Quantum Fields from Random Fields

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Quantization days 2020

November 5, 2020

Table of contents

1. Random fields and Quantum fields
   - Two slides of motivation
   - Axioms for Random Fields
2. Simple example(s)
   - Gaussian fields $\to$ Free Quantum fields in ≥ 0 space dimensions
3. (The theorems)
   - Wightman: Quantum $\to$ Random
   - Osterwalder–Schrader: Random $\to$ Quantum

Random fields and Quantum fields

Random fields:

- **Probability measure** on infinite dimensional space of scalar functions $\varphi : \mathbb{R}^n \to \mathbb{R}$.
- Usually of the form

$$\mathbb{E}F = \frac{1}{Z_A} \int F(\varphi) e^{-A(\varphi)} D\varphi$$

- Important observables: $F = \prod_i \varphi(y_i)$, $F = e^{\varphi(y)}$.

Quantum fields:

- **Hilbert space** $\mathscr{H}$ with a Fock space structure $\mathscr{H} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n$.
- Family of self-adjoint operators $\Phi(t, \overrightarrow{x}) : \mathscr{H} \to \mathscr{H}$, $(t, \overrightarrow{x}) \in \mathbb{R} \times \mathbb{R}^d$.
- Distinguished vector $\Omega \in \mathscr{H}$ called the vacuum.
- Important observables: inner products $\langle \Omega, \prod_i \Phi(t_i, \overrightarrow{x_i}) \Omega \rangle_{\mathscr{H}}$.

On top of these we will need regularity, symmetry and "positivity" properties.

Imaginary time

A story in physics roughly says that the imaginary time inner products

$$\langle \Omega, \prod_{j=1}^{k} \Phi(-it_j, \overrightarrow{x_j})\Omega \rangle =: S((t_1, \overrightarrow{x_1}), \ldots, (t_k, \overrightarrow{x_k}))$$

seem to be correlation functions of a Gibbs-type probability measure i.e.

$$S((t_1, \overrightarrow{x_1}), \ldots, (t_k, \overrightarrow{x_k})) = \frac{1}{Z_A} \int \prod_{j=1}^{k} \varphi(t_j, \overrightarrow{x_j}) e^{-A(\varphi)} D\varphi$$

for some Action Functional A

$$A(\varphi) = \int_{\mathbb{R}^{1+d}} \left((\dot{\varphi}(t, \overrightarrow{x})^2 + |\nabla \varphi(t, \overrightarrow{x})|^2 + V(\varphi(t, \overrightarrow{x}))\right) dt\, d^d \overrightarrow{x}$$

Can you go in the other direction? Can you analytically continue probabilistic correlation functions from real time to imaginary time?

Axioms for field theories

Random fields:

Random field φ with probability distribution μ

- Euclidean symmetry: $\mathbb{E}e^{\varphi(x)} = \mathbb{E}e^{\varphi(Tx)}$, where T can be rotation, translation, reflection
- Reflection positivity
- Regularity: $\mathbb{E}\prod_i \varphi(y_i)$ is a Tempered distribution + more
- Cluster property: $\varphi(y_1)$ becomes independent of $\varphi(y_2)$ as $|y_1 - y_2| \to \infty$.

Quantum fields:

- Hilbert space $\mathscr{H}$ and a unique vacuum vector $\Omega \in \mathscr{H}$.
- Self-adjoint field operators $\Phi(t, \overrightarrow{x}) : \mathscr{H} \to \mathscr{H}$ s.t. $(t, \overrightarrow{x}) \mapsto \langle \psi_1, \Phi(t, \overrightarrow{x})\psi_2 \rangle_{\mathscr{H}}$ is a Tempered Distribution.
- Poincare symmetry (Lorentz + translations)
- Causality and "Energy-momentum relation": $E^2 - |\overrightarrow{p}|^2 \geq 0$

The Quantum Field axioms were introduced by Lars Gårding and Arthur Wightman (published 1964).

