NOTES ON THE 2D YANG-MILLS MASTER FIELD AND FREE PROBABILITY THEORY

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1. Motivation

The study of the large-N limit of Yang-Mills theory has attracted many attentions after 't Hooft's observation [tH] that the $1/N$-expansion of the non-Gaussian term in the perturbative method simplifies considerably, for the reason that only planar Feynmann graphs dominate.

In this note we exhibit the mathematically well-established $N \to \infty$ limit (the master field) of the Yang-Mills theory on the plane where the ratio N/β is fixed. In this regime, we show first that the Wilson loop observable is asymptotically deterministic (Section 4). In order to consider the holonomy along a reasonably large family of loops simultaneously, the limit field is described using the very natural framework of free probability introduced and studied by Voiculescu (see [Voi2] or Terry Tao's blog for a survey). Basic notions of free probability theory are gathered without proof in Section 5. Free probability has long been expected to play a central role in the master field and the first mathematically rigorous description of the master field using free probability is due to Anshelevich and Sengupta [AS] and Thierry Lévy [Le] independently (see section 6).

2. Preliminaries

We first recall some elements in the set-up. The Lie algebra $\mathfrak{u}(N)$ of $U(N)$ consist of skew-hermitian matrices, is equipped with the inner product

$$\langle \xi, \zeta \rangle = \operatorname{tr}(\xi^* \zeta) = -\operatorname{tr}(\xi\zeta).$$

It is viewed as the inner product on the tangent space at I. We define the Riemannian metric on the Lie group $U(N)$ by left-translation, namely for $g \in U(N)$, $\xi, \zeta \in \mathfrak{u}(N)$, $g\xi, g\zeta$ are in the tangent space of $U(N)$ at g, we define their inner product by

$$\langle g\xi, g\zeta \rangle_g := \langle \xi, \zeta \rangle = \langle \xi, \zeta \rangle_I$$

where the subscript indicates the base point of the tangent space.

Let $(T_a)_{a \le \dim[\mathfrak{u}(N)]}$ be an orthonormal basis of the $\mathbb{R}$-vector space $\mathfrak{u}(N)$, namely $\langle T_a, T_b \rangle = \delta_{a,b}$.

Lemma 2.1. *For any $N \times N$ matrices X and Y,*

$$\sum_{a}^{dim[\mathfrak{u}(N)]} \operatorname{tr}(XT_a)\operatorname{tr}(T_aY) = -\operatorname{tr}(XY); \tag{1}$$

$$\sum_{a}^{dim[\mathfrak{u}(N)]} T_a^2 = -NI. \tag{2}$$

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Proof. For X and Y in $\mathfrak{u}(N)$, (1) just follows from the Parseval identity. The general case follows from the fact that any arbitrary complex matrix X is a complex linear combination of skew-hermitian matrices $i(X+X^*)$ and $(X-X^*)$, and (1) is complex linear in X and Y.

Let E_{ij} be the matrix with entry 1 at (i,j) and 0 elsewhere. We apply (1),

$$\left(\sum_a T_a^2\right)_{ij} = \sum_a^{dim[\mathfrak{u}(N)]} \sum_{k=1}^N (T_a)_{ik}(T_a)_{kj} = \sum_{k=1}^N \sum_a^{dim[\mathfrak{u}(N)]} \operatorname{tr}(E_{ki}T_a)\operatorname{tr}(E_{jk}T_a)$$
$$= \sum_{k=1}^N -\operatorname{tr}(E_{jk}E_{ki}) = \sum_{k=1}^N -\operatorname{tr} E_{ji} = -N\delta_{i,j},$$

which proves (2). □

The Laplace-Beltrami operator (Laplacian) on the Riemannian manifold $U(N)$ is given by

$$\Delta = \sum_a^{dim[\mathfrak{u}(N)]} \partial_{T_a}^2,$$

where we write the same T_a for the left-invariant vector field $g \mapsto gT_a$. The heat kernel is the solution to

$$\frac{\partial \mathscr{Q}_t(x)}{\partial t} = \frac{1}{2}\Delta_x \mathscr{Q}_t(x), \quad \mathscr{Q}_0 = \delta_I.$$

