Stochastic quantization and Yang-Mills

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November 6, 2020

a little fact:

1. Probability measure: $\mu = \frac{1}{Z} e^{-V(x)} dx$
2. Stochastic process: $dX_t = -\frac{1}{2} V'(X) dt + dB_t$

$$\mu \text{ is invariant under } X_t$$

quantum field theory

1. Functional integral formulation: $\text{Prob}(\Phi) \propto e^{-\mathcal{S}(\Phi)}$.
2. Stochastic quantization formulation: $\partial_t \Phi = -\frac{\delta \mathcal{S}}{\delta \Phi} + \xi$

ξ: space-time white noise

Examples:

1. GFF: $\mathcal{S}(\Phi) = \int \frac{1}{2} |\nabla \Phi|^2 \, dx \quad \Rightarrow \partial_t \Phi = \Delta \Phi + \xi$
2. $\mathcal{S}(\Phi) = \int \frac{1}{2} |\nabla \Phi|^2 + \frac{1}{4} \Phi^4 \, dx \quad \Rightarrow \partial_t \Phi = \Delta \Phi - \Phi^3 + \xi$

SPDE: one slide tutorial

Linear: $\partial_t \Phi = \Delta\Phi + \xi$. Theorem: $\xi \in \mathcal{C}^{-\frac{d+2}{2}-}$ and $\Phi \in \mathcal{C}^{-\frac{d-2}{2}-}$

Corollary: $d \geq 2$, Φ is distribution. (GFF is invariant.)

Nonlinear: $\partial_t u = \Delta u + u^2 + \xi$ in $d = 2$. *FORMAL*!

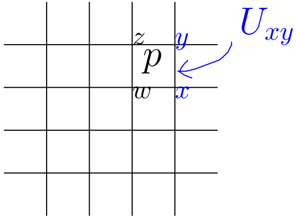
$$\partial_t u_\varepsilon = \Delta u_\varepsilon + u_\varepsilon^2 - ? + \xi_\varepsilon$$

Let $u_\varepsilon = \Phi_\varepsilon + v_\varepsilon$. (Recall $\Phi \in \mathcal{C}^{0-}$)

$$\partial_t v_\varepsilon = \Delta v_\varepsilon + \Phi_\varepsilon^2 - ? + 2\Phi_\varepsilon v_\varepsilon + v_\varepsilon^2$$

1. $\Phi_\varepsilon^2 - \mathbf{E}(\Phi_\varepsilon^2)$ converges in $\mathcal{C}^{0-}$.
2. Solve v by classical PDE: $v \in \mathcal{C}^{2-}$.
3. Theorem: renormalized equ has a limit solution $u = \Phi + v$.

Definition of Yang-Mills model



$$U(p) = U_{xy} U_{yz} U_{zw} U_{wx}$$

$$\mathrm{Prob}(U) = e^{\sum_p S(U(p))} d\sigma_{\text{Haar}}/Z$$

(S is real-valued, $S(yxy^{-1}) = S(x)$)

$U \sim e^{\varepsilon A}$

$$A = \sum_{i=1}^{d} A_i dx_i$$

(each A_i value in Lie algebra i.e. skew-Hermitian)

$$F_A = dA + [A, A]$$

$$\mathrm{Prob}(A) \propto e^{-\|F_A\|^2}$$

Gauge symmetry:

1. Lattice measure is invariant $U_{xy} \mapsto g(x) U_{xy} g(y)^{-1}$
2. In continuum formally invariant $A \mapsto gAg^{-1} - gdg^{-1}$

Wilson loop observables

Lattice path γ, holonomy $\mathrm{hol}_\gamma = \prod_{\{x,y\}\in\gamma} U_{xy}$.
Loop γ, Wilson loop $W_\gamma = Tr(\mathrm{hol}_\gamma)$.

Smooth path $\gamma : [0,1] \to \mathbf{R}^d$, $\mathrm{hol}_\gamma = F(1)$ where $dF = F \cdot A(d\gamma)$
Smooth loop, $W_\gamma = Tr(\mathrm{hol}_\gamma)$.

W_γ is gauge invariant.

Challenges of YM:

1. make sense measure or observables in continuum
   (related: gauge fixing, Gribov ambiguity, ghosts etc.)
2. large scale behavior:
   i.e. construct measure on whole $\mathbf{R}^d$, decay of loop correlations
   (related: confinement, area law, mass gap..)
3. many other problems, e.g. large N.

Earlier works in continuum:

[Gross, Driver, Sengupta, Levy..] 2d YM observables.
Many results in abelian cases (80s-00s).