The Random Field axioms were introduced by Konrad Osterwalder and Robert Schrader (published 1973-1975)

Equivalence of the axioms proven by Osterwalder and Schrader in 1975.

Axioms for random fields: Reflection positivity

Let φ be a random field with probability distribution μ supported in Schwartz distributions $\mathscr{S}'(\mathbb{R}^n) = \mathscr{S}'(\mathbb{R} \times \mathbb{R}^{n-1})$.

- $\mathscr{E} := L^2(\mathscr{S}'(\mathbb{R}^n), d\mu)$.
- $\mathscr{E}_\pm := L^2(\mathscr{S}'(\mathbb{R}_\pm \times \mathbb{R}^{n-1})d\mu)$, the subspace consisting observables supported in $\mathscr{S}'(\mathbb{R}_\pm \times \mathbb{R}^{n-1})$.
  Intuitively: $\mathscr{E}_+$ consists of the observables F that depend on $\varphi(t, \overrightarrow{x})$ only for $t > 0$.
- $\Theta : \mathscr{E}_+ \to \mathscr{E}_-$ by $(\Theta F)(\varphi) = \overline{F(\theta\varphi)}$ with $\theta\varphi(t, \overrightarrow{x}) = \varphi(-t, \overrightarrow{x})$.
- $\langle F, G\rangle_{\mathscr{E}_+} := \int F(\varphi)(\Theta G)(\varphi)d\mu(\varphi) = \mathbb{E}F(\varphi)(\Theta G)(\varphi)$.

1. **Reflection Positivity** : For all $F \in \mathscr{E}_+$ we have $\langle F, F\rangle_{\mathscr{E}_+} \geq 0$.

Suffices to consider F belonging to

$$\mathscr{A}_+ := \{F(\varphi) = \sum_{k=1}^{N} c_k e^{i(\varphi, f_k)} \mid c_k \in \mathbb{C}, f_k \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^{n-1})\} \subset \mathscr{E}_+.$$

$(\varphi, f_k) := \int \varphi(y) f_k(y) d^n y$.

Axioms for random fields

- **Reflection Positivity:** For all $F \in \mathscr{E}_+$ we have $\langle F, F \rangle_{\mathscr{E}} \geq 0$.
- **Euclidean symmetry:** μ is invariant under Euclidean symmetries: $\mathbb{E}e^{\varphi(x)} = \mathbb{E}e^{\varphi(Tx)}$, where T can be Rotation, Reflection, Translation.
- **Cluster property:** Let $(s_i, \overrightarrow{x_i}) \in \mathbb{R} \times \mathbb{R}^{n-1}$

$$\lim_{t \to \infty} \mathbb{E} \prod_{i=1}^{k} \varphi(s_i - t, \overrightarrow{x_i}) \prod_{j=k+1}^{l} \varphi(s_j, \overrightarrow{x_j}) = \mathbb{E} \prod_{i=1}^{k} \varphi(s_i, \overrightarrow{x_i}) \mathbb{E} \prod_{j=k+1}^{l} \varphi(s_j, \overrightarrow{x_j})$$

- **Regularity(*):** The correlation functions

$$(y_1, \ldots, y_k) \mapsto \mathbb{E} \prod_{i=1}^{k} \varphi(y_i), \quad y_i \in \mathbb{R}^n,$$

are tempered distributions in the region of non-coinciding points ($y_i \neq y_j$ for $i \neq j$).
Also need an additional **growth estimate** when $k \to \infty$ and when $s \to 0$.

