Two-dimensional random conformal geometry combines techniques in complex analysis and stochastic analysis to study conformally invariant random objects in the plane. One of the central objects — Schramm–Loewner evolution (SLE) — is a one-parameter family of random fractal planar curves without self-crossing. Some of those random curves arise as the scaling limits of interfaces in two-dimensional critical lattice models and therefore give a new mathematical perspective to study 2D conformal field theory. Without invoking discrete models, the entire family of SLE curves can be defined directly in the continuum and is uniquely characterized by two seemingly weak properties: conformal invariance and domain Markov property (will be explained below). The fact that there is only a one free parameter in the definition of SLE is rooted in the universality of Brownian motion — one of the most fundamental concept in probability theory.

In this article we will explain that a natural quantity — Loewner energy — arising from the large deviations of SLE and associated to each Jordan curve, is connected to the Kähler geometry of universal Teichmüller space. Universal Teichmüller space $T(1)$ is an infinite-dimensional complex Banach manifold and a homogeneous space which contains all the Teichmüller spaces of hyperbolic surfaces. There are several equivalent ways to describe the universal Teichmüller space, one is by identifying it with a family of Jordan curves (i.e. quasicircles). Universal Teichmüller space is endowed with many structures, but none of them seems to be directly related to stochastic processes. However, it has an essentially unique homogeneous Kähler metric, whose Kähler potential — defined on the Weil–Petersson universal Teichmüller space $T_0(1) \subset T(1)$ — turns out to be exactly the Loewner energy.

The goal of this article is to introduce Loewner energy and universal Liouville action gently and give the intuition behind the identity between them. We will also give a brief outline of the proof.
How does a random simple curve on the plane look like?

Examples of such simple random curves are interfaces in a critical lattice model, or a simple random walk after erasing all the loops chronologically, or a uniformly sampled self-avoiding walk, all on a grid of infinitely fine mesh (by taking the scaling limit where the mesh size goes to zero). Since the random curve has to avoid its past trajectory, it cannot be a Markov process. (For a Markov process, such as the 2D Brownian motion, the future path only depends on the current location. 2D Brownian motion hits itself infinitely many times on arbitrarily small time interval.)

To make it more tractable, we first assume that the random simple curve connects two distinct boundary points (prime ends) $a, b$ in a simply connected domain $D \subset \mathbb{C}$. We call such a simple curve a chord in $(D; a, b)$ and oriented from $a$ to $b$. This setup will allow us to progressively slit open the curve starting from $a$, so that the remaining part of the curve will be a chord connecting the endpoint of the slit and $b$.

One advantage of working in two-dimension is that there is a rich family of conformal maps, given by biholomorphic functions (so that we may use single-variable complex analysis). In particular, the Riemann mapping theorem states that there exists a conformal map $\phi$ sending the simply connected domain $D$ to the upper half-plane $H = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$, which maps $a$ to $0 \in \partial H = \mathbb{R} \cup \{\infty\}$ and $b$ to $\infty$. Another choice of such conformal map has to be of the form $\lambda \phi$ for some $\lambda > 0$.

We further assume that the random chord satisfies two properties:

- **Conformal invariance:** The random chord $\gamma$ in $\mathbb{H}$ connecting $0$ to $\infty$ should have the same law as its image under the map $z \mapsto \lambda z$ for every $\lambda > 0$. This then allows us to define the random chord in $(D; a, b)$ as the preimage of $\phi$.

- **Domain Markov property:** Chords of $(\mathbb{H}; 0, \infty)$ have a parametrization (called the capacity parametrization). If we slit the random chord $\gamma$ in $(\mathbb{H}; 0, \infty)$ from $\gamma(0) = 0$ up to $\gamma(s)$ and map conformally $(\mathbb{H} \setminus \gamma[0, s]; \gamma(s), \infty)$ to $(\mathbb{H}; 0, \infty)$, then the image of $\gamma[s, \infty)$ should have the same law as $\gamma$ and be independent of $\gamma[0, s]$, for every $s > 0$.

