# From the random geometry of conformally invariant systems to the Kähler geometry of universal Teichmüller space

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Two-dimensional random conformal geometry combines techniques in complex analysis and stochastic analysis to study conformally invariant random objects in the plane. One of the central notions — the Schramm–Loewner evolution (SLE) — is a one-parameter family of random fractal planar curves without self-crossing. For some parameter values, SLE curves are scaling limits of interfaces in two-dimensional critical lattice models giving a new mathematical perspective to study 2D conformal field theory. Without invoking discrete models, the family of SLE curves can be defined directly in the continuum and is uniquely characterized by two seemingly weak properties: *conformal invariance* and *domain Markov property* (which will be explained below). The fact that there is only one free parameter in the definition of SLE is rooted in the universality of Brownian motion, one of the most fundamental concepts in stochastic analysis.

This article will explain that a natural quantity associated with each Jordan curve — Loewner energy — whose definition arises from SLE, is connected to the Kähler geometry of universal Teichmüller space. Universal Teichmüller space $T(1)$ is an infinite-dimensional complex Banach manifold and a homogeneous space which contains all Teichmüller spaces of hyperbolic surfaces. There are several equivalent ways to describe universal Teichmüller space, one of which is by identifying it with a particular family of Jordan curves — quasicircles. Universal Teichmüller space is endowed with many geometric structures; in particular, it has an essentially unique homogeneous Kähler metric. None of those geometric structures is directly related to stochastic processes. However, its Kähler potential, defined on the connected component of the circle (i.e., Weil–Petersson Teichmüller space) and called the universal Liouville action, surprisingly coincides with the Loewner energy.

The profound reason behind this identity, which is hard to believe to be mere coincidence, is still largely mysterious. This article aims to give the background on Loewner energy, the universal Liouville action, and the intuition behind the proof of the identity. We will also discuss the implications of this link and the connections between Loewner energy and other mathematical subjects.

## Conformally invariant simple random curves in the plane

Critical lattice models provide several important examples of conformally invariant random self-avoiding curves. We mention loop-erased random walk, spin cluster interfaces in the critical Ising model, the level line of a discrete Gaussian free field, etc. We would like to describe what lattice size scaling limits of such curves may look like. Physicists predict that conformal symmetries emerge for a wide-range of well-chosen statistical mechanics models [2, 17]. For instance, the simple random walk in $\mathbb{Z}^2$ converges in the scaling limit to the planar Brownian motion, whose trajectory is indeed conformally invariant up to reparametrization. However, since we deal with self-avoiding paths, any scaling limit should still be non self-crossing and cannot be a Markov process. (The future path of a Markov process only depends on the current location. For instance, Brownian motion hits itself infinitely many times on arbitrarily small time interval.)

By considering properties expected from the discrete models as predicted by physics, O. Schramm formulated a mathematical description of random simple curves with conformal symmetries [22]. To make it more tractable, he first assumed that the random simple curve connects two distinct boundary points (prime ends) $a, b$ in a simply connected domain $D \subsetneq \mathbb{C}$. We call such a simple curve a *chord* in $(D; a, b)$ and oriented from $a$ to $b$. This setup will allow us to progressively slit open the curve starting from $a$, so that the remaining part of the curve will be a chord connecting the endpoint of the slit and $b$.

One advantage of working in two dimensions is that there is a rich family of conformal maps, given by biholomorphic functions (so that we may use single-variable complex analysis). In particular, the Riemann mapping theorem states that there exists a conformal map $\varphi$ sending the simply connected domain $D$ to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \colon \operatorname{Im}(z) > 0\}$, which maps $a$ to $0 \in \partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ and $b$ to $\infty$. Another choice of such conformal map has to be of the form $\lambda\varphi$ for some $\lambda > 0$.

Schramm pointed out that the random simple chord has to satisfy two properties:

- *Conformal invariance:* The random chord $\gamma$ in $\mathbb{H}$ connecting 0 to $\infty$ should have the same law as its image under the map $z \mapsto \lambda z$ for every $\lambda > 0$. This then allows us to define the random chord in $(D; a, b)$ as the preimage under $\varphi$.
- *Domain Markov property:* Chords of $(\mathbb{H}; 0, \infty)$ have a parametrization by capacity (explained below). If we slit the random chord $\gamma$ in $(\mathbb{H}; 0, \infty)$ from $\gamma(0) = 0$ up to $\gamma(s)$ and map conformally $(\mathbb{H} \smallsetminus \gamma[0, s]; \gamma(s), \infty)$ to $(\mathbb{H}; 0, \infty)$, then the image of $\gamma[s, \infty)$ should have the same law as $\gamma$ and be independent of $\gamma[0, s]$, for every $s > 0$.

Indeed, many scaling limits of the interfaces in critical lattice models are proved to satisfy these axioms, e.g., [22, 23, 26]. However, we do not need to invoke discrete models if we use these two properties as the axiomatic definition of the random chord of interest in the continuum. Schramm noticed that the Loewner transform [15] provides a perfect tool to describe it. The *Loewner transform* encodes each (deterministic) chord in $(\mathbb{H}; 0, \infty)$ into a continuous real-valued function. More precisely, we say that $\gamma$ is *parametrized by capacity* if the unique conformal map $g_t : \mathbb{H} \smallsetminus \gamma[0, t] \to \mathbb{H}$, normalized such that the expansion at infinity is given by $g_t(z) = z + o(1)$, has in fact the next term in the expansion being $2t/z$,

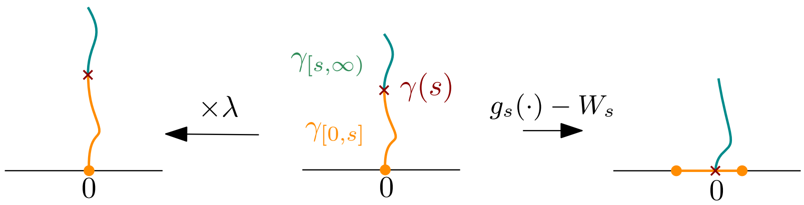


Figure 1: The left arrow illustrates a scaling map, and the right arrow illustrates the uniformizing conformal map from the slit domain onto the upper half-plane. The law of the random chord should be identical in these three pictures.

namely,

$$g_t(z) = z + \frac{2t}{z} + o\Big(\frac{1}{z}\Big).$$

The image of the tip $\gamma(t)$ of $\gamma[0,t]$ under $g_t$ is denoted as $W_t \in \mathbb{R}$. We denote by $T \in [0,\infty]$ the total capacity of $\gamma$. As $t$ varies in $[0,T)$, $t \mapsto W_t$ is continuous and starting from $W_0 = 0$. We call $W$ the *Loewner driving function* of $\gamma$.