Axioms for random fields: Regularity

- **Regularity(*):** The functions $(y_1, \ldots, y_k) \mapsto \mathbb{E} \prod_{i=1}^{k} \varphi(y_i)$ are tempered distributions.
  OS showed that the following suffices: Denote $S_2(y_2 - y_1) = \mathbb{E}\varphi(y_1)\varphi(y_2)$. Assume

$$\left| \int S_2(y) f(y) d^n y \right| \leq \|\tilde{f}\|,$$

  where $\|\tilde{f}\|$ is a Schwartz norm of the Laplace transform of f.
  Analogous bounds for higher order correlations S_k.
  Not practical to work with!
  OS also used something like

$$\left| \, S_2(s, \overrightarrow{x}) \right| \; \leq C_2 \left((1 + |\overrightarrow{x}|)(1 + |s|)(1 + |s|^{-1}) \right)^{N_2}$$

  i.e. S_2 is tempered and has a power law divergence as $s \to 0$.
  Analogous bounds for S_k with $C_k \simeq (k!)^p$.
  Glimm–Jaffe book: Bound for the Laplace transform

$$\left| \mathbb{E} e^{(\varphi, f)} \right| \leq e^{c(\|f\|_{L_1} + \|f\|_{L_p}^p)}, \quad 1 \leq p \leq 2.$$

Constructing a Quantum field from a Random field

We are going to construct a Quantum field from a random Gaussian field.

We will choose $n = 1$, i.e. the random field lives on $\mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}$.

Leads to a quantum field in 1 time and 0 space dimensions.

The construction consists of

1. We start with a Gaussian field φ with a covariance kernel $G(t,s) = \frac{e^{-\omega|t-s|}}{2\omega}$.
2. We show that the probability distribution μ of φ is Reflection Positive.
3. We then construct a Hilbert space $\mathscr{H}$ from a subspace of $L^2(\mathscr{S}'(\mathbb{R}), d\mu)$.
4. We show that $\mathscr{H}$ is unitarily equivalent with $L^2(\mathbb{R}, e^{-\omega x^2} dx)$.
5. We show that the translation semigroup on $L^2(\mathscr{S}'(\mathbb{R}), d\mu)$ maps to a translation semigroup on $L^2(\mathbb{R}, e^{-\omega x^2} dx)$ and compute its generator (the Hamiltonian).
6. The resulting system will look like a Quantum Harmonic Oscillator.

Reconstruction: The simplest example

- We set $n = 1$ and μ a **Gaussian measure** on $\mathscr{S}'(\mathbb{R})$ with covariance

$$G(t,s) = \frac{e^{-\omega|t-s|}}{2\omega}, \quad \omega > 0.$$

  Note: This is the Green function of the operator $-\partial_t^2 + \omega^2$. Thus think of:

$$d\mu(\varphi) \simeq e^{-\frac{1}{2}\int \varphi(t)(-\partial_t^2+\omega^2)\varphi(t)dt} D\varphi$$

- We denote the Gaussian field given by this measure by $\varphi \in \mathscr{S}'(\mathbb{R})$. Thus $\mathbb{E}[\varphi(t)\varphi(s)] = \int \varphi(t)\varphi(s)d\mu(\varphi) = G(t,s)$.
- Recall $(\Theta F)(\varphi) = \overline{F(\theta\varphi)}$, $(\theta\varphi)(t) = \varphi(-t)$. Reflection Positivity is the condition:

$$\mathbb{E}(\Theta F)(\varphi)F(\varphi) \geq 0$$

  where $F \in L^2(\mathscr{S}'(\mathbb{R}_+), d\mu)$ and we can take $F(\varphi) = e^{(\varphi,f)}$ with $f \in \mathscr{S}(\mathbb{R}_+)$.
- Is μ Reflection Positive?

Reflection Positivity for φ

- Keep in mind: $\mathbb{E}\varphi(t)\varphi(s) = G(t,s) = \frac{e^{-\omega|t-s|}}{2\omega}$
- Fact: For Gaussians the Reflection Positivity

$$\mathbb{E}(\Theta F)(\varphi)F(\varphi) \geq 0$$

  is equivalent to

$$\int (\theta f)(t)f(s)G(t,s)dtds \geq 0$$

  for all $f \in \mathscr{S}(\mathbb{R}_+)$.
- We have:

$$\begin{aligned}
\int (\theta f)(t)f(s)G(t,s)dtds &= \frac{1}{2\omega}\int f(-t)f(s)e^{-\omega|t-s|}dtds \\
&= \frac{1}{2\omega}\int_{(-\infty,0)} dt \int_{(0,\infty)} ds\, f(-t)f(s)e^{-\omega|t-s|}dtds \\
&= \frac{1}{2\omega}\int_{(-\infty,0)} f(-t)e^{\omega t}dt \int_{(0,\infty)} f(s)e^{-\omega s}ds \\
&\geq 0\,.
\end{aligned}$$