The Brownian motion on $U(N)$ at time t starting from I has the density $\mathscr{Q}_t(x)$ (by definition) with respect to the unit Haar measure (denoted as dx), which is also the solution to Itô's equation

$$d\tilde{g}(t) = \tilde{g}(t)dW(t) - \frac{1}{2}\tilde{g}(t)(dW^*(t)dW(t)), \quad \tilde{g}(0) = I \tag{3}$$

where W is a standard Brownian motion on $\mathfrak{u}(N)$. We may check easily that $\tilde{g}(t) \in U(N)$. In fact, applying Itô's formula, we have

$$d(\tilde{g}(t)^*\tilde{g}(t)) = d(\tilde{g}(t)^*)\tilde{g}(t) + \tilde{g}(t)^* d(\tilde{g}(t))$$
$$= \left(-\tilde{g}(t)dW(t) - \frac{1}{2}\tilde{g}(t)dW^*(t)dW(t)\right)^* \tilde{g}(t)$$
$$+ \tilde{g}(t)^*\left(-\tilde{g}(t)dW(t) - \frac{1}{2}\tilde{g}(t)dW^*(t)dW(t)\right) + dW^*(t)dW(t)$$
$$= \left(-IdW(t)^* - \frac{N}{2}Idt\right) + \left(-IdW(t) - \frac{N}{2}Idt\right) + NIdt = 0$$

since $W(t)^* = -W(t)$. We have therefore $\tilde{g}(t)^*\tilde{g}(t) = \tilde{g}(0)^*\tilde{g}(0) = I$.

It is equivalent to define $\tilde{g}(t)$ by the path ordered product

$$\mathscr{P}\left(\exp\int_0^t dW\right) = \lim_{\varepsilon\to 0} \exp(W_\varepsilon - W_0)\exp(W_{2\varepsilon} - W_\varepsilon)\cdots\exp(W_{\lfloor t\varepsilon^{-1}\rfloor\varepsilon} - W_{\lfloor t\varepsilon^{-1}\rfloor\varepsilon-\varepsilon}).$$

3. U(N) 2D-Yang-Mills measure

We recall first the construction of the $U(N)$ Yang-Mills measure on $\mathbb{R}^2$. In particular, the holonomy h_l (the parallel transport along a loop l) has the law of the Brownian motion on $U(N)$ at time proportional to the area $|l|$ enclosed by the loop l. Notice that the description is specific to the continuum theory, which is due to the infinite divisibility of the parallel transport.

The Yang-Mills measure in $\mathbb{R}^2$ on the principal $U(N)$-connection is given by

$$d\mu^{YM}(A) = Z^{-1}\exp(-\beta\|f\|_2^2)df, \tag{4}$$

where $f = \partial_y A_x : \mathbb{R}^2 \to \mathfrak{u}(N)$ is (minus) the curvature for the connection $A = A_x dx + A_y dy$ with axial gauge fixing ($A_y \equiv 0$ and $A_x(x,0) \equiv 0$), and

$$\|f\|_2^2 = \int_{\mathbb{R}^2} \langle f(z), f(z) \rangle \, dz^2.$$

The way to make sense of (4) is to take

$$f = \frac{1}{\sqrt{2\beta}} \sum_a \alpha_a T_a,$$

where (α_a) is a family of independent white noise on $\mathbb{R}^2$. In particular, for a cross-vertical (horizontal like) path $c(t) = (t, y(t))$ from $[0,T]$ to $\mathbb{R}^2$, define

$$\begin{aligned} \tilde{M}(t) &:= \int_0^t A(c(s))dc(s) = \int_0^t A_x(c(s))ds = \int_0^t \int_0^{y(s)} \partial_y A_x(s,y) dy ds \\ &= \frac{1}{\sqrt{2\beta}} \sum_a T_a \int_0^t \int_0^{y(s)} \alpha_a(s,y) dy ds = \frac{1}{\sqrt{2\beta}} \sum_a T_a B_a(t), \end{aligned} \tag{5}$$

where (B_a) is an independent family of Brownian motions with clock the area $|c[0,t]|$ enclosed between the path $c[0,t]$, the x-axis, the vertical segments $[(0,0),(0,y(0))]$ and $[(t,0),(t,y(t))]$. Therefore $\tilde{M}$ is a time changed Brownian motion on $\mathfrak{u}(N)$.

