in terms of a *good* basis
D. Freed, B.Driver

• an almost complex structure on LG A. Pressley, then the Riemannian tensor can be computed using a Kählerian metric on LG
I.Shigekawa, S. Taniguchi

• the Ricci is defined as $\mathbf{Ric} = \mathbf{dd}^* + \mathbf{d}^*\mathbf{d} - \nabla^*\nabla$.

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