Construction of the Hilbert space

- We construct a Hilbert space out of

$$\mathscr{A}_+ := \{\sum_{k=1}^{N} c_k e^{i(\varphi, f_k)} \mid c_k \in \mathbb{C}, f_k \in C_0^\infty(\mathbb{R}_+)\} \subset L^2(\mathscr{S}'(\mathbb{R}_+), d\mu)$$

  Set

$$\langle F, G\rangle_{\mathscr{A}_+} = \mathbb{E}(\Theta G)(\varphi)F(\varphi).$$

- This is **positive semidefinite** 1 by Reflection Positivity!
- The quotient $\mathscr{A}_+/\mathscr{N}$, where $\mathscr{N} = \{F \in \mathscr{A}_+ \mid \langle F, F\rangle = 0\}$, is an **inner product space.**
- Then define $\mathscr{H}_+ :=$ completion of $\mathscr{A}_+/\mathscr{N}$.
- $\mathscr{H}_+$ is automatically a **Hilbert space.**

Decomposition of φ

- $G(t,s) = \frac{e^{-\omega|t-s|}}{2\omega}$, the Green function of $(-\partial_t^2 + \omega^2)$.
- First we **decompose** the covariance kernel

$$G(t,s) = G_+(t,s) + G_-(t,s) + G_0(t,s)$$

- Set

$$G_+(t,s) = \big(G(t,s) - G(t,-s)\big)\mathbf{1}_{t,s\geq 0}$$
$$G_-(t,s) = \big(G(t,s) - G(t,-s)\big)\mathbf{1}_{t,s\leq 0}$$

  Method of images! $G_\pm$ are the Dirichlet Green functions of $(-\partial_t^2 + \omega^2)$ on $\mathbb{R}_\pm$.
- $G_0 := G - G_+ - G_-$. Follows that $G_0(t,s) = \frac{e^{-\omega(|t|+|s|)}}{2\omega}$.
- $G_\pm$ are **positive** as operators: $\int f(t)f(s)G_\pm(t,s)dtds \geq 0$.
- Clearly also G_0 is positive.
  $\Longrightarrow$ φ decomposes into **independent** parts $\varphi = \varphi_+ + \varphi_- + \varphi_0$.
- Here $\varphi_0(t) = e^{-\omega|t|}x$, $x \sim \mathcal{N}(0, \frac{1}{2\omega})$, so the probability distribution μ_0 of φ_0 is $\frac{\sqrt{\omega}}{\sqrt{\pi}}e^{-\omega x^2}dx$.

Unitary equivalence with $L^2(\mathbb{R}, e^{-\omega x^2}dx)$

- φ decomposes into **independent** parts $\varphi = \varphi_+ + \varphi_- + \varphi_0$ with

$$\mathbb{E}\varphi_\pm(t)\varphi_\pm(s) = G_\pm(t,s), \quad \mathbb{E}\varphi_0(t)\varphi_0(s) = G_0(t,s) = \frac{e^{-\omega(|t|+|s|)}}{2\omega}$$

- We have $\varphi_0(t) = e^{-\omega|t|}x$ with $x \sim \mathcal{N}(0, \frac{1}{2\omega})$.
  Thus $\mathbb{E}_0\overline{F_1(\varphi_0)}F_2(\varphi_0) = \langle F_1, F_2\rangle_{L^2(\mathbb{R}, e^{-\omega x^2}dx)}$.
- The decomposition implies

$$\mathbb{E}F(\varphi) = \mathbb{E}_+\mathbb{E}_-\mathbb{E}_0 F(\varphi_+ + \varphi_- + \varphi_0).$$