The (stochastic version of the) parallel transport along the path c satisfies the Stratonovich stochastic differential equation

$$dh_{c[0,t]} = h_{c[0,t]} A(c(t))dc(t) = h_{c[0,t]} \circ d\tilde{M}(t), \quad h_{c[0]} = I$$

whose solution is the path ordered product

$$h_{c[0,t]} = \mathscr{P}\left(\exp \int_0^t d\tilde{M}\right) = \lim_{\varepsilon \to 0} \exp(M_\varepsilon - M_0) \exp(M_{2\varepsilon} - M_\varepsilon) \cdots \exp(M_{\lfloor t\varepsilon^{-1} \rfloor \varepsilon} - M_{\lfloor t\varepsilon^{-1} \rfloor \varepsilon - \varepsilon})$$

and has the law of the Brownian motion on $U(N)$ at time $|c[0,t]| / 2\beta$.

We define the parallel transport (the holonomy) along a basic loop $l = c_1 c_2 \cdots c_k$, which is a concatenation of vertical lines (along which the parallel transport is the identity) and cross-vertical paths by

$$h_l = h_{c_1} \cdots h_{c_k}.$$

The following two facts are not immediate from what is described above and we will take them for granted (or see Nina's notes which are to be extended in the future).

- When l is simple, h_l has the law of the Brownian motion on $U(N)$ at time $|l| / 2\beta$, where $|l|$ is again the area enclosed by l.
- If two loops have disjoint interior, then their holonomies are independent.

4. Expectation and variance of a Wilson loop variable

We will denote $\mathrm{tr}_N(\cdot) = \mathrm{tr}(\cdot)/N$ the normalized trace. In this section, we prove that the Wilson loop observable is asymptotically deterministic. It is illuminating to see first an alternative computation of $E(tr_N(h_l))$ than in the last lecture.

Theorem 4.1. *Let $S = |l|$ denote the area enclosed by the simple basic loop l. The expectation of the Wilson loop variable $\mathrm{tr}_N h_l$ is given by*

$$E(\mathrm{tr}_N h_l) = \exp\left(-\frac{N}{4\beta} S\right).$$

Proof. We consider the Wilson loop expectation as a function of S, where β and N are fixed. Since the holonomy h_l has density $\mathcal{Q}_{S/2\beta}(x)$ with respect to the Haar measure dx, by taking the derivative with respect to S, we have

$$\begin{aligned}\frac{\partial E(\operatorname{tr}_N h_l)}{\partial S} &= \int_{U(N)} \operatorname{tr}_N(x)\frac{\partial \mathcal{Q}_{S/2\beta}(x)}{\partial S}dx = \frac{1}{4\beta}\int_{U(N)} \operatorname{tr}_N(x)\Delta_x \mathcal{Q}_{S/2\beta}(x)dx \\ &= \frac{1}{4\beta}\int_{U(N)} \Delta_x \operatorname{tr}_N(x)\mathcal{Q}_{S/2\beta}(x)dx.\end{aligned}$$

Since $\Delta_x = \sum_a \partial^2_{T_a}$, we have

$$\Delta_x \operatorname{tr}_N(x) = \sum_a \partial^2_{T_a} \operatorname{tr}_N(x) = \sum_a \operatorname{tr}_N(xT_a^2) = \operatorname{tr}_N(x\sum_a T_a^2) = -N\operatorname{tr}_N(x)$$

by applying (2). Therefore, $E(\operatorname{tr}_N(h_c))$ satisfies the ordinary differential equation

$$\frac{\partial E(\operatorname{tr}_N h_l)}{\partial S} = -\frac{N}{4\beta}E(\operatorname{tr}_N h_l),$$

with the initial condition 1 when $S = 0$. We conclude that $E(\operatorname{tr}_N h_l) = \exp\left(-\frac{N}{4\beta}S\right)$. □

Notice that $E(\operatorname{tr}_N h_l)$ only depends on the ratio $\lambda := N/2\beta$. If we fix this ratio and let N goes to infinity, we see that the Wilson loop variable becomes deterministic:

Theorem 4.2. *The variance of* $\operatorname{tr}_N(h_l)$ *converges to* 0 *as* $N \to \infty$.