![Figure 1: The left arrow illustrates a scaling map, and the right arrow illustrates the uniformizing conformal map from the slit domain onto the upper half-plane. The law of the random chord should be identical in these three pictures.](image)

These two properties are natural as they are satisfied by many curves in statistical mechanics lattice models whose scaling limit has conformal symmetry. However, we do not need to invoke discrete models if we use these two properties as the axiomatic definition of the random chord of interest. Indeed, O. Schramm [10] uses the Loewner transform to define such a random chord. The Loewner transform encodes each (deterministic) chord...
in \((\mathbb{H}; 0, \infty)\) into a continuous real-valued function \(t \mapsto W_t, \mathbb{R}_+ \to \mathbb{R}\) with \(W_0 = 0\), called \textit{Loewner driving function}. More precisely, under the capacity parametrization of the chord \(\gamma\), the conformal map

\[ g_t : \mathbb{H} \setminus \gamma[0, t] \to \mathbb{H} \]

which has the expansion

\[ g_t(z) = z + \frac{2t}{z} + o\left(\frac{1}{z}\right) \]

as \(z \to \infty\)

maps the tip of \(\gamma[0, t], \gamma(t)\), to a real number denoted as \(W_t\) (capacity parametrization means that the coefficient of \(1/z\) in the asymptotic expansion of \(g_t\) is given by \(2t\)).

It is not hard to check that the two properties of the random simple chord above translate into the properties of the random driving function \(W\):

\begin{itemize}
  \item \(W\) has the same law as the function \(t \mapsto \lambda W_{\lambda - 2t}\), for every \(\lambda > 0\);
  \item for every \(s > 0\), \(t \mapsto W_{t+s} - W_t\) has the same law as \(W\) and is independent of \(W|_{[0,s]}\).
\end{itemize}

It turns out the only possible random continuous function satisfying these properties is of the form \(\sqrt{\kappa}B\), where \(B\) is the standard Brownian motion and \(\kappa \geq 0\). In fact, the second property guarantees \(W\) to be a continuous Lévy process. The classification of continuous Lévy processes tells us that it is of the form \(t \mapsto \sqrt{\kappa}B_t + at\), for some \(a \in \mathbb{R}\). One may view the classification as the manifestation of two most fundamental theorems in probability theory: the law of large number (which explains the occurrence of the deterministic drift \(t \mapsto at\)), central limit theorem (which explains the occurrence of the Gaussian process \(\sqrt{\kappa}B\)). Then, the first property then implies that \(a = 0\).

\textit{Schramm–Loewner evolution} \(\text{SLE}_\kappa\) in \((\mathbb{H}; 0, \infty)\) is the random curve whose driving function is \(\sqrt{\kappa}B\), for \(\kappa \geq 0\). More precisely, when \(\kappa \leq 4\), \(\text{SLE}_\kappa\) is indeed the random simple chord with driving function \(\sqrt{\kappa}B\) in the sense described above; for \(\kappa > 4\), the Loewner driving function does not define a growing slit \(\gamma[0, t]\) but a growing compact set \(K_t \subset \mathbb{H}\). In this case, \(\text{SLE}_\kappa\) is referred to the curve which carves out progressively the boundary of \(K_t\) (when \(\kappa \geq 8\), \(\text{SLE}_\kappa\) is a random space-filling curve) [8]. In this article, we will only consider the case where \(\kappa \to 0^+\), so we do not enter into further details to discuss the \(\kappa > 4\) case. As we explained above, SLEs are the only random chords which satisfy the conformal invariance and domain Markov property. The \(\text{SLE}_\kappa\) in another domain \((D; a, b)\) is defined as the preimage of \(\text{SLE}_\kappa\) in \((\mathbb{H}; 0, \infty)\) by any conformal map \(\varphi : D \to \mathbb{H}\) sending respectively \(a, b\) to \(0, \infty\).

We note that when \(\kappa = 0\), \(\text{SLE}_0\) in \((\mathbb{H}; 0, \infty)\) is the imaginary axis and \(g_t(z) = \sqrt{z^2 + 4t}\) is the corresponding conformal map \(\mathbb{H} \setminus i[0, 2\sqrt{t}] \to \mathbb{H}\).