- Unitary equivalence with $L^2(\mathbb{R}, e^{-\omega x^2}dx)$:

$$\begin{aligned}
\langle F, G\rangle_{\mathscr{H}_+} &= \mathbb{E}(\Theta G)(\varphi)F(\varphi) \\
&= \mathbb{E}_+\mathbb{E}_-\mathbb{E}_0(\Theta G)(\varphi_- + \varphi_0)F(\varphi_+ + \varphi_0) \\
&= \mathbb{E}_0\left[\mathbb{E}_-\overline{G(\theta(\varphi_- + \varphi_0))}\mathbb{E}_+F(\varphi_+ + \varphi_0)\right] \\
&= \mathbb{E}_0\left[\overline{\mathbb{E}_+G(\varphi_+ + \varphi_0)}\mathbb{E}_+F(\varphi_+ + \varphi_0)\right] \\
&= \langle \mathbb{E}_+G(\varphi_+ + \cdot), \mathbb{E}_+F(\varphi_0 + \cdot)\rangle_{L^2(\mathbb{R}, e^{-\omega x^2}dx)}
\end{aligned}$$

We used $\theta\varphi_- \overset{law}{=} \varphi_+$ and $\theta\varphi_0 = \varphi_0$.

Unitary equivalence with $L^2(\mathbb{R}, e^{-\omega x^2} dx)$

- We got

$$\langle F, G\rangle_{\mathscr{H}_+} = \langle \mathbb{E}_+ G(\varphi_+ + \cdot), \mathbb{E}_+ F(\varphi_0 + \cdot)\rangle_{L^2(\mathbb{R}, e^{-\omega x^2} dx)}$$

- Thus the map $U : \mathscr{H}_+ \to L^2(\mathbb{R}, e^{-\omega x^2} dx)$ given by

$$(UF)(\varphi_0) = \mathbb{E}_+ F(\varphi_+ + \varphi_0)$$

  is an **isometry** .
- The range of U is dense:

$$\begin{aligned} Ue^{i(\varphi,f)} &= e^{i(\varphi_0,f)}\mathbb{E}_+ e^{i(\varphi_+,f)} = e^{i(\varphi_0,f)} e^{-\frac{1}{2}\mathbb{E}(\varphi_+,f)^2} \\ &= e^{ix\int f(t)e^{-\omega t}dt} e^{-\frac{1}{2}(f,G_+ f)} \end{aligned}$$

  Thus range of U contains all vectors of the form $e^{i\alpha x}$, $x \sim \mu_0$. This is a dense set in $L^2(\mathbb{R}, e^{-\omega x^2} dx)$. U preserves inner product and has dense range $\implies$ U is **unitary.**
- $U\mathscr{H}_+ = L^2(\mathbb{R}, e^{-\omega x^2} dx)$.

Time translation semigroup

On $\mathscr{H}_+$ we can define the time translation

$$T_t e^{i(\varphi,f)} := e^{i(\varphi,f_t)}, \quad f_t(s) := f(s-t)$$

$T(t)$ is **self-adjoint:**

$$\begin{aligned}
\langle T(t)e^{i(\varphi,f)}, e^{i(\varphi,g)}\rangle_{\mathscr{H}_+} &= \mathbb{E}\Theta(e^{i(\varphi,f_t)})e^{i(\varphi,g)} \\
&= \mathbb{E}e^{-i(\varphi,\theta f_t)}e^{i(\varphi,g)} \\
&= \mathbb{E}e^{-i(\varphi,(\theta f)_{-t})}e^{i(\varphi,g)} \\
&= \mathbb{E}e^{-i(\varphi,\theta f)}e^{i(\varphi,g_t)} \\
&= \langle e^{i(\varphi,f)}, T(t)e^{i(\varphi,g)}\rangle_{\mathscr{H}_+}
\end{aligned}$$