Proof. We compute $E(\operatorname{tr}^2_N(h_l))$ as in the proof of Theorem 4.1. More precisely, we need to know the value of $\Delta_x(\operatorname{tr}^2_N(x))$:

$$\begin{aligned}\Delta_x(\operatorname{tr}^2_N(x)) &= \sum_a \partial^2_{T_a}\operatorname{tr}^2_N(x) = \sum_a \partial_{T_a}\left(2\operatorname{tr}_N(x)\operatorname{tr}_N(xT_a)\right) \\ &= 2\sum_a \operatorname{tr}^2_N(xT_a) + \operatorname{tr}_N(x)\operatorname{tr}_N(xT_a^2) \\ &= \frac{2}{N^2}\sum_a \operatorname{tr}(xT_a)\operatorname{tr}(T_a x) + 2\operatorname{tr}_N(x)\operatorname{tr}_N(x\sum_a T_a^2) \\ \text{using (1) and (2)} &= -\frac{2}{N}\operatorname{tr}_N(x^2) - 2N\operatorname{tr}^2_N(x).\end{aligned}$$

Notice that $|\operatorname{tr}_N(x^2)|$ is bounded by 1. Therefore $E(\operatorname{tr}^2_N h_l)$ satisfies the differential equation

$$\frac{\partial E(\operatorname{tr}^2_N h_l)}{\partial S} = -\frac{N}{2\beta}E(\operatorname{tr}^2_N h_l) + R(N,S)$$

where $|R(N,S)| \le 1/N$ for all S and N. The initial condition when $S = 0$ is again given by $E(\operatorname{tr}^2_N h_l) = 1$. It is then not hard to see that as $N \to \infty$, for fixed S, $E(\operatorname{tr}^2_N h_l)$ converges to the solution of

$$f'(S) = -\lambda f(S), \quad f(0) = 1,$$

namely $f(S) = \exp(-\lambda S)$. Therefore the variance,

$$E(\operatorname{tr}^2_N h_l) - [E(\operatorname{tr}_N h_l)]^2 \xrightarrow[N/2\beta=\lambda]{N\to\infty} f(S) - [\exp(-\lambda S/2)]^2 = 0$$

as claimed. □

The *master field* is heuristically a (deterministic) connection A_∞ over an infinite dimensional principal bundle $P_\infty \to \mathbb{R}^2$, such that for any "reasonable" loop $l \subset \mathbb{R}^2$, the Wilson observable $\operatorname{tr}_N(h_l)$ converges as $N \to \infty$ to the trace of the holonomy u_l of the master field, as N goes to ∞ when $\lambda = N/2\beta$ being fixed. This also includes the convergence of the moments $\operatorname{tr}_N(h_l^k)$, which can be considered as the Wilson observable of the loop obtained from l winding k times.

In other words, we want to show that the **spectral measure** of h_l converges to the spectral measure of u_l.

Moreover, the master field should describe the interaction between different loops in the limit. Due to the non-commutative nature of $U(N)$, it concerns not only the joint law of the spectral measures of the holonomies, but the **mixed moments** $\mathrm{tr}_N(h_{l_1}^{k_1}h_{l_2}^{k_2}\cdots)$, where the loops $(l_1, l_2, \cdots)$ may intersect, overlap and repeat.

Free probability theory provides the suitable framework to describe these moments.

In particular, the holonomies of two loops with disjoint interior are independent in the $U(N)$ theory, become freely independent in the master field as $N \to \infty$ [AS].

5. Free probability spaces

Before we give the construction of the master field in the next section (Theorem 6.5), let us review some notions of free probability theory. The facts are stated without proof.

In probability theory, we usually try to forget the sample space Ω as much as possible and are mostly concerned only about events and their probabilities. The random variables and expectations are considered as derived concepts. Free probability theory introduced and developed by Dan Voiculescu [Voi3] takes the abstraction further, and views the (commutative) algebra of random variables $\mathscr{O}$ and their expectations τ as being the foundational concept, ignoring the presence of the original sample space, the algebra of events, and the probability measure. This concept allows one to generalize probability theory to a non-commutative algebra of "random variables". We will see that it allows one to more easily take certain types of limits, such as the large N limit of $N \times N$ random matrices.

