Quantization days

Geometric quantisation

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- Can measure kinetic energy: $K = \frac{1}{2}mv^2$.
- Evolution: mechanical energy $H = K + V$ preserved (V some potential).

Phase-space

Classical phase-space = smooth manifold M (plus structure).
Points parametrise states.

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Here $M = \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$.
Conceptually $M = T\,Q$, tangent bundle: velocity vector tangent to trajectories $\mathbf{q} = \mathbf{q}(t)$.

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(Here Riemannian structure on M.)

State-observable duality

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In general *expected values* $\langle f \rangle_x$ of observables. There's a pairing (think: distributions).

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Equivalent to eqs. of motion *à la* Hamilton:

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}\,, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}\,.$$

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Define a vector field $X_H = X_H(\mathbf{q}, \mathbf{p})$ on M.

Symmetries

Association $H \mapsto X_H$. (Spoiler: *Poisson bracket*).

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Solve the Cauchy problem

$$\begin{cases} (\dot{\mathbf{q}}, \dot{\mathbf{p}}) = X_H(\mathbf{q}, \mathbf{p}), \\ \big(\mathbf{q}(0), \mathbf{p}(0)\big) = (\mathbf{q}_0, \mathbf{p}_0). \end{cases}$$

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Get one-parameter group $\varphi^H \colon \mathbb{R} \to \mathrm{Diff}(M)$, acting on observables:

$$U_t f(\mathbf{q}, \mathbf{p}) = f\big(\varphi_t^H(\mathbf{q}, \mathbf{p})\big).$$

Infinitesimal symmetries

Observables evolve: $f_t = U_t f$, $t \in \mathbb{R}$.

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$$\frac{\partial f_t}{\partial t} = \frac{\partial f_{s+t}}{\partial s}\bigg|_{s=0} = \frac{\partial (f_t \circ \varphi_s^H)}{\partial s}\bigg|_{s=0} = \mathrm{grad}(f_t) \cdot X_H \,.$$

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So:

$$\dot{f}_t = X_H(f_t) = \{ H, f_t \} := \frac{\partial H}{\partial p}\frac{\partial f_t}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial f_t}{\partial p} \,.$$

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This we abstract/generalise.

Time evolution

With an operation $\{\cdot,\cdot\}\colon C^\infty(M)\times C^\infty(M)\to C^\infty(M)$ (t.b. axiomatised).

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Equivalent to *Liouville* picture: states vary via Liouville eqs.
In general: study evolution of exp. values $(f,x)\mapsto\langle f\rangle_x$.

Poisson structure

Definition

A Poisson manifold is a pair $\big(M, \{\,\cdot,\cdot\,\}\big)$, where M is a smooth manifold and $\{\,\cdot,\cdot\,\}\colon C^\infty(M) \wedge C^\infty(M) \to C^\infty(M)$ a Lie bracket satisfying Leibnitz's rule:

$$\{\,H, fg\,\} = f\{\,H, g\,\} + \{\,H, f\,\}\,g\,, \qquad \text{for } H, f, g \in C^\infty(M)\,.$$

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Classical systems = Poisson manifolds with a Hamiltonian.

Phase-space

Still want states-observables duality and evolution.

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Example (Quantum point-particle in euclidean space)

Lebesgue measure $\mathrm{d}\mathbf{q}$ on $Q = \mathbb{R}^3$, $\mathcal{H} = L^2(Q, \mathbb{C})$, and

$$\langle \psi \mid \psi' \rangle := \int_Q \psi \overline{\psi'} \, \mathrm{d}\mathbf{q} \, .$$

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Multiplication $\mu_f\colon \psi \mapsto f\psi$ for $f \in L^\infty_{\mathrm{loc}}(Q)$, domain

$$\mathcal{D}(\mu_f) = \{\, \psi \in \mathcal{H} \mid f\psi \in \mathcal{H} \,\} \subseteq \mathcal{H}\,.$$

(Selfadjoint iff. $f(x) \in \mathbb{R}$ for dq-a.e. $x \in Q$.)