Time translation semigroup

- $T(t)$ is a **contraction:**

$$\|T(t)F\|^2 = \langle T(t)F, T(t)F\rangle = \langle F, T(2t)F\rangle \leq \|F\|\|T(2t)F\|$$
$$\implies \|T(t)F\| \leq \|F\|^{1/2}\|T(2t)F\|^{1/2}$$
$$\implies \|T(t)F\| \ \leq \|F\|^{1/2+1/4}\|T(4t)F\|^{1/4}$$
$$\implies \|T(t)F\| \leq \|F\|^{1/2+\ldots+1/2^k}\|T(2^k t)F\|^{1/2^k}$$

Translation invariance implies

$$\begin{aligned}\|T(2^n t)F\|^2 = \langle F, T(2^{n+1})F\rangle &= \mathbb{E}(\Theta F)T(2^{n+1}t)F \\ &\leq (\mathbb{E}(\Theta F)^2)^{1/2}(\mathbb{E}(T(2^{n+1}t)F)^2)^{1/2}\end{aligned}$$
$$\mathbb{E}(T(2^{n+1}t)e^{i(\varphi,f)})^2 = \mathbb{E}e^{2i(\varphi,f_{2^{n+1}t})} = \mathbb{E}e^{2i(\varphi,f)} = \mathbb{E}(e^{i(\varphi,f)})^2$$

Conclusion: $\|T(t)F\|_{\mathscr{H}_+} \leq \|F\|_{\mathscr{H}_+}$

- **Hille–Yosida:** $T(t) = e^{-tH}$ where H is **positive** and **self-adjoint** operator.

Time translation on $L^2(\mathbb{R}, e^{-\omega x^2} dx)$

- $e^{-tH} : \mathscr{H}_+ \to \mathscr{H}_+$,
  $U : \mathscr{H}_+ \to L^2(\mathbb{R}, e^{-\omega x^2} dx)$, $U = \mathbb{E}_+$
- Time translation semigroup on $L^2(\mathbb{R}, e^{-\omega x^2} dx)$ by $e^{-t\tilde{H}} := Ue^{-tH}U^{-1}$
- We want to compute $e^{-t\tilde{H}} Ue^{i(\varphi,f)}$.

$$\begin{aligned} e^{-t\tilde{H}} Ue^{i(\varphi,f)} = Ue^{-tH} e^{i(\varphi,f)} = Ue^{i(\varphi,f_t)} &= e^{i(\varphi_0,f_t)} \mathbb{E}_+ e^{i(\varphi_+,f_t)} \\ &= e^{ix\int f_t(s)e^{-\omega s} ds} e^{-\frac{1}{2}(f_t, G_0 f_t)} \end{aligned}$$

- What is essential is the functional dependence on x of the RHS. Take $F = e^{i\varphi(0)\alpha}$ for $\alpha \in \mathbb{C}$. Then $UF = e^{ix\alpha}$. Now

$$\begin{aligned} e^{-t\tilde{H}} Ue^{i\varphi(0)\alpha} &= Ue^{-tH} e^{i\varphi(0)\alpha} \\ &= Ue^{i\varphi(t)\alpha} \\ &= e^{-\frac{\alpha^2}{2} G_+(t,t)} e^{i\varphi_0(t)\alpha} \\ &= e^{-\frac{\alpha^2}{2}(\frac{1}{2\omega} - \frac{1}{2\omega} e^{-2\omega t})} e^{ie^{-\omega t} x\alpha} \end{aligned}$$

Time translation on $L^2(\mathbb{R}, e^{-\omega x^2}dx)$

The Hamiltonian $\tilde{H}$ can now be evaluated

$$\begin{aligned}
-\tilde{H}Ue^{i\varphi(0)\alpha} &= \frac{d}{dt}\Big|_{t=0} e^{-t\tilde{H}}Ue^{i\varphi(0)\alpha} \\
&= \frac{d}{dt}\Big|_{t=0} e^{-\frac{\alpha^2}{2}(\frac{1}{2\omega}-\frac{1}{2\omega}e^{-2\omega t})}e^{ie^{-\omega t}x\alpha} \\
&= \big(-\frac{\alpha^2}{2} - ix\alpha\omega\big)e^{-\frac{\alpha^2}{2\omega}}Ue^{i\varphi(0)\alpha} \\
&= \big(-\tfrac{1}{2}\tfrac{d^2}{dx^2} - \omega x\tfrac{d}{dx}\big)Ue^{i\varphi(0)\alpha}
\end{aligned}$$