It is a simple exercise to check that the two properties of the random simple chord above translate into the properties of the random driving function $W$:

- $W$ has the same law as the function $t \mapsto \lambda W_{\lambda^{-2}t}$, for every $\lambda > 0$;
- for every $s > 0$, $t \mapsto W_{t+s} - W_s$ has the same law as $W$ and is independent of $W|_{[0,s]}$.

It turns out the only possible random continuous function satisfying these properties is of the form of $\sqrt{\kappa}B$, where $B$ is the standard Brownian motion and $\kappa \ge 0$. In fact, the second property guarantees $W$ to be a continuous Lévy process. The classification of continuous Lévy processes tells us that it is of the form $t \mapsto \sqrt{\kappa}B_t + at$, for some $a \in \mathbb{R}$. One may view the classification as the manifestation of two most fundamental theorems in probability theory: the law of large numbers (which explains the occurrence of the deterministic drift $t \mapsto at$), central limit theorem (which explains the occurrence of the Gaussian process $\sqrt{k}B$). The first property then implies that $\lambda^{-1}a = a$, which shows $a = 0$.

*Schramm–Loewner evolution* $\mathrm{SLE}_\kappa$ in $(\mathbb{H};0,\infty)$ is the random curve whose driving function is $\sqrt{\kappa}B$, for $\kappa \ge 0$. More precisely, when $\kappa \le 4$, $\mathrm{SLE}_\kappa$ is indeed the random simple chord with driving function $\sqrt{\kappa}B$ in the sense described above; for $\kappa > 4$, the Loewner driving function does not define a growing slit $\gamma[0,t]$ but a growing compact set $K_t \subset \overline{\mathbb{H}}$. In this case, $\mathrm{SLE}_\kappa$ is referred to the curve which carves out progressively the boundary of $K_t$ (when $\kappa \ge 8$, $\mathrm{SLE}_\kappa$ is a random space-filling curve) [20]. In this article, we will only consider the case where $\kappa \le 4$, so we do not enter into further details to discuss the $\kappa > 4$ case. As we explained above, SLEs are the only random chords which satisfy the conformal invariance and domain Markov property. The $\mathrm{SLE}_\kappa$ in another domain $(D;a,b)$ is defined as the preimage of $\mathrm{SLE}_\kappa$ in $(\mathbb{H};0,\infty)$ under any conformal map $\varphi : D \to \mathbb{H}$ sending respectively $a, b$ to $0, \infty$.

We note that when $\kappa = 0$, $\mathrm{SLE}_0$ in $(\mathbb{H};0,\infty)$ is the imaginary axis and $g_t(z) = \sqrt{z^2 + 4t}$ is the corresponding conformal map $\mathbb{H} \smallsetminus \mathfrak{i}[0, 2\sqrt{t}] \to \mathbb{H}$.

## Loewner energy and SLE

We will now slowly move out from the probability world by only looking at a deterministic functional that arises from SLE. Schilder's theorem states that the large deviation rate function of $(\sqrt{\kappa}B)_{\kappa\to 0+}$ is given by the Dirichlet energy

$$I(W) := \frac{1}{2}\int_0^\infty \dot{W}_t^2 \, \mathrm{d}t,$$

where $\dot{W}_t := \mathrm{d}W_t/\mathrm{d}t$. Roughly speaking, the large deviation rate function is the exponential decay rate (as $\kappa \to 0+$) of the probability of $\sqrt{\kappa}B$ staying close to a given a deterministic real-valued function $W$ (it is a rare event when $W$ is not the constant 0 function):

$$-\kappa \log \mathbb{P}(\sqrt{\kappa}B \text{ stays close to } W) \sim_{\kappa\to 0+} I(W).$$

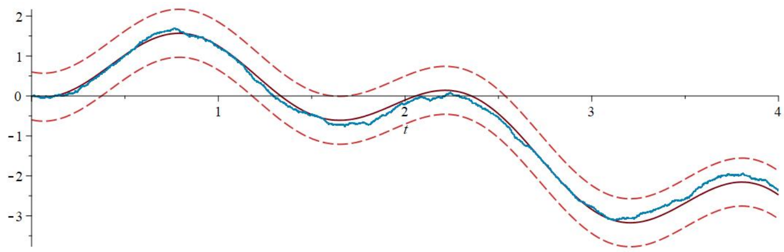


Figure 2: An illustration of the rare event of $\sqrt{\kappa}B$ being close to a deterministic function $W$ (whose graph is the solid red line). The blue curve is a simulation of $\sqrt{\kappa}B$ with $\kappa = 0.1$ over 10000 steps.

Then it is not surprising that using an appropriate topology (proved in [18] for the Hausdorff metric) that a similar large deviation principle holds for $\mathrm{SLE}_{0+}$: given a chord $\gamma$ in the domain $(D; a, b)$,

$$-\kappa \log \mathbb{P}(\mathrm{SLE}_\kappa \text{ in } (D;a,b) \text{ stays close to } \gamma) \sim_{\kappa\to 0+} I^C_{D;a,b}(\gamma). \tag{1}$$

Here $I^C_{D;a,b}(\gamma) := I(W)$ is called the *chordal Loewner energy* of $\gamma$, where $W$ is the driving function of $\varphi(\gamma)$ and $\varphi$ is any conformal map sending $(D; a, b)$ onto $(\mathbb{H}; 0, \infty)$.

We can also generalize the Loewner energy to Jordan curves (simple loops) and denote it as $I^L$. This generalization will show more symmetries and we will focus on the energy for Jordan curves from now on. It is a generalization of the chordal energy because of the property:

$$I^L(\gamma \cup \mathbb{R}_+) = I^C_{\mathbb{C}\smallsetminus\mathbb{R}_+;0,\infty}(\gamma) \tag{2}$$

for every simple chord $\gamma$ in $(\mathbb{C} \smallsetminus \mathbb{R}_+; 0, \infty)$. More precisely, let $\gamma : [0, 1] \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be a continuously parametrized Jordan curve with $\gamma(0) = \gamma(1)$. For every $\varepsilon > 0$, $\gamma[\varepsilon, 1]$ is a chord connecting $\gamma(\varepsilon)$ to $\gamma(1)$ in the simply connected domain $\hat{\mathbb{C}} \smallsetminus \gamma[0, \varepsilon]$.