Our results in $d = 2$ (and $d = 3$ in progress):

We construct stochastic quantization / Langevin dynamic of YM

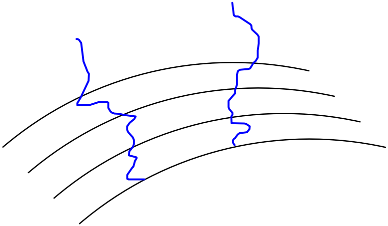
$$\partial_t A = -d_A^* F_A - d_A d^* A + \xi$$

for which $e^{-\|F_A\|^2}$ is formally invariant; more explicitly

$$\partial_t A_i = \Delta A_i + [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] + \xi_i$$

In particular

1. We built the orbit space $\{A\}/\mathcal{G}$
2. We define the Markov process.
3. Gauge invariance of this process.



Stochastic: [Bern-Halpern-Sadun-Taubes'87] (physics)

Deterministic: Donaldson, Rade, Waldron, S.J.Oh etc.

Space of orbits (state space) "$\{A\}/\mathcal{G}$"

Questions to address:

? Want holonomies, but, distributions can't integrate along curves. (However, 2d GFF $\in \mathcal{C}^{0-}$ can.)

? Is "$\{A\}/\mathcal{G}$" reasonable (e.g. Polish) for measure theory?

Theorem. Let $d = 2$, $\alpha = 1^-$

We constructed a space Ω^1_α of (Lie algebra valued) 1-forms s.t.

1. For each $A \in \Omega^1_\alpha$, one can define $\mathrm{hol}_\gamma(A)$
2. $$\underbrace{\mathcal{C}^{\frac{\alpha}{2}}}_{\text{continuous}} \subset \Omega^1_\alpha \subset \underbrace{\mathcal{C}^{\alpha-1}}_{\text{distributions}}$$
3. $\Omega^1_\alpha/\mathcal{G}^\alpha$ is Polish
4. GFF $\in \Omega^1_\alpha$

Sketch of proof.

1. Start with functions from $\{$line segments$\}$ to $\mathbf{R}$.

$$\|A\|_{\alpha-gr} = \sup_{\ell} \frac{|A(\ell)|}{|\ell|^{\alpha}} \qquad (\alpha = 1^{-})$$

2. Changes are small when slightly turning directions

$$\|A\|_{\alpha-vee} = \sup_{\ell,\bar{\ell}} \frac{|A(\ell) - A(\bar{\ell})|}{\mathrm{Area}(\ell,\bar{\ell})^{\frac{\alpha}{2}}}$$

3. $\Omega^1_\alpha =$ closure of smooth 1-forms under above two norms.

Sketch of proof (continued).

4. For a triangle P, one has $|A(\partial P)| \leq \|A\|_\alpha \mathrm{Area}(P)^{\frac{\alpha}{2}}$
5. Approximate holonomy along smooth γ by line segments
6. $\partial_t A = \Delta A + \xi$ and GFF in Ω^1_α (rebuild Schauder + Kolmogorov)

SPDE (linear and trivial example)

2D stochastic abelian YM: $A = (A_1, A_2)$, $F_A = dA$ and $d = \text{curl}$.

$$\mathcal{S}(A) = \int_{\mathbf{T}^2} (dA)^2 dx$$
$$\partial_t A = -d^* dA + \xi$$

Gauge invariance: $\tilde{A}(t,x) := A(t,x) + df(x)$ satisfies same equation.
Problem: not elliptic. (Recall $\Delta = -d^* d - dd^*$.)

“Donaldson-DeTurck trick”:

A calculation: Let $\partial_t B = \Delta B + \xi$ and $A_t := B_t + \int_0^t dd^* B_s ds$
Then, $\partial_t A = \Delta B + \xi + dd^* B = -d^* dB + \xi = -d^* dA + \xi$.

- time-dependent gauge transformations.

SPDE: 1st try to nonlinear case

[Shen'18] abelian YM couple Higgs: $\int_{\mathbf{T}^2} (\mathrm{Curl} A)^2 + |D^A \Phi|^2 \, dx$