I.e. $\tilde{H} = \frac{1}{2}\frac{d^2}{dx^2} + \omega x\frac{d}{dx}$ on $L^2(\mathbb{R}, e^{-\omega x^2}dx)$.
A unitary map $V : L^2(\mathbb{R}, e^{-\omega x^2}dx) \to L^2(\mathbb{R}, dx)$ is given by $(Vf)(x) = e^{-\frac{1}{2}\omega x^2}f(x)$. Then

$$V\tilde{H}V^{-1} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\omega^2x^2 - \frac{1}{2}\omega.$$

The Hamiltonian of the **Quantum Harmonic Oscillator.**

Oscillator

- We ended up with a Quantum mechanical system

$$H : L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$
$$(Hf)(x) = (-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\,\omega^2 x^2 - \frac{1}{2}\,\omega)f(x)$$

- The vacuum state $\Omega \in L^2(\mathbb{R})$ is the state satisfying $H\Omega = 0$. The solution is $\Omega(x) = (V1)(x) = e^{-\frac{1}{2}\omega x^2}$.
- What are the functions $\langle \Omega, \prod_i \Phi(t_i)\Omega \rangle$? We were supposed to have something like

$$\langle \Omega, \prod_j \Phi(-it_j)\Omega \rangle = \mathbb{E} \prod_j \varphi_0(t_j)$$

Quantum field axioms also say that

$$\Phi(t) = e^{itH}\Phi(0)e^{-itH}$$

I.e. $\Phi(-it) = e^{tH}\Phi(0)e^{-tH}$.
Here $\Phi(0)$ corresponds to the multiplication operator $x : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(xf)(x) = xf(x)$.

Two-point function

- $\Phi(-it) = e^{tH}\Phi(0)e^{-tH}$.
- $\Phi(0)$ is given by $x : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(xf)(x) = xf(x)$.
- Then for $s > t > 0$

$$\begin{aligned}
\langle \Omega, \Phi(-it)\Phi(-is)\Omega\rangle_{L^2(\mathbb{R})} &= \langle \Omega, e^{-tH}\ \Phi(0)e^{-(s-t)H}\Phi(0)e^{sH}\ \Omega\rangle_{L^2(\mathbb{R})} \\
&= \langle \Omega, \Phi(0)e^{-(s-t)H}\ \Phi(0)\Omega\rangle_{L^2(\mathbb{R})} \\
&= \mathbb{E}_0\varphi_0(0)\varphi_0(s-t) \\
&= G_0(s,t) \\
&= \frac{e^{-\omega(s+t)}}{2\omega}.
\end{aligned}$$

End result

- We started with a Gaussian measure with covariance operator $G = (-\partial_t^2 + \omega)^{-1}$ and Covariance Kernel $G(t,s) = \frac{e^{-\omega|t-s|)}}{2\omega}$.
- Using Reflection Positivity we got the Hilbert space $\mathscr{H}_+ \overset{U}{\mapsto} L^2(\mathbb{R}, e^{-\omega x^2} dx) \overset{V}{\mapsto} L^2(\mathbb{R}, dx)$ and on $L^2(\mathbb{R}, dx)$ we got the Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\omega^2 x^2 - \frac{1}{2}\omega$$

- A vacuum state $\Omega = e^{-\omega x^2}$
- Imaginary time field operator $\Phi(-it) = e^{tH}\,\Phi(0)e^{-tH}$ with $\Phi(0)$ the multiplication operator $f(x) \mapsto xf(x)$ on $L^2(\mathbb{R})$.
- Imaginary time Wightman function $\langle \Omega, \Phi(-it)\Phi(-is)\Omega \rangle_{L^2(\mathbb{R})} = G_0(t,s)$.
- How to get real time Wightman functions?