**Definition 1.** The *rooted loop Loewner energy* of $\gamma$ rooted at $\gamma(0)$ is defined as

$$I^L(\gamma, \gamma(0)) := \lim_{\varepsilon\to 0} I^C_{\hat{\mathbb{C}}\smallsetminus\gamma[0,\varepsilon];\gamma(\varepsilon),\gamma(0)}(\gamma[\varepsilon, 1]). \tag{3}$$

It turns out that the definition does not depend on the choice of the orientation of the curve [30] nor on its root [21]. Therefore, we omit the root in the notation and (2) can be seen by putting the root at $\infty$ and orient the curve from $\infty \to 0$ along $\mathbb{R}_+$. The independence of the Loewner energy from the parametrization is not obvious from the definition, since the chordal energies $I^C$ are defined using the Loewner driving function which depends strongly on the past of the curve. This independence suggests that there must be an intrinsic expression of the Loewner energy which does not use any parametrization of the Jordan curve. The answer is indeed given by the identity with the universal Liouville action that will be the subject of the next section.

We remark that since the Loewner energy is defined via uniformizing maps (to define the driving function), it is invariant under conformal automorphisms of $\hat{\mathbb{C}}$ (i.e., Möbius transformations $z \mapsto \frac{az+b}{cz+d}$). Moreover, the Loewner energy is zero if and only if $\gamma$ is a circle and Möbius transformations map a circle to a circle. Therefore, the Loewner energy may be viewed as a quantity measuring the roundness of the unparametrized Jordan curve.

The loop energy is, as one may guess, related to the loop version of $\mathrm{SLE}_\kappa$, constructed in [3, 32, 34] and called *$SLE_\kappa$ loop measure* $\mu^\kappa$. In this case, one may even not have to let $\kappa \to 0+$. A recent work [11] shows that for a fixed $\kappa \le 4$,

$$\lim_{\varepsilon \to 0} \frac{\mu^\kappa\left(O_\varepsilon(\gamma)\right)}{\mu^\kappa\left(O_\varepsilon(S^1)\right)} = \exp\left(\frac{c(\kappa)}{24} I^L(\gamma)\right), \tag{4}$$

where $c(\kappa) := (6-\kappa)(3\kappa-8)/2\kappa$ is the *central charge* (terminology coming from the connection between SLE and conformal field theory) of $\mathrm{SLE}_\kappa$, $S^1$ is the unit circle, and $O_\varepsilon$ is a sort of $\varepsilon$-neighborhood of $\gamma$ in the loop space. In physics terminology, $(c(\kappa)/24)I^L$ is the *action*, or the *Onsager–Machlup functional*, of the $\mathrm{SLE}_\kappa$ loop measure. On may also check that as $\kappa \to 0+$, $c(\kappa) \sim -24/\kappa$ and we recover the asymptotics similar to (1).

## Identity with the Universal Liouville action

We will focus on the Loewner energy for Jordan curves (and forget about its relation to SLE for a moment). The following theorem gives an equivalent expression the Loewner energy in terms of only two conformal maps. Since the Loewner energy is invariant under Möbius transformations, without loss of generality, we may assume that the Jordan curve $\gamma$ does not pass through $\infty$.

**Theorem 2** (See [31, Thm. 1.4])**.** *Let $\Omega$ (resp., $\Omega^*$) denote the component of $\hat{\mathbb{C}} \smallsetminus \gamma$ which does not contain $\infty$ (resp., which contains $\infty$) and $f$ (resp., $g$) be a conformal map from the unit disk $\mathbb{D} = \{z \in \hat{\mathbb{C}} \colon |z| < 1\}$ onto $\Omega$ (resp., from $\mathbb{D}^* = \{z \in \hat{\mathbb{C}} \colon |z| > 1\}$ onto $\Omega^*$). We assume further that $g(\infty) = \infty$. The Loewner energy of $\gamma$ satisfies*

$$I^L(\gamma) = \frac{1}{\pi}\int_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 \mathrm{d}^2 z + \frac{1}{\pi}\int_{\mathbb{D}^*} \left|\frac{g''}{g'}\right|^2 \mathrm{d}^2 z + 4\log\left|\frac{f'(0)}{g'(\infty)}\right| =: \frac{\mathbf{S_1}(\gamma)}{\pi}, \tag{5}$$

*where $g'(\infty) := \lim_{z\to\infty} g'(z)$ and $\mathrm{d}^2 z$ is the Euclidean area measure.*

The quantity $\mathbf{S_1}$ is introduced in [27] under the name *universal Liouville action*. Its value does not depend on the choice of $f$ and $g$ as long as $g(\infty) = \infty$. A Jordan curve for which $\int_{\mathbb{D}} |f''/f'|^2 \,\mathrm{d}^2 z$ is finite is called a *Weil–Petersson quasicircle.* It turns out that $\int_{\mathbb{D}} |f''/f'|^2 \,\mathrm{d}^2 z$ is finite if and only if $\int_{\mathbb{D}^*} |g''/g'|^2 \,\mathrm{d}^2 z$ is finite. Hence, we have:

**Corollary 3.** *A Jordan curve $\gamma$ has finite Loewner energy if and only if $\gamma$ is a Weil–Petersson quasicircle.*

Weil–Petersson quasicircles have well-defined arclength and have Hausdorff dimension one. Therefore, finite Loewner energy curves are more regular than $\mathrm{SLE}_\kappa$ curves, which have Hausdorff dimension $(1 + \kappa/8) \wedge 2$ by [1]. This is not surprising as functions with finite Dirichlet energy are also more regular than a Brownian path.

As mentioned above, the universal Liouville action does not depend on any special point on the curve $\gamma$ and has the advantage of involving only two conformal maps. Whereas to define the Loewner energy through the driving function, one has to study the whole family of uniformizing conformal mappings of the slit domains. However, the way of considering the Jordan curve as a progressively growing slit (which closes up on itself) allowed us to relate the Loewner energy to SLE curves.

## Weil–Petersson Teichmüller space

The universal Liouville action arises from a very different context — Teichmüller theory. Teichmüller spaces are introduced by Teichmüller to parametrize the family of complex structures on a surface using quasiconformal mappings. In particular, the Teichmüller space of a genus $g \geq 2$ closed surface is homeomorphic to $\mathbb{R}^{6g-6}$. We are interested in the universal Teichmüller space which contains all Teichmüller spaces of surfaces of negative Euler characteristics, hence the name *universal*, which can also be represented by the class of quasicircles.