\textbf{Dirichlet energy and Loewner energy}

We will now slowly move out from the probability world by only looking at a functional that arises from SLE. Schilder’s theorem states that the large deviation rate function of \((\sqrt{\kappa}B)_{\kappa \to 0^+}\) is given by the Dirichlet energy

\[ I(W) := \frac{1}{2} \int_0^\infty W_t^2 \, dt, \]
where \( \dot{W}_t := dW_t / dt \). In physics terms, \( I(\cdot) \) is also called the action functional of Brownian motion. Roughly speaking, the large deviation principle means

\[
P(\sqrt{\kappa} B \approx W) \sim_{\kappa \to 0^+} \exp \left( - \frac{I(W)}{\kappa} \right).
\]

So it is not surprising that under appropriate topology (proved in [7] for the Hausdorff metric) that a similar large deviation principle holds for SLE\(_{\kappa^+}\):

\[
P(\text{SLE}_\kappa \approx \gamma) \sim_{\kappa \to 0^+} \exp \left( - \frac{I_{\text{C}}(\gamma)}{\kappa} \right). \tag{1}
\]

Here both the SLE curve and \( \gamma \) are chords in the domain \((D; a, b)\), and \( I_{\text{C}}(\gamma) := I(W) \) is called the chordal Loewner energy of \( \gamma \), where \( W \) is the driving function of \( \varphi(\gamma) \) and \( \varphi \) maps conformally \((D; a, b)\) onto \((\mathbb{H}; 0, \infty)\).

We now generalize the Loewner energy to Jordan curves (simple loops) on the Riemann sphere \( \hat{C} = \mathbb{C} \cup \{\infty\} \) and denote it as \( I_L \). This generalization has the property

\[
I_{\hat{C}\setminus R^+;0,\infty}(\gamma) = I_L(\gamma \cup R^+)
\]

for every simple chord \( \gamma \) in \((C \setminus R^+; 0, \infty)\). More precisely, let \( \gamma : [0, 1] \to \hat{C} \) be a continuously parametrized Jordan curve with \( \gamma(0) = \gamma(1) \). For every \( \varepsilon > 0 \), \( \gamma[\varepsilon, 1] \) is a chord connecting \( \gamma(\varepsilon) \) to \( \gamma(1) \) in the simply connected domain \( \hat{C} \setminus \gamma[0, \varepsilon] \). The rooted loop Loewner energy of \( \gamma \) rooted at \( \gamma(0) \) is defined as

\[
I_L(\gamma, \gamma(0)) := \lim_{\varepsilon \to 0} I_{\hat{C}\setminus \gamma[0,\varepsilon]\setminus \gamma(0)}(\gamma[\varepsilon, 1]). \tag{3}
\]

It turns out that the definition does not depend on the choice of the orientation of the curve [16] nor on its root [9]. Therefore, we omit the root in the notation and (2) can be seen by taking the root at \( \infty \) and orient the curve from \( \infty \to 0 \) along \( R_+ \). The independence from the parametrization is not obvious from the definition, since the chordal energies \( I_C \) are defined using the Loewner driving function which depends strongly on the past of the curve. This independence suggests that there must be an intrinsic expression of the Loewner energy which does not use any parametrization of the Jordan curve. The answer is indeed given by an identity with the universal Liouville action that we will explain in the next section.

We remark that since the Loewner energy is defined via uniformizing maps (to define the driving function), it is invariant under conformal automorphisms of \( \hat{C} \) (namely, Möbius transformations \( z \mapsto \frac{az + b}{cz + d} \)). Moreover, the Loewner energy is zero if and only if \( \gamma \) is a circle and Möbius transformations act transitively on the family of circles. Therefore, the Loewner energy may be viewed as a quantity measuring the roundness of the unparametrized Jordan curve.

**Identity with the Universal Liouville action**

The following theorem gives an equivalent expression of the Loewner energy of a Jordan curve. Since the Loewner energy is invariant under Möbius transformations, without loss of generality, we may assume that the Jordan curve \( \gamma \) does not pass through \( \infty \).
Theorem 1 (See [17, Thm. 1.4]). Let \( \Omega \) (resp. \( \Omega^* \)) denote the component of \( \hat{\mathbb{C}} \setminus \gamma \) which does not contain \( \infty \) (resp. which contains \( \infty \)) and \( f \) (resp. \( g \)) be a conformal map from the unit disk \( \mathbb{D} = \{ z \in \hat{\mathbb{C}} : |z| < 1 \} \) onto \( \Omega \) (resp. \( \mathbb{D}^* = \{ z \in \hat{\mathbb{C}} : |z| > 1 \} \) onto \( \Omega^* \)). We assume further that \( g(\infty) = \infty \). The Loewner energy of \( \gamma \) satisfies

\[
I^L(\gamma) = \frac{1}{\pi} \int_{\mathbb{D}} \left| f''(z) \right|^2 \, d^2z + \frac{1}{\pi} \int_{\mathbb{D}^*} \left| \frac{g''(z)}{g'(z)} \right|^2 \, d^2z + 4 \log \frac{|f'(0)|}{g'(\infty)} =: S_1(\gamma),
\]

where \( g'(\infty) = \lim_{z \to \infty} g'(z) \) and \( d^2z \) is the Euclidean area measure.