1. Discretize (by lattice gauge theory to preserve gauge symmetry)
2. Renormalize (without breaking gauge symmetry)
3. Take limit (the gauge invariant dynamic)

$$(D_j^{A^\varepsilon} \Phi^\varepsilon)(x) = \varepsilon^{-1} \Big(e^{-i\varepsilon\lambda A^\varepsilon(x, x+\mathbf{e}_j)} \Phi^\varepsilon(x + \mathbf{e}_j) - \Phi^\varepsilon(x) \Big)$$

$$\begin{cases} \partial_t A_1^\varepsilon(e) = \nabla_2^\varepsilon \nabla_2^\varepsilon A_1^\varepsilon(e) - \nabla_1^\varepsilon \nabla_2^\varepsilon A_2^\varepsilon(e) \\ \qquad\qquad + \varepsilon^{-1} \lambda \, \mathrm{Im} \Big(e^{-i\varepsilon\lambda A_1^\varepsilon(e)} \Phi^\varepsilon(e_+) \bar{\Phi}^\varepsilon(e_-) \Big) + \xi_1^\varepsilon(e) \\ \partial_t \Phi^\varepsilon(x) = \varepsilon^{-2} \Big(\sum_{\mathbf{e}} e^{-i\varepsilon\lambda A^\varepsilon(x, x+\mathbf{e})} \Phi^\varepsilon(x + \mathbf{e}) - 4\Phi^\varepsilon(x) \Big) + \zeta^\varepsilon(x) \end{cases}$$

Theorem [S.'18]: $(A^\varepsilon, \Phi^\varepsilon)$ converges as $\varepsilon \to 0$.

No renormalization for A! (Φ needs a $-C_\varepsilon \Phi_\varepsilon$ renormalization)

Non-abelian case

[Chandra-Chevyrev-Hairer-S. '20] Let $\xi^\varepsilon := \xi * \rho^\varepsilon$ (smooth mollification!)

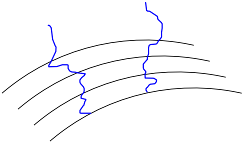
$$\partial_t A^\varepsilon_\mu = \Delta A^\varepsilon_\mu + \sum_{\nu=1}^{d} [A^\varepsilon_\nu, 2\partial_\nu A^\varepsilon_\mu - \partial_\mu A^\varepsilon_\nu + [A^\varepsilon_\nu, A^\varepsilon_\mu]] \underline{\underline{- C^\varepsilon A^\varepsilon_\mu + \xi^\varepsilon_\mu}}$$

where $C^\varepsilon \sim \frac{1}{\varepsilon}$ in $d = 3$ and $C^\varepsilon \sim O(1)$ in $d = 2$.

Gauge invariance lost for $\varepsilon > 0$. For gauge transformation g,

$$\xi_\varepsilon \mapsto g\xi_\varepsilon g^{-1} \quad \text{and} \quad C^\varepsilon A^\varepsilon \mapsto C^\varepsilon (A^\varepsilon - gdg^{-1})$$

Theorem. $\exists$ unique choice of C^ε, s.t. limit is gauge invariant



$$\partial_t A^\varepsilon = -d^*_{A^\varepsilon} F_{A^\varepsilon} - d_{A^\varepsilon} d^* A^\varepsilon + \xi^\varepsilon - C^\varepsilon A^\varepsilon \qquad \text{init.cond. } A^\varepsilon(0)$$

If g_ε are (time-dependent) gauge transformations solving

$$g_\varepsilon^{-1} \partial_t g_\varepsilon = -D^*_{A^\varepsilon}(g_\varepsilon^{-1} dg_\varepsilon) \qquad \text{init.cond. } g_\varepsilon(0)$$

then $B^\varepsilon(t) := g_\varepsilon(t) \circ A^\varepsilon(t)$ satisfies

$$\partial_t B^\varepsilon = -d^*_{B^\varepsilon} F_{B^\varepsilon} - d_{B^\varepsilon} d^* B^\varepsilon + \underline{\underline{g_\varepsilon \xi_\varepsilon g_\varepsilon^{-1} - C^\varepsilon (B^\varepsilon - g_\varepsilon dg_\varepsilon^{-1})}}$$

with initial condition $B^\varepsilon(0) = g_\varepsilon(0) \circ A^\varepsilon(0)$.

Key idea: Calculate renormalization for $g_\varepsilon \xi_\varepsilon g_\varepsilon^{-1}$. Conclusion: $\exists 1$ choice C_ε, s.t. in the limit, $(B, B(0)) \overset{law}{=} (A, B(0))$.

Hidden details...

1. Positive existing time (as a process on the orbit space)

2. 3D: $\mathrm{hol}_\gamma(\mathcal{F}_\delta A)$ (F_δ is deterministic YM flow)
   Orbit space defined by $A \sim \bar{A}$ iff $\mathcal{F}_\delta A \sim \mathcal{F}_\delta \bar{A}$ for some $\delta > 0$

Hidden details...

3. New framework of regularity structures
   Goal: vector-space-valued SPDEs coordinate-independently. Use category theory to transplant Hairer's original framework to new setting.
4. Couple gauge fields with other fields

Unclear at the moment...

– couple fermionic fields

– manifolds rather than torus.

– construct long time / full space solutions.

– correlation decay / mass gap etc.

– 4D.

Thank you for your attention!