Analytic continuation & more

Quantum fields:

- Hilbert space $\mathscr{H}$ and a vacuum $\Omega \in \mathscr{H}$.
- Self-adjoint field operators $\Phi(t, \overrightarrow{x}) : \mathscr{H} \to \mathscr{H}$ s.t. $(t, \overrightarrow{x}) \mapsto \langle \psi_1, \Phi(t, \overrightarrow{x}) \psi_2 \rangle_{\mathscr{H}}$ is a Tempered Distribution.
- Poincare symmetry
- Causality and $E^2 - |\overrightarrow{p}|^2 \geq 0$

Random fields:
Random field φ with probability distribution μ

- Euclidean symmetry
- Reflection positivity
- Regularity
- Cluster property

Random to Quantum: $S_2(t_1 - t_2, \overrightarrow{x_1} - \overrightarrow{x_2}) := \mathbb{E}\varphi(t_1, \overrightarrow{x_1})\varphi(t_2, \overrightarrow{x2})$

1. Reflection positivity gives a Hilbert space and e^{-tH}. The semigroup extends to $e^{(-t+is)H}$ for $t \geq 0$ and this semigroup is **holomorphic for $t > 0$.**
2. This yields an analytic continuation of $S_2(\tau, \overrightarrow{x})$ to $\Re(\tau) > 0$.
3. This + regularity $\implies$ $S_2(is, \overrightarrow{x})$ is a tempered distribution and the support of its Fourier transform belongs to the "future light cone".
4. The regularity assumptions ensure that this works also for S_k, $k > 2$.
5. Cluster property $\implies$ uniqueness of vacuum.

Free fields in higher dimensions

- Gaussian field φ on $\mathbb{R}^n$ with a covariance operator $(-\Delta + m^2)^{-1}$.
- We have $G(y_1, y_2) = G(|y_1 - y_2|)$ and the Fourier transform satisfies

$$G(y_1, y_2) = \int \frac{1}{p^2 + m^2} e^{ip\cdot(y_1 - y_2)} \frac{d^n p}{(2\pi)^n}$$

After using the rotational symmetry and computing, we get

$$G(y_1, y_2) = \int \frac{e^{-|y_1 - y_2|\sqrt{|k|^2 + m^2}}}{2\sqrt{|k|^2 + m^2}} \frac{d^{n-1} k}{(2\pi)^{n-1}}$$

Compare with $G(t, s) = \frac{e^{-\omega|t-s|}}{2\omega}$

- Now $\varphi = \varphi_+ + \varphi_- + \varphi_0$ Where $\varphi_\pm$ are Dirichlet free fields with mass m^2 on $\mathbb{R}_\pm \times \mathbb{R}^{n-1}$.
- When $m = 0$ the time-zero field φ_0 is the Harmonic Extension of $\varphi|_{t=0}$ to $\mathbb{R}^n$.
- Leads to a Hilbert space with a Hamiltonian of an infinite collection of independent quantum harmonic oscillators.

References

1. Barry Simon: $P(\varphi)_2$ Euclidean (Quantum) Field Theory. 1974
2. James Glimm, Arthur Jaffe: Quantum Physics: A Functional Integral Point of View. 1981 (2nd Ed. 1987)
3. Christian Anderson. Defining physics at imaginary time: reflection positivity for certain Riemannian manifolds. (2013 thesis)
4. Slava Rychkov. Seminar talk available on YouTube "CFT Osterwalder-Schrader Theorem"
5. Konrad Osterwalder and Robert Schrader. Axioms for Euclidean Green's functions I-II. (CMP 1973 & 1975)

The 1975 OS-paper contains a proof of equivalence of the Random field axioms and the Quantum field axioms.