The quantity \( S_1 \) is introduced in [13] under the name universal Liouville action. Its value does not depend on the choice of \( f \) and \( g \). A curve for which \( \int_{\mathbb{D}} |f''(z)|^2 \, d^2z \) is finite is called a Weil–Petersson quasicircle. It turns out that \( \int_{\mathbb{D}} |f''(z)|^2 \, d^2z \) is finite if and only if \( \int_{\mathbb{D}^*} |g''(z)/g'(z)|^2 \, d^2z \) is finite. Hence, we have:

Corollary 2. A Jordan curve \( \gamma \) has finite Loewner energy if and only if \( \gamma \) is a Weil–Petersson quasicircle.

As mentioned above, the universal Liouville action does not depend on any special point on the curve \( \gamma \) and has the advantage of involving only two conformal maps. Whereas to define the Loewner energy through the driving function, one has to study the whole family of uniformizing conformal mappings of the slit domains. However, the way of considering the Jordan curve as a progressively growing slit (which closes up on itself) allowed us to relate to SLE curves.

**Weil–Petersson Teichmüller space**

The universal Liouville action appeared in [13] in a very different context that we now discuss briefly its background.

We first identify a Jordan curve \( \gamma \) with a homeomorphism of the unit circle \( S^1 = \partial \mathbb{D} = \partial \mathbb{D}^* \) as follows. By Carathéodory theorem, any conformal map \( f : \mathbb{D} \to \Omega \) (resp. \( g : \mathbb{D}^* \to \Omega^* \)) extends continuously to a homeomorphism between the closures \( \overline{\mathbb{D}} \to \overline{\Omega} \) (resp. \( \overline{\mathbb{D}^*} \to \overline{\Omega^*} \)). In particular, \( f \) and \( g \) restricted to \( S^1 \) define two homeomorphisms \( S^1 \to \gamma \). The welding homeomorphism compares these two homeomorphisms and is defined as the circle homeomorphism \( \varphi := g^{-1} \circ f |_{S^1} \).

The converse operation — solving the conformal welding problem — consists of finding a Jordan curve \( \gamma \) whose welding homeomorphism is a given circle homeomorphism \( \varphi \). We note that if \( \gamma \) is a solution, then \( A \circ \gamma \) is also a solution (by replacing \( f \) by \( A \circ f \) and \( g \) by \( A \circ g \)), where \( A \) is a Möbius transformation of \( \hat{\mathbb{C}} \). Solution may not exist, and if exist, it may not be unique (up to post-composition by Möbius transformations), see, e.g., [1]. However, classical results in quasiconformal mappings (Beurling-Ahlfors) shows that if the circle homeomorphism is quasisymmetric, then the solution to the conformal welding problem is unique up to post-composition by Möbius transformations. The corresponding Jordan curves are called quasicircles. Let

\[
\text{QS}(S^1) := \left\{ \varphi \in \text{Hom}(S^1) : \exists M > 1, \forall \theta \in \mathbb{R}, \forall t \in (0, \pi), \frac{1}{M} \leq \frac{\varphi(e^{i(\theta + t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta - t)})} \leq M \right\}
\]
denote the group of quasisymmetric circle homeomorphisms. To fix a normalization, we assume that the conformal map \( f \) satisfies \( f(0) = 0, f'(0) = 1 \) and \( f''(0) = 0 \) and put no condition on \( g \) except that \( g(\mathbb{D}^*) = \hat{C} \setminus \mathcal{J}(\mathbb{D}) \). In other words, we consider the welding homeomorphism to be in the homogeneous space \( \text{Möb}(S^1) \setminus \text{QS}(S^1) \), where \( \text{Möb}(S^1) \) is the group of Möbius transformations preserving \( S^1 \) (since \( g \) can be replaced by \( g \circ B \) for any \( B \in \text{Möb}(S^1) \)). The homogeneous space \( T(1) := \text{Möb}(S^1) \setminus \text{QS}(S^1) \) is called the universal Teichmüller space for the reason that all Teichmüller spaces of hyperbolic surfaces are embedded in \( T(1) \).