More precisely, we first identify a Jordan curve $\gamma$ with a homeomorphism of the unit circle $S^1 = \partial\mathbb{D} = \partial\mathbb{D}^*$ as follows. By Carathéodory theorem, any conformal map $f : \mathbb{D} \to \Omega$ (resp., $g : \mathbb{D}^* \to \Omega^*$) extends continuously to a homeomorphism between the closures $\overline{\mathbb{D}} \to \overline{\Omega}$ (resp., $\overline{\mathbb{D}^*} \to \overline{\Omega^*}$). In particular, $f$ and $g$ restricted to $S^1$ define two homeomorphisms $S^1 \to \gamma$. The *welding homeomorphism* compares these two homeomorphisms and is defined as the circle homeomorphism $\varphi := g^{-1} \circ f|_{S^1}$.

The converse operation — solving the conformal welding problem — consists of finding a Jordan curve $\gamma$ and corresponding conformal maps $f$ and $g$, whose welding homeomorphism $g^{-1} \circ f|_{S^1}$ is a given circle homeomorphism $\varphi$. We note that if $\gamma$ is a solution, then $A \circ \gamma$ is also a solution (by replacing $f$ by $A \circ f$ and $g$ by $A \circ g$), where $A$ is any Möbius transformation of $\hat{\mathbb{C}}$. Solution may not exist, and if one exists, it may not be unique (up to post-composition by Möbius transformations), see, e.g., [5]. The complete characterization of circle homeomorphisms for which there is a unique solution to conformal welding is an open question, [7] shows that the question is indeed hard and the corresponding Jordan curves form a set which is *not* even Borel measurable.

However, classical results in quasiconformal mappings by Beurling and Ahlfors [4] show that if the circle homeomorphism is *quasisymmetric*, then the solution to the conformal welding problem exists and is unique up to post-composition by Möbius transformations.

The corresponding Jordan curves are called *quasicircles.* Let

$$\mathrm{QS}(S^1) := \left\{\varphi \in \mathrm{Hom}(S^1) \colon \exists M > 1,\, \forall \theta \in \mathbb{R},\, \forall t \in (0,\pi),\, \frac{1}{M} \le \left|\frac{\varphi(e^{\mathfrak{i}(\theta+t)}) - \varphi(e^{\mathfrak{i}\theta})}{\varphi(e^{\mathfrak{i}\theta}) - \varphi(e^{\mathfrak{i}(\theta-t)})}\right| \le M\right\}$$

denote the group of quasisymmetric circle homeomorphisms. Let $\varphi \in \mathrm{QS}(S^1)$. We fix a normalization to the solution of welding problem for $\varphi$ by assuming that the conformal map $f$ satisfies $f(0) = 0, f'(0) = 1$ and $f''(0) = 0$ and put no condition on $g$ (except that $g(\mathbb{D}^*) = \hat{\mathbb{C}} \smallsetminus \overline{f(\mathbb{D})}$). In other words, we consider the welding homeomorphism to be in the homogeneous space $\mathrm{Möb}(S^1)\backslash \mathrm{QS}(S^1)$, where $\mathrm{Möb}(S^1)$ is the group of Möbius transformations preserving $S^1$ (since $g$ can be replaced by $g \circ B$ for any $B \in \mathrm{Möb}(S^1)$). The homogeneous space $T(1) := \mathrm{Möb}(S^1)\backslash \mathrm{QS}(S^1)$ is called the *universal Teichmüller space.*

Universal Teichmüller space has a structure of infinite dimensional complex Banach manifold such that the group $\mathrm{QS}(S^1)$ acts on the right on $T(1)$ holomorphically. One wonders whether it can be further equipped with a *Kähler metric*, namely, a symplectic form $\omega(\cdot,\cdot)$ and Riemannian metric $\langle\cdot,\cdot\rangle$ that are invariant under the right action and compatible with the complex structure (encoded in an operator $J$ on the tangent bundle such that $J^2 = -\mathrm{Id}$ which plays the role of "multiplication by $\mathfrak{i}$"). Being compatible means $\omega(\cdot, J(\cdot)) = \langle\cdot,\cdot\rangle$. This question has been addressed by string theorists [8] and [33] who consider only the *smooth part* $\mathrm{Möb}(S^1)\backslash \mathrm{Diff}^\infty(S^1)$ of the universal Teichmüller space without having to worry about any convergence issue on this infinite dimensional manifold. It turns out there is a *unique* Kähler metric up to a scaling factor.

Let us explain briefly how to derive this Kähler metric (also ignoring the convergence question). Concretely, the tangent space at $[\mathrm{Id}] \in T(1)$ consists of vector fields $\mathrm{Vect}(S^1)$ on $S^1$ with Fourier expansion:

$$v = \sum_{n \neq \pm 1, 0} v_n e_n := \sum_{n\neq \pm 1,0} v_n e^{\mathfrak{i}n\theta}\frac{\partial}{\partial\theta} \quad \text{satisfying } \overline{v}_n = v_{-n}.$$

We have omitted the Fourier modes for $n \in \{\pm 1, 0\}$ as they generate the Lie algebra of $\mathrm{Möb}(S^1)$.

The *almost complex structure* $J : \mathrm{Vect}(S^1) \to \mathrm{Vect}(S^1)$ (such that $J^2 = -I$) induced from the complex structure is given by the Hilbert transform [16]:

$$Jv = \mathfrak{i}\sum_{n=2}^{\infty} v_n e_n - \mathfrak{i}\sum_{n=-\infty}^{-2} v_n e_n.$$

The family $\{e_n := e^{\mathfrak{i}n\theta}\partial/\partial\theta\}_{n\neq\pm 1,0}$ generates the complexification of $\mathrm{Vect}(S^1)$:

$$\mathrm{Vect}^{\mathbb{C}}(S^1) = \{\sum_{n\neq\pm 1,0} u_n e_n \mid u_n \in \mathbb{C}\},$$

with the Lie bracket

$$[e_m, e_n] = \mathfrak{i}(n-m)e_{n+m}.$$

**Theorem 4** (See [8])**.** *Up to a scaling factor, there is a unique homogeneous Kähler metric on* $\mathrm{Möb}(S^1)\backslash \mathrm{Diff}^\infty(S^1)$*. The symplectic form (closed and non-degenerate* 2*-form) is given* *by* $\forall u, v \in$ *Vect*$(S^1)$

$$\omega(u,v) = -\omega(v,u) = -\alpha \operatorname{Im}\left(\sum_{n=2}^{\infty}(n^3-n)u_n\overline{v_n}\right),$$