Definition 5.1. An *algebraic probability space/free probability space* is a couple $(\mathscr{O}, \tau)$, where $\mathscr{O}$ is an unital $*$-algebra (having a neutral element $\mathbf{1}$ for multiplication) over $\mathbb{C}$, namely

- $x^{**} = (x^*)^* = x$;
- $(x+y)^* = x^* + y^*$;
- $(xy)^* = y^*x^*$;
- $(\lambda x)^* = \overline{\lambda}x^*$, for all $\lambda \in \mathbb{C}$,

and $\tau : \mathscr{O} \to \mathbb{C}$ is $\mathbb{C}$-linear such that

- $\tau(\mathbf{1}) = 1$;
- $\tau(x^*x) \geq 0$;
- $\tau(x^*) = \overline{\tau(x)}$.

The elements of $\mathscr{O}$ are "random variables", and τ is the "expectation". Some examples:

(1) **The classical probability theory**: $\mathscr{O} = L^{\infty-}(\Omega, \mathbb{C}) := \cap_{p\geq 1} L^p(\Omega)$ is the algebra of random variable with finite moments for all $p \in \mathbb{N}$; $\tau = E$, $*$ is complex conjugation. One may also take $\mathscr{O} = L^\infty(\Omega, \mathbb{C})$, but it will exclude interesting elements such as the Gaussians.
(2) **Deterministic matrices**: This is a non-commutative example where $\mathscr{O} = M_N(\mathbb{C})$, $\tau(X) = \mathrm{tr}_N(X)$ (the expectation of eigenvalues under the empirical measure on the spectrum), $X^* = \overline{{}^tX}$.
(3) **Random matrices**: $\mathscr{O} = L^{\infty-}(\Omega) \otimes M_N(\mathbb{C})$ whose elements are random matrices with entries having all moments, $\tau(X) = E(\mathrm{tr}_N(X))$, $X^* = \overline{{}^tX}$.

The philosophy: in free probability space, one can compute the moments $\tau(X^k)$ and mixed moments $\tau(X^kY^mX^n\cdots)$, etc., and in fact, "only the moments matter" (this later allows us to compare different values of N in the random matrices).

The *distribution* of $X \in \mathscr{O}$ is the linear functional:

$$\mathbb{C}[X] \to \mathbb{C} : P \mapsto \tau(P(X)).$$

Similarly, the *joint distribution* of $X_1, \cdots, X_k$ is defined as the data given by their mixed moments.

We say that $X \in \mathscr{O}$ is *bounded* if the *spectral radius* of X

$$\rho(X) := \lim_{k\to\infty} \left|\tau(X^{2k})\right|^{1/2k} < \infty.$$

Theorem 5.2. *If X is bounded and self-adjoint ($X = X^*$), then $\exists!\ \mu_X$ on $[-\rho(X), \rho(X)]$ such that for all polynomials P,*

$$\tau(P(X)) = \int_{[-\rho(X),\rho(X)]} P(x) d\mu_X(x).$$

Many notions and features in the classical probability theory have their counterpart in free probability theory. We list a few of them below. We will see that even with the few axioms we have, which drop out the commutativity of the random variables, universality results like the central limit theorem is still present. And the semicircle law plays the role of the Gaussians in the classical theory.

Definition 5.3 (Convergence in distribution)**.** Let $(\mathscr{O}_N, \tau_N)_{N\in\{1,2,\cdots,\infty\}}$ be algebraic probability spaces, and $X_{N,1}, \cdots X_{N,k}$ elements of $\mathscr{O}_N$. We say that $(X_{N,1}, \cdots X_{N,k})$ *converges to* $(X_{\infty,1}, \cdots X_{\infty,k})$ *in moments/in distribution* if

$$\tau_N(X_{N,i_1} X_{N,i_2} \cdots X_{N,i_m}) \xrightarrow[N\to\infty]{} \tau_\infty(X_{\infty,i_1} X_{\infty,i_2} \cdots X_{\infty,i_m})$$

for all $m \geq 1$ and all sequence $(i_1, \cdots, i_m)$ in $[1, k]$.