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The expected value of the observable $T\colon \mathcal{D}(T) \to \mathcal{H}$ in the state $\psi \in \mathbb{P}\big(\mathcal{D}(T)\big)$ is

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Note $\langle T \rangle_{e^{i\theta}\psi} = \langle T \rangle_\psi$, $\theta \in \mathbb{R}$.

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(It preserves selfadjoint-ness.)

Equivalent to Schrödinger picture:

$$\dot{\psi} = -\frac{i}{\hbar}\widehat{H}\psi\,.$$

Symmetries

Again evolution operators $U_t \in U(\mathcal{H})$:

$$U_t = \exp\left(-\frac{it}{\hbar}\widehat{H}\right), \qquad \text{whence} \qquad T_t = U_t^\dagger T U_t\,.$$

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Mathematically: compare with $\big(M, \{\,\cdot,\cdot\,\}\big)$. Obtain $\{\,\cdot,\cdot\,\}$ from $\{\,\cdot,\cdot\,\}_\hbar$ as $\hbar \to 0$.

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Thus $A \simeq A_0[\![\hbar]\!]$:

$$T \in A \Longrightarrow T = \sum_{k \geqslant 0} f_k \hbar^k, \qquad f_k \in A_0\,.$$

Products and commutators

Expand product of A_0 in A:

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Lives inside $\hbar A \subseteq A$: divide by $\hbar$.

Proposition

For $f, g \in A_0$ *define*

$$\{ f, g \}_A := \frac{1}{\hbar}[f, g] + \hbar A \,.$$

Then $\{ \cdot, \cdot \}_A : A_0 \wedge A_0 \to A_0$ *is a Poisson bracket.*

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Formalises $\frac{1}{\hbar}[f, g] - \{ f, g \} = O(\hbar)$.

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Theorem (Kontsevich: formality, 1997)

These always exist for $A_0 = C^\infty(M, \mathbb{C})$.

That's not all.

Goals and obstacles

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Construct quantum phase-spaces from Poisson manifolds: viewpoint of *geometric quantisation*.
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Theorem (Groenewold–Van Hove: no go, 1946–1951)

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Theorem (Groenewold–Van Hove: no go, 1946–1951)

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Still geo. quant. made significant advances (esp. in Mathematics).

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A complex line bundle on M is a smooth map $\pi\colon L \to M$ s.t.

- $L_x = \pi^{-1}(x) \simeq \mathbb{C}$ as vector spaces;
- for all $x \in M$ there is a nbhd. $U \subseteq M$ with a diffeo. $\pi^{-1}(U) \xrightarrow{\varphi_U} U \times \mathbb{C}$, and $\pi = \mathrm{pr}_U \circ \varphi_U$;
- if $U, V \subseteq M$ are like that then $\varphi_V \circ \varphi_U^{-1}\colon (U \cap V) \times \mathbb{C} \circlearrowleft$ is fibrewise linear.

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- if $U, V \subseteq M$ are like that then $\varphi_V \circ \varphi_U^{-1}\colon (U \cap V) \times \mathbb{C} \circlearrowleft$ is fibrewise linear.

Sections = maps $\psi\colon M \to L$ with $\pi \circ \psi = \mathrm{Id}_M$.

Back to favourite example

Example (Free point-particle)

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Denote $\Omega^0(M, L) = \left\{ \psi \in C^\infty(M, L) \,\middle|\, \psi \circ \pi = \mathrm{Id}_M \right\}$.

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Define

$$\langle \psi \mid \psi' \rangle = \int_M h(\psi, \psi')\nu\,,$$

$\nu \in \Omega^{\mathrm{top}}(M)$ volume form.

Example (Cont.)

Take $h\colon L \times L \to \mathbb{C}$ with

$$h\big((x,\eta),(x',\eta')\big) := \eta\overline{\eta'}\,, \qquad x,x' \in M, \eta,\eta' \in \mathbb{C}\,.$$

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Something to adjust... see later.

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Acts on $\Omega^0(M, L)$, extend to $\mathcal{D}(T_f) \subseteq \mathcal{H}$.

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Compatibility follows from sesquilinearity of h.