*where* $\alpha \in \mathbb{R}_+$. *The symmetric* 2*-tensor* $\langle\cdot,\cdot\rangle$ *that is compatible with* $\omega$ *and* $J$*, in the sense*

$$\langle u, v\rangle = \omega(u, Jv) = \alpha \operatorname{Re}\left(\sum_{n=2}^{\infty}(n^3-n)u_n\overline{v_n}\right)$$

*is positive and definite. This metric is called the* Weil–Petersson metric*.*

*Proof.* Assume that $\omega$ is a homogeneous symplectic form. Since $\omega$ is closed,

$$\begin{aligned}0 = \mathrm{d}\omega(e_m, e_n, e_p) = {} & e_m(\omega(e_n, e_p)) + e_n(\omega(e_p, e_m)) + e_p(\omega(e_m, e_n)) \\ & - \omega([e_m, e_n], e_p) - \omega([e_n, e_p], e_m) - \omega([e_p, e_m], e_n).\end{aligned}$$

By homogeneity, the first three terms on the right-hand side vanish and we have

$$\omega([e_m, e_n], e_p) + \omega([e_n, e_p], e_m) + \omega([e_p, e_m], e_n)) = 0. \tag{6}$$

Moreover, $\omega$ has kernel spanned by $e_{-1}, e_0$ and $e_1$. From these constraints we can determine $\omega$ as follows.

By taking $p = 0$, (6) gives that $(n+m)\omega(e_m, e_n) = 0$. Therefore $\omega(e_m, e_n) = 0$ when $m \neq -n$. We let $a_m := \omega(e_m, e_{-m})$. Take $p = -m-1$, $n = 1$, (6) gives

$$(1-m)a_{m+1} + (m+2)a_m = 0$$

which implies there exists $\alpha \in \mathbb{C}$ such that $a_m = \mathrm{i}\alpha(m^3-m)/2$ for all $m \geq 2$ (hence, for all $m \in \mathbb{Z}$).

When $u, v \in \mathrm{Vect}(S^1)$, we have $u_{-m} = \overline{u_m}$ and similarly for $v$. Therefore,

$$\omega(u,v) = \frac{\mathrm{i}\alpha}{2}\sum_{m\neq \pm 1, 0}(m^3-m)u_m v_{-m} = -\alpha \operatorname{Im}\left(\sum_{m=2}^{\infty}(m^3-m)u_m\overline{v_m}\right)$$

and the compatible symmetric tensor

$$\langle u, v\rangle := \omega(u, Jv) = \alpha \operatorname{Re}\left(\sum_{m=2}^{\infty}(m^3-m)u_m\overline{v_m}\right).$$

We obtain $\alpha > 0$ from the assumption that $\langle\cdot,\cdot\rangle$ is positive definite. □

Takhtajan and Teo [27] defined and extended rigorously the infinite-dimensional Kähler manifold structure and the Weil–Petersson metric to $T(1)$. In fact, the subspace of $u \in \mathrm{Vect}(S^1)$ such that $\langle u, u\rangle < \infty$ coincides with the $H^{3/2}$ Sobolev space of vector fields (which is strictly smaller than the tangent space of $T(1)$ which is given by the space of Zygmund vector fields). The image of $H^{3/2}$ under the right action by $\mathrm{QS}(S^1)$ on $T(1)$ defines a tangent subbundle.

**Theorem 5** (See [27]). *The connected component $T_0(1)$ of the integral manifold containing $[\mathrm{Id}] \in T(1)$ — called* the Weil–Petersson Teichmüller space *— is a complete, infinite-dimensional Kähler–Einstein manifold with negative curvatures. Moreover, $[\varphi] \in T_0(1)$ if and only if $\varphi = g^{-1} \circ f|_{S^1}$ where $\int_{\mathbb{D}} |f''/f'|^2 \,\mathrm{d}^2 z < \infty$.*

Therefore, $T_0(1)$ is the completion of $\text{Möb}(S^1)\backslash \text{Diff}^\infty(S^1)$. Moreover, a Jordan curve is associated with an element in $T_0(1)$ via conformal welding if and only if it is a Weil–Petersson quasicircle. There are many equivalent characterizations of the elements in $T_0(1)$, see, e.g., [6] for an extensive summary. In particular, Shen [25] showed that $[\varphi] \in T_0(1)$ if and only if $\log \varphi'$ is in the Sobolev space $H^{1/2}$. Kähler metrics admit locally defined Kähler potential. Here, the Weil–Petersson metric has a globally defined potential given by the universal Liouville action:

**Theorem 6** (See [27, Cor. II.4.2]). *The universal Liouville action $\mathbf{S}_1 : T_0(1) \to \mathbb{R}_+$ is a Kähler potential for the Weil–Petersson metric. That is, up to a positive scaling factor,*

$$\mathrm{i}\partial\bar{\partial}\mathbf{S}_1 = \omega,$$

*where $\omega$ is the symplectic form in Theorem 4.*

## How do we come up with the identity?

We now explain how we could guess the identity between the Loewner energy and the universal Liouville action in Theorem 2 using ideas from random conformal geometry. The first step of the proof is to show the following identity for a curve passing through $\infty$.

**Theorem 7** (See [31, Thm. 1.1]). *If $\gamma$ is a Jordan curve passing through $\infty$, then*

$$I^L(\gamma) = \frac{1}{\pi}\int_{\mathbb{H}} |\nabla \log |f'|\,|^2 \mathrm{d}^2 z + \frac{1}{\pi}\int_{\mathbb{H}^*} |\nabla \log |g'|\,|^2 \mathrm{d}^2 z = \frac{1}{\pi}\int_{\mathbb{H}} \left|\frac{f''}{f'}\right|^2 \mathrm{d}^2 z + \frac{1}{\pi}\int_{\mathbb{H}^*} \left|\frac{g''}{g'}\right|^2 \mathrm{d}^2 z$$

*where $f$ and $g$ map conformally the upper half-plane $\mathbb{H}$ and the lower half-plane $\mathbb{H}^*$ onto $H$ and $H^*$, the two components of $\mathbb{C} \smallsetminus \gamma$ respectively, while fixing $\infty$.*

This theorem can be viewed as the finite energy analog of the quantum zipper coupling between SLE and Gaussian free field (GFF) [12, 24] that we now explain. We do not make rigorous statements and only argue heuristically here.