Definition 5.4 (Semicircle law)**.** The centered *semicircle law* of variance σ^2 is the distribution $\mathrm{SC}_\sigma := \mathbb{C}[X] \to \mathbb{C}$ defined by

$$P \mapsto \frac{1}{2\pi\sigma^2} \int_{-2\sigma}^{2\sigma} P(t)\sqrt{4\sigma^2 - t^2} dt.$$

Example 5.5 (Wigner's matrices)**.** Let $E_a := iT_a$ be a basis of the space of $N \times N$ Hermitian matrices, say, consisting of elements of the form of either $(E_a)_{jk} = (E_a)_{kj} = 1/2$ for certain $k \neq j$ and 0 elsewhere, or $(E_a)_{jk} = -(E_a)_{kj} = i/2$ or $(E_a)_{jj} = 1$ for a single j. Let ν_a be a family of $\mathscr{N}(0, 1)$ distributed real random variables. The Wigner's random Hermitian matrix

$$X_N := \frac{1}{\sqrt{N}} \sum_a \nu_a E_a$$

is an element of $(L^{\infty-}(\Omega) \otimes M_N, \tau_N)$ where $\tau_N(\cdot) = E(\mathrm{tr}_N(\cdot))$. Wigner's semicircle law states that the empirical law of X_N on the spectrum converges almost surely to the semicircle law with variance 1. Therefore X_N converges in moments to the distribution SC_1.

Definition 5.6 (Free independence)**.** Let $X_1, X_2, \cdots, X_k$ be random variables of $(\mathscr{O}, \tau)$. They are *freely independent* (or just *free*) if

$$\tau\left[\Big(P_1(X_{i_1}) - \tau(P_1(X_{i_1}))\Big)\Big(P_2(X_{i_2}) - \tau(P_2(X_{i_2}))\Big)\cdots\Big(P_n(X_{i_n}) - \tau(P_n(X_{i_n}))\Big)\right] = 0,$$

whenever $i_1, \cdots, i_n \in [1, k]$ are consecutively distinct, and $P_i \in \mathbb{C}[X]$.

A sequence of k-tuples $(X_{N,1}, X_{N,2}, \cdots, X_{N,k})_{N\geq 1}$ of elements in $(\mathscr{O}_N, \tau_N)$ is *asymptotically free* if for all polynomials P_i as above,

$$\tau_N\left[\Big(P_1(X_{N,i_1}) - \tau_N(P_1(X_{N,i_1}))\Big)\cdots\Big(P_n(X_{N,i_n}) - \tau_N(P_n(X_{N,i_n}))\Big)\right] \xrightarrow{N\to\infty} 0.$$

From the definition one can easily deduce if X and Y are freely independent, then

$$\tau(XY) = \tau\left((X - \tau(X))(Y - \tau(Y))\right) + \tau(X)\tau(Y) = \tau(X)\tau(Y).$$

Similarly if moreover $\tau(X) = \tau(Y) = 0$ and $\tau(X^2) = \tau(Y^2) \neq 0$,

$$\tau(XYXY) = 0 \neq \tau(X^2)\tau(Y^2) = \tau(X^2Y^2).$$

We see that it is quite different from the notion of independence in classical probability. In particular, it is easy to check that if X_1 and X_2 commute, being freely independent implies that at least one of them is constant $\in \mathbb{C}$.

Theorem 5.7 (Free central limit theorem)**.** *Let $X \in \mathscr{O}$ with $\tau(X) = 0$ and $\tau(X^2) = \sigma^2$, $X_1, X_2, \cdots$ freely independent copies of X. Then*

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\text{in moments}} SC_{\sigma^2}.$$

It is worth mentioning that adding freely independent variables has a much stronger averaging effect than in the classical CLT, such that the limiting distribution is compactly supported. It does not cover the classical CLT, since if a family of iid commutative random variables is freely independent, then its elements are all constant (therefore 0, since they are centered) from the remark above.