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Curvature

Definition

The curvature of the connection ∇ is the 2-form $F_\nabla \in \Omega^2(M, \mathbb{C})$ defined by

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Upshot: control curvature.

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Important class of Poisson manifolds.

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This we abstract/generalise.

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And prequantum operators via ∇.

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Let's see about symmetry.

About symmetry

Need to study

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$$\int_M (\mathcal{L}_X F)\nu = 0\,, \qquad F = h(\psi, \psi')\,.$$

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Proof.

$$\mathcal{L}_{X_f}\omega = \iota_{X_f} d\omega + d\langle \omega, X_f \wedge \cdot \rangle = d(df) = 0.$$

□

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Corollary (Liouville thm.)

Hamiltonian flows preserve the Liouville volume:

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Integral of exact form vanishes by Stokes ($\partial M = \varnothing$).

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Let $\kappa > 0$ be an integer.

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Later: semiclassical limit in Berezin–Toeplitz quantisation.

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Then choices parametrised by $H^1(M, U(1))$.

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$$[X_i, Y_j] = \delta_{ij}\,, \qquad \left[X_i, X_j\right] = \left[Y_i, Y_j\right] = 0\,.$$

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Representation $A \to \mathrm{Op}(\mathcal{H})$ via

$$X_i \longmapsto \mu_{q_i}\,, \qquad Y_i \longmapsto -\nabla_{\partial_{q_i}} = -\partial_{q_i}\,.$$

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Not irreducible: $T_i = \mu_{p_i}$ and $T_i' = \partial_{p_i}$ commute with representation.

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In turn $T_x \Lambda^{\perp_\omega} = \{ u \in T_x M \mid \omega_x(u, v) = 0 \text{ for } v \in T_x \Lambda \}$.

Real polarisations

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Correct the quantum phase-space.

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Crucial point: works *much* more generally.

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Take $P_{(\mathbf{q},\mathbf{p})}$ = complexification of $T_{(\mathbf{q},\mathbf{p})}\Lambda_{\mathbf{q}}$.

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More general: $\mathbb{S}^2$ has no real polarisation (hairy ball Thm.)

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For us: they have Kähler polarisation.

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Definition

A Kähler manifold is a triple (M, ω, I), where (M, ω) is symplectic and I is a complex structure s.t.

$$\langle \omega, IX \wedge IY \rangle = \langle \omega, X \wedge Y \rangle \,, \qquad \langle g, X \otimes X \rangle := \langle \omega, X \wedge IX \rangle > 0 \,,$$

for vector fields X, Y on M.

For us: they have Kähler polarisation.

Geo. quant. Hilbert space $\mathcal{H} = \mathcal{H}_I$ for them.

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So

$$\mathcal{H}_I = \left\{ \psi \in H^0(M, L) \,\middle|\, \int_M h(\psi, \psi)\nu < +\infty \right\} .$$

Identification sections/functions

Example (Cont.)

Holomorphic frame for $L = M \times \mathbb{C}$:

$$\sigma := \exp\left(-\frac{1}{2\hbar}|z|^2\right) .$$

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With annihilation/creation:

$$a_i^* = \mu_{z_i} , \qquad a_i = \partial_{z_i} .$$

Segal–Bargmann space II

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Hence $\langle \partial_{z_i} f \mid f \rangle = \langle f \mid \frac{z_i}{\hbar} f \rangle$.

Toeplitz operators

Problem: for $f \in L^\infty_{\mathrm{loc}}(M, \mathbb{C})$ we have $\mu_f \colon \mathcal{H}_I \to L^2(M, \mathbb{C})$, not landing into $\mathcal{H}_I$.

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Also called Berezin–Toeplitz quantisation.

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- $\|T_f^{(\hbar)}\| \leqslant \|f\|_{L^\infty}$, *and* $\|T_f^{(\hbar)}\| \to \|f\|_{L^\infty}$ *as* $\hbar \to 0$.

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Hence $f * g$ defined by $T_f T_g = T_{f*g}$ yields a $\hbar$-deformation.

THANK YOU FOR YOUR ATTENTION!