A quantum surface is a domain $D$ equipped with a Liouville quantum gravity ($\sqrt{\kappa}$-LQG) measure, defined using a regularization of $e^{\sqrt{\kappa}\Phi}\mathrm{d}^2 z$, where $\sqrt{\kappa} \in (0, 2)$, and $\Phi$ is a Gaussian field with the covariance of a free boundary GFF. GFF is a random real-valued Schwartz distribution defined on the two-dimensional domain $D$ and is the analog of Brownian motion by replacing the time interval by $D$. Schilder's theorem for Gaussian measures shows that the large deviation rate function for $\sqrt{k}\Phi$, as $\kappa \to 0+$, is the Dirichlet energy on $D$, defined for all $\varphi \in W^{1,2}_{loc}$ as

$$\mathcal{D}_D(\varphi) := \frac{1}{4\pi}\int_D |\nabla \varphi|^2 \mathrm{d}^2 z \in [0, \infty].$$

If $\mathcal{D}_D(\varphi) < \infty$, we say that $\varphi \in \mathcal{E}(D)$.

We use the following dictionary illustrating the analogy between the concepts in random conformal geometry (left column) and their large deviation counterparts (right column).

| SLE/GFF with $\kappa \to 0+$ | Finite energy |
|---|---|
| $\mathrm{SLE}_\kappa$ loop | Jordan curve $\gamma$ with $I^L(\gamma) < \infty$ i.e., a Weil–Petersson quasicircle |
| Free boundary GFF $\sqrt{\kappa}\Phi$ on $\mathbb{H}$ (on $\mathbb{C}$) | $2u \in \mathcal{E}(\mathbb{H})$ $(2\varphi \in \mathcal{E}(\mathbb{C}))$ |
| $\sqrt{\kappa}$-LQG on quantum plane $\approx e^{\sqrt{\kappa}\Phi} \mathrm{d}^2 z$ | measure on $\mathbb{C}$: $e^{2\varphi}\, \mathrm{d}^2 z$, $\varphi \in \mathcal{E}(\mathbb{C})$ |
| $\sqrt{\kappa}$-LQG on quantum half-plane on $\mathbb{H}$ | measure on $\mathbb{H}$: $e^{2u}\, \mathrm{d}^2 z$, $u \in \mathcal{E}(\mathbb{H})$ |
| Quantum zipper coupling: " $\mathrm{SLE}_\kappa$ cuts an independent quantum plane $e^{\sqrt{\kappa}\Phi}\mathrm{d}^2 z$ into two independent quantum half-planes $e^{\sqrt{\kappa}\Phi_1}, e^{\sqrt{\kappa}\Phi_2}$ " | A Weil–Petersson quasicircle $\gamma$ cuts $\mathbb{C}$ with measure $e^{2\varphi}\mathrm{d}^2 z$, where $\varphi \in \mathcal{E}(\mathbb{C})$, into half-planes with measure $e^{2u}\mathrm{d}^2 z$ and $e^{2v}\mathrm{d}^2 z$ with $u \in \mathcal{E}(\mathbb{H}), v \in \mathcal{E}(\mathbb{H}^*)$, and $I^L(\gamma) + \mathcal{D}_{\mathbb{C}}(2\varphi) = \mathcal{D}_{\mathbb{H}}(2u) + \mathcal{D}_{\mathbb{H}^*}(2v)$ |

In the last line, two domains $D$ and $D'$ equipped with a measure are considered equivalent if there exist a conformal map $D \to D'$ such that the measure on $D'$ equals the pushforward of the measure on $D$. In particular, if a Jordan curve $\gamma$ cuts $\mathbb{C}$ into two domains $H$ and $H^*$ as above and $f$ and $g$ are the conformal maps in Theorem 7, then $e^{2\varphi}\mathrm{d}^2 z$ on $H$ and on $H^*$ are equivalent, respectively, to $e^{2u}\mathrm{d}^2 z$ on $\mathbb{H}$ and $e^{2v}\mathrm{d}^2 z$ on $\mathbb{H}^*$ where

$$u = \varphi \circ f + \log |f'|, \quad v = \varphi \circ g + \log |g'|.$$

The identity $I^L(\gamma) + \mathcal{D}_{\mathbb{C}}(2\varphi) = \mathcal{D}_{\mathbb{H}}(2u) + \mathcal{D}_{\mathbb{H}^*}(2v)$ is more general than Theorem 7 (which corresponds to the case where $\varphi \equiv 0$) and we argue heuristically as follows. From the quantum zipper coupling, one expects that using an appropriate choice of topology and for small $\kappa$,

$$\begin{aligned} &\text{“}\mathbb{P}(\mathrm{SLE}_\kappa \text{ loop stays close to } \gamma,\ \sqrt{\kappa}\Phi \text{ stays close to } 2\varphi) \\ &= \mathbb{P}(\sqrt{\kappa}\Phi_1 \text{ stays close to } 2u,\ \sqrt{\kappa}\Phi_2 \text{ stays close to } 2v)\text{”}. \end{aligned} \tag{7}$$

We obtain from the independence between SLE and $\Phi$, the large deviation principle of SLE (1), and the large deviation principle for GFF (Schilder's theorem for Gaussian measure)

$$\begin{aligned} &\text{“} \lim_{\kappa \to 0} -\kappa \log \mathbb{P}(\mathrm{SLE}_\kappa \text{ stays close to } \gamma,\ \sqrt{\kappa}\Phi \text{ stays close to } 2\varphi) \\ &= \lim_{\kappa \to 0} -\kappa \log \mathbb{P}(\mathrm{SLE}_\kappa \text{ stays close to } \gamma) + \lim_{\kappa \to 0} -\kappa \log \mathbb{P}(\sqrt{\kappa}\Phi \text{ stays close to } 2\varphi) \\ &= I^L(\gamma) + \mathcal{D}_{\mathbb{C}}(2\varphi)\text{”}. \end{aligned}$$

On the other hand the independence between $\Phi_1$ and $\Phi_2$ gives

$$\begin{aligned} &\text{“} \lim_{\kappa \to 0} -\kappa \log \mathbb{P}(\sqrt{\kappa}\Phi_1 \text{ stays close to } 2u,\ \sqrt{\kappa}\Phi_2 \text{ stays close to } 2v) \\ &= \mathcal{D}_{\mathbb{H}}(2u) + \mathcal{D}_{\mathbb{H}^*}(2v)\text{”}. \end{aligned}$$

We obtain the identity $I^L(\gamma) + \mathcal{D}_{\mathbb{C}}(2\varphi) = \mathcal{D}_{\mathbb{H}}(2u) + \mathcal{D}_{\mathbb{H}^*}(2v)$ from (7). Theorem 7 follows by taking $\varphi \equiv 0$.