Just as in the proof of the classical central limit theorem one uses the characteristic functions to show the convergence, the free analog proven by Voiculescu uses the R-transform [Voi1]. In particular, if X and Y are freely independent, then $R_{X+Y}(s) = R_X(s) + R_Y(s)$, and the semicircle law's R-transform has the nice linear expression: $R_{SC_{\sigma^2}}(s) = \sigma^2 s$. It implies that

Lemma 5.8. *The sum of freely independent elements of distribution SC_{σ^2} and $SC_{(\sigma')^2}$ has the distribution $SC_{\sigma^2+(\sigma')^2}$.*

The occurrence of the same semicircle law as for Wigner's random matrices is not a coincidence. In fact, it also gives an alternative proof of Wigner's semicircle law from the infinite divisibility of the Wigner's matrices [Voi3].

Definition 5.9 (Free Brownian motion)**.** A *free Brownian motion* in $(\mathscr{O}, \tau)$ is a path $F : [0,T] \to \mathscr{O}$, where $T > 0$, for which

(1) each $F(t)$ is self-adjoint ($F(t)^* = F(t)$), and $F(0) = 0$;
(2) if $t_0 \le t_1 \le \cdots \le t_n$ in $[0,T]$ then
$$F(t_0),\ F(t_1) - F(t_0),\ \cdots\ F(t_n) - F(t_{n-1})$$
are freely independent;
(3) there is a strictly increasing continuous function $S_F : [0,T] \to [0,\infty)$, with $S_F(0) = 0$ such that if $0 \le s < t$ then $F(t) - F(s)$ has the centered semicircle law with variance $S_F(t) - S_F(s)$.

The function S_F is the *clock* of the free Brownian motion.

The free Brownian motion can be constructed explicitly in a particular free probability space [Sp], where $\mathscr{O}$ consists of bounded operators on the Fock space $\mathscr{F}$ and $\tau(X) = \langle X1, 1\rangle_{\mathscr{F}}$, where $1 \in \mathbb{C} \subset \mathscr{F}$ is the vacuum state. As we will only be interested by the spectral measures (and mixed moments), we choose to leave the underlying space abstract. Readers may consult [Sp, AS] for the (interesting) construction, where we see the free analog of the white noise, etc.

6. The 2D-continuum Master field

For simplicity, we take $\beta = N/2$. We will first give the construction of the free holonomy, using the analogous construction to the $U(N)$ case. It shows that this is the most natural candidate for the holonomy in the master field, and the convergence of the $U(N)$ holonomies to the free holonomies is indeed proven by Anshelevich and Sengupta (Theorem 6.5).

We recall the definition of the time-dependent Wigner's random matrices (see Example 5.5):

$$X_N(t) := \frac{1}{\sqrt{N}} \sum_a B_a(t) E_a,$$

where $(B_a(\cdot))$ is a family of independent Brownian motion with the same clock $S : [0,T] \to [0,\infty)$, namely $\mathbf{Var}(B_a(t)) = S(t)$ for all a and (E_a) is the basis of $N \times N$ Hermitian matrices as in Example 5.5. Voiculescu observed that a family of independent Wigner's random matrices are asymptotically free as $N \to \infty$. From the independence of increments of the Brownian motion, it implies:

Theorem 6.1 ([Voi3], Thm. 2.2)**.** *The process $t \mapsto X_N(t)$ in free probability space $(L^{\infty-}(\Omega) \otimes M_N(\mathbb{C}), E(\mathrm{tr}_N(\cdot)))$ converges in distribution (in the sense of finite dimensional marginals with respect to t) to the free Brownian motion F with clock S as $N \to \infty$.*

Comparing $X_N(t)$ to $\tilde{M}(t)$ in (5), we see that (i times) the free Brownian motion is exactly the large-N limit of ($1/\sqrt{N}$ times) the Brownian motion on the Lie algebra $\mathfrak{u}(N)$.