One technical difficulty to make this argument rigorous lies in choosing the right topologies so that these three equations in quote marks hold. However, once we have guessed the identity in Theorem 7, the actual proof is purely analytic and straightforward (without mentioning SLE or GFF) that we discuss now. For interested readers, we remark that starting from Theorem 2, more identities around the Loewner energy expanding the dictionary above between random conformal geometry and finite energy objects, are explored in [28, 29]. The proofs there are also entirely analytic and such a dictionary has provided inspiration for proving theorems on both sides.

## Outline of the proof of Theorem 2

To show Theorem 2 (where the Jordan curve does not pass through $\infty$), we prove first Theorem 7 without invoking SLE or GFF. For this,

- we show first the identity holds when the curve is of the form of $\mathbb{R}_+ \cup \eta$, where $\eta$ is a chord in $(\mathbb{C} \smallsetminus \mathbb{R}_+; 0, \infty)$ with driving function $W : \mathbb{R}_+ \to \mathbb{R}$. More specifically, we treat the following cases:
  - when $W_t = at$, for $t \in [0, T]$ and $W_t = aT$ for $t \geq T$ (the computation is technical but straightforward in this case);
  - when $W$ is piecewise linear;
  - when $W$ satisfies $I(W) < \infty$, we approximate $W$ by piecewise linear functions.
- We deduce the identity for curves of the form $[M, \infty] \cup \eta$ where $\eta$ is a chord in $(\mathbb{C} \smallsetminus [M, \infty); M, \infty)$. Then we let $M \to \infty$.

The second step aims at giving a more symmetric description of the Loewner energy by viewing the Jordan curve in the sphere $S^2$, so that the point $\infty$ plays no special role. We equip $S^2$ with a Riemannian metric $g = e^{2\varphi} g_0$, conformally equivalent to the round metric $g_0$ (the metric induced from $S^2 \subset \mathbb{R}^3$). Let $\gamma \subset S^2$ be a *smooth* Jordan curve dividing $S^2$ into two components $D_1$ and $D_2$. Denote by $\Delta_{D_i,g}$ the Laplace–Beltrami operator with Dirichlet boundary condition on $(D_i, g)$. We introduce the functional $\mathcal{H}(\cdot, g)$ on the space of smooth Jordan curves:

$$\mathcal{H}(\gamma, g) := \log \det{}'_\zeta(-\Delta_{S^2,g}) - \log \mathrm{vol}_g(S^2) - \log \det{}_\zeta(-\Delta_{D_1,g}) - \log \det{}_\zeta(-\Delta_{D_2,g}), \quad (8)$$

where $\det_\zeta$ denotes the zeta-regularized determinant.

**Theorem 8** (See [31, Thm. 7.3])**.** *We have the following results:*

*(i) The functional $\mathcal{H}$ is conformally invariant, i.e., $\mathcal{H}(\cdot, g) = \mathcal{H}(\cdot, g_0)$;*

*(ii) Let $\gamma$ be a smooth Jordan curve on $S^2$. We have the identity*

$$I^L(\gamma) = 12\mathcal{H}(\gamma, g) - 12\mathcal{H}(\mathcal{C}, g) = 12 \log \frac{\det_\zeta(-\Delta_{\mathbb{D}_1,g})\det_\zeta(-\Delta_{\mathbb{D}_2,g})}{\det_\zeta(-\Delta_{D_1,g})\det_\zeta(-\Delta_{D_2,g})}, \quad (9)$$

*where $\mathcal{C}$ is any circle and $\mathbb{D}_1$ and $\mathbb{D}_2$ are the two components of the complement of $\mathcal{C}$.*

Let us make some remarks:

- Since the Loewner energy is nonnegative, (ii) implies that circles minimize $\mathcal{H}(\cdot, g)$ among all smooth Jordan curves. This result was proved previously by [10] using variational method.
- We assumed the curve $\gamma$ to be smooth so that $\det_\zeta(-\Delta)$ is well-defined. This assumption can possibly be weakened.
- The Polyakov–Alvarez formula compares $\det_\zeta(-\Delta_{\mathbb{D}_i,g})$ to $\det_\zeta(-\Delta_{D_i,g})$ and involves conformal maps from $\mathbb{D}_i$ to $D_i$. From this we deduce the result by comparing (9) to the expression in Theorem 7.

Finally, for a smooth Jordan curve which does not pass through $\infty$, we use Theorem 8 and Polyakov–Alvarez formula again to deduce the identity in Theorem 2. The identity for an arbitrary bounded Jordan curve follows from an approximation argument by smooth Jordan curves.

## Further discussions

We mention that there are other identities between the Loewner energy and mathematical objects that are seemingly far away. We now give two examples. Theorem 2 suggests there should be a link between SLE and these objects. However, the link is widely unknown.

**Grunsky operator and Coulomb gas.** With each Jordan curve $\gamma$ is associated a *Grunsky operator* $G_\gamma$, defined explicitly using the coefficients of a conformal map $g$ from $\mathbb{D}^*$ to $\Omega^*$. See, e.g., [19]. It is a classical object in geometric function theory and many properties of $\gamma$ are equivalent to the operator-theoretic properties of $G_\gamma$. It was shown in [27] that $\gamma$ is Weil–Petersson if and only if $G_\gamma$ is Hilbert–Schmidt which is equivalent to $\det(\mathrm{Id} - G_\gamma^* G_\gamma)$ being well-defined. More surprisingly, Theorem 2 and [27] together imply that

$$I^L(\gamma) = -12 \log \det(\mathrm{Id} - G_\gamma^* G_\gamma).$$

Using this identity, [13, 14] showed that the Loewner energy appears in the constant term of large $n$ asymptotics of the free energy of $n$-particle Coulomb gas confined on $\gamma$ and in $\Omega$.