The free multiplicative Brownian motion is defined analogously to the Brownian motion on $U(N)$ (3):

Definition 6.2 (Free multiplicative Brownian motion)**.** Consider a free Brownian motion $b : [0,T] \to \mathscr{O}$ with clock S and $b(0) = 0$. The *free multiplicative Brownian motion* is the process $u^b : [0,T] \to \mathscr{O}$ satisfying the stochastic differential equation

$$du^b(t) = u^b(t) i db(t) - \frac{1}{2} u^b(t) dS(t), \tag{6}$$

with the initial condition $u^b(0) = I$. The law of $u^b(t)$ is the *multiplicative semicircle law (MSC)* of parameter $S(t)$.

The existence and uniqueness of the solution to (6) is guaranteed by the results of Kümmerer and Speicher [KS].

Theorem 6.3. *If b and b' are two free Brownian motions that are freely independent. Then u^b and $u^{b'}$ are also freely independent.*

Definition 6.4 (Free parallel transport)**.** Let $c(t) = (t, y(t)) : [0,T] \to \mathbb{R}^2$ be a cross-vertical path, the *free parallel transport* along c is defined as $t \mapsto u^F(t)$, where F is the free Brownian motion with clock $|c[0,t]|$ (the area between the path $c[0,t]$ and the x-axis).

We write $u_c := u^F(T)$, it has the law of MSC of parameter $|c[0,T]|$.

A *basic loop* $l = c_1 c_2 \cdots c_n$ is a loop composed of finitely many paths c_k which are either vertical or cross-vertical, such that c_1 starts at $(0,0)$. The free holonomy is defined as

$$u_l = u_{c_1} u_{c_2} \cdots u_{c_n}.$$

As we have not explicitly written down what are the free holonomies (but only their distribution), it may be nontransparent what the product means, although this can be made precise with the explicit construction mentioned at the end of Section 5.

There is another equivalent way to get around of this issue: choosing the same base point for all loops allows one to decompose a general loop into concatenations of elementary lassos (which have interior bounded by two vertical segments and two cross-vertical paths, and may be extended by a back-tracking path that connects to the base point $(0,0)$). The free holonomy of an elementary lasso has the distribution of the MSC of parameter the enclosed area (it is non-trivial but should be the same argument as in the $U(N)$-case). Two disjoint lassos have holonomies that are freely independent (essentially from Theorem 6.3). Then we define the holonomy of a general loop as the corresponding product of the holonomy of the lassos. There is no ambiguity in how to decompose the loop, as the product of two freely independent MSC gives an MSC which adds up their parameters.

Theorem 6.5 ([AS] Thm 6.1). *Let $l_1, \cdots, l_n$ be basic loops, with finitely many mutual intersection points. Then*

$$(h_{l_1}, h_{l_1}^*, \cdots, h_{l_n}, h_{l_n}^*) \to (u_{l_1}, u_{l_1}^*, \cdots, u_{l_n}, u_{l_n}^*), \quad \text{as } N \to \infty$$

in distribution, where u_l denotes the free holonomy around a loop l and h_l the $U(N)$-valued stochastic holonomy.

The proof consists of three steps:

(1) show that for an elementary lasso l, (h_l, h_l^*) converges to (u_l, u_l^*) in distribution by explicit moment computation;
(2) For lassos with disjoint interior, the joint law converges to the freely independent family of free holonomies since they are the multiplicative Brownian motions generated by freely independent Brownian motions;
(3) decompose $l_1, \cdots, l_n$ into products of simple disjoint lassos and the result follows immediately since the moments of moments are also moments.

Corollary 6.6. *Suppose l is a basic simple closed curve. Then u_l has multiplicative semicircle law with parameter given by the area enclosed by the loop.*

A priori, the distribution of u_l can be obtained from computing the moments of h_l and taking the $N \to \infty$ limit without referring to the free Brownian motion, since we only care about its distribution at a certain time proportional to its area. However, as the master field is supposed to describe all the loop holonomies simultaneously, the multiplicative Brownian motion $t \mapsto u_{c[0,t]}$, where we consider $c[0,t]$ as the loop bounded by $[(0,0),(t,0)]$, $[(0,0),c(0)]$, $c[0,t]$ and $[(t,0),c(t)]$, is already encoded in the data of the master field.

We also chose to present as we did for aesthetic reason, since the free Brownian motion approach is perfectly parallel to the $U(N)$ theory, and I find it hard to give an equally beautiful description of the free holonomy without this process.

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