**Renormalized volume.** We saw that the Loewner energy is invariant under Möbius transformations of $\hat{\mathbb{C}}$. Since Möbius transformations extend to isometries of the hyperbolic 3-space $\mathbb{H}^3$ (whose boundary at infinity is identified with the Riemann sphere $\hat{\mathbb{C}}$), it is natural to wonder if there is a geometric quantity in $\mathbb{H}^3$ that is equal to the Loewner energy (question raised in [6]). The answer is positive. It is shown recently in [9] that the Loewner energy of $\gamma$ equals the renormalized volume of the 3-manifold $N_\gamma$ in $\mathbb{H}^3$ bounded by the two Epstein surfaces $\Sigma$ and $\Sigma^*$ (these are surfaces in $\mathbb{H}^3$ determined and bounded by $\gamma$):

$$I^L(\gamma) = \frac{4}{\pi}\left(\mathrm{vol}(N_\gamma) - \frac{1}{2}\int_{\Sigma \cup \Sigma^*} H \,\mathrm{d}A\right),$$

where $H$ is the mean curvature on $\Sigma$ and $\Sigma^*$ and $\mathrm{d}A$ is the area form induced from (the hyperbolic metric in) $\mathbb{H}^3$. The result uses the fact that $I^L$ is a Kähler potential for the Weil–Petersson metric.

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## References


[1] V. Beffara. The dimension of the SLE curves. *Ann. Probab.*, 36(4):1421–1452, 2008.

[2] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Phys. B*, 241(2):333–380, 1984.

[3] S. Benoist and J. Dubédat. An $\mathrm{SLE}_2$ loop measure. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(3):1406–1436, 2016.

[4] A. Beurling and L. Ahlfors. The boundary correspondence under quasiconformal mappings. *Acta Math.*, 96:125–142, 1956.

[5] C. J. Bishop. Conformal welding of rectifiable curves. *Math. Scand.*, 67(1):61–72, 1990.

[6] C. J. Bishop. Weil-Petersson curves, $\beta$-numbers, and minimal surfaces. *preprint*, 2019.

[7] C. J. Bishop. Conformal removability is hard. *preprint*, 2020.

[8] M. J. Bowick and S. G. Rajeev. String theory as the Kähler geometry of loop space. *Phys. Rev. Lett.*, 58(6):535–538, 1987.

[9] M. Bridgeman, K. Bromberg, F. Vargas-Pallete, and Y. Wang. Universal Liouville action as a renormalized volume and its gradient flow. *arXiv:2311.18767*, 2023.

[10] D. Burghelea, T. Kappeler, P. McDonald, and L. Friedlander. On the functional logdet and related flows on the space of closed embedded curves on $S^2$. *J. Funct. Anal.*, 120(2):440–466, 1994.

[11] M. Carfagnini and Y. Wang. Onsager Machlup functional for $\mathrm{SLE}_\kappa$ loop measures, 2023.

[12] B. Duplantier, J. Miller, and S. Sheffield. Liouville quantum gravity as a mating of trees. *arXiv preprint: 1409.7055*, 2014.

[13] K. Johansson. Strong Szegő theorem on a Jordan curve. In *Toeplitz operators and random matrices—in memory of Harold Widom*, volume 289 of *Oper. Theory Adv. Appl.*, pages 427–461. Birkhäuser/Springer, Cham, [2022] ©2022.

[14] K. Johansson and F. Viklund. Coulomb gas and the Grunsky operator on a Jordan domain with corners, 2023.

[15] K. Loewner. Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I. *Math. Ann.*, 89(1-2):103–121, 1923.

[16] S. Nag and A. Verjovsky. $\mathrm{Diff}(S^1)$ and the Teichmüller spaces. *Comm. Math. Phys.*, 130(1):123–138, 1990.

[17] A. Z. Patashinskii. Fluctuation theory of phase transitions. 1979.

[18] E. Peltola and Y. Wang. Large deviations of multichordal $\mathrm{SLE}_{0+}$, real rational functions, and zeta-regularized determinants of Laplacians. *J. Eur. Math. Soc.*, published online, 2023.

[19] C. Pommerenke. *Univalent functions.* Vandenhoeck & Ruprecht, Göttingen, 1975. With a chapter on quadratic differentials by Gerd Jensen, Studia Mathematica/Mathematische Lehrbücher, Band XXV.

[20] S. Rohde and O. Schramm. Basic properties of SLE. *Ann. of Math. (2)*, 161(2):883–924, 2005.

[21] S. Rohde and Y. Wang. The Loewner energy of loops and regularity of driving functions. *Int. Math. Res. Not. IMRN*, 2021(10):7715–7763, 2021.

[22] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[23] O. Schramm and S. Sheffield. Harmonic explorer and its convergence to $\mathrm{SLE}_4$. *Ann. Probab.*, 33(6):2127–2148, 2005.

[24] S. Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.*, 44(5):3474–3545, 2016.

[25] Y. Shen. Weil-Petersson Teichmüller space. *Amer. J. Math.*, 140(4):1041–1074, 2018.

[26] S. Smirnov. Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model. *Ann. of Math. (2)*, 172(2):1435–1467, 2010.

[27] L. A. Takhtajan and L.-P. Teo. Weil-Petersson metric on the universal Teichmüller space. *Mem. Amer. Math. Soc.*, 183(861):viii+119, 2006.

[28] F. Viklund and Y. Wang. Interplay between Loewner and Dirichlet energies via conformal welding and flow-lines. *Geom. Funct. Anal.*, 30(1):289–321, 2020.

[29] F. Viklund and Y. Wang. The Loewner-Kufarev Energy and Foliations by Weil-Petersson Quasicircles. *arXiv preprint: 2012.05771*, 2020.

[30] Y. Wang. The energy of a deterministic Loewner chain: reversibility and interpretation via $\mathrm{SLE}_{0+}$. *J. Eur. Math. Soc. (JEMS)*, 21(7):1915–1941, 2019.

[31] Y. Wang. Equivalent descriptions of the Loewner energy. *Invent. Math.*, 218(2):573–621, 2019.

[32] W. Werner. The conformally invariant measure on self-avoiding loops. *J. Amer. Math. Soc.*, 21(1):137–169, 2008.

[33] E. Witten. Coadjoint orbits of the Virasoro group. *Comm. Math. Phys.*, 114(1):1–53, 1988.

[34] D. Zhan. SLE loop measures. *Probab. Theory Related Fields*, 179(1-2):345–406, 2021.