Universal Teichmüller space has a structure of infinite dimensional complex Banach manifold (local charts given by the Bers embedding) and the group \( \text{QS}(S^1) \) acts on the right on \( T(1) \) holomorphically. One wonders whether it can be further equipped with a Kähler metric, namely, a symplectic form and Riemannian metric that are invariant under the right action and compatible with the complex structure. This question has been addressed by string theorists [3] and [18] who consider only the smooth part \( \text{Möb}(S^1) \setminus \text{Diff}^\infty(S^1) \) of the universal Teichmüller space without worrying about any convergence issue on this infinite dimensional manifold. It turns out there is a unique Kähler metric on \( S \) up to a scaling factor.

Let us explain briefly how the Kähler structure is derived (also ignoring the convergence question). Concretely, the tangent space at \([\text{Id}]\) consists of vector fields \( \text{Vect}(S^1) \) on \( S^1 \) with Fourier expansion:

\[
v = \sum_{n \neq \pm 1, 0} v_n e_n := \sum_{n \neq \pm 1, 0} v_n e^{i \theta} \frac{\partial}{\partial \theta} \quad \text{satisfying } \overline{v}_n = v_{-n}.
\]

The almost complex structure \( J : \text{Vect}(S^1) \to \text{Vect}(S^1) \) (such that \( J^2 = -I \)) induced from the complex structure from the Bers embedding is given by the Hilbert transform [6]:

\[
Jv = i \sum_{n=2}^{\infty} v_n e_n - i \sum_{n=-\infty}^{-2} v_n e_n.
\]

The family \( \{ e_n := e^{i \theta} \partial / \partial \theta \}_{n \neq \pm 1, 0} \) generates the complexification of \( \text{Vect}(S^1) \):

\[
\text{Vect}^C(S^1) = \{ \sum_{n \neq \pm 1, 0} u_n e_n \mid u_n \in \mathbb{C} \},
\]

with the Lie bracket

\[
[e_m, e_n] = i(n - m)e_{n+m}.
\]

**Theorem 3** (See [3]). Up to a scaling factor, there is a unique homogeneous Kähler metric on \( \text{Möb}(S^1) \setminus \text{Diff}^\infty(S^1) \). The symplectic form (closed and non-degenerate 2-form) is given by

\[
\omega(u, v) = -\omega(v, u) = -\alpha \text{Im} \left( \sum_{n=2}^{\infty} (n^3 - n) u_n \overline{v}_n \right),
\]

for all \( u, v \in \text{Vect}(S^1) \), and for some \( \alpha \in \mathbb{R}_+ \). The Riemannian metric \( g \) compatible with \( \omega \) and \( J \), in the sense that

\[
g(u, v) := \omega(u, Jv) = -\alpha \text{Im} \left( \sum_{n=2}^{\infty} -i(n^3 - n) u_n \overline{v}_n \right),
\]
\[ \sum_{n=2}^{\infty} (n^3 - n) u_n v_n \]

is positive and definite. This metric is called the Weil–Petersson metric.

**Proof.** Assume \( \omega \) is a homogeneous symplectic form. It is closed, therefore

\[
d\omega(e_m, e_n, e_p) = e_m(\omega(e_n, e_p)) + e_n(\omega(e_p, e_m)) + e_p(\omega(e_m, e_n)) \]

\[
- (\omega([e_m, e_n], e_p) + \omega([e_n, e_p], e_m) + \omega([e_p, e_m], e_n)) = 0.
\]

By homogeneity, \( e_m(\omega(e_n, e_p)) \) vanishes and we have

\[
\omega([e_m, e_n], e_p) + \omega([e_n, e_p], e_m) + \omega([e_p, e_m], e_n) = 0. \tag{5}
\]

Moreover, \( \omega \) has kernel spanned by \( e_{-1}, e_0 \) and \( e_1 \) (i.e. the Lie algebra of \( \text{M"{o}b}(S^1) \)). From these constraints we can determine \( \omega \) as follows.

By taking \( p = 0 \), (5) gives that

\[
(n + m) \omega(e_m, e_n) = 0.
\]

Therefore \( \omega(e_m, e_n) \) can be nonzero only when \( m = -n \). We set \( \omega(e_m, e_{-m}) =: a_m \). Take \( p = -m - 1, n = 1 \), (5) gives

\[
(1 - m)a_{m+1} + (m + 2)a_m = 0
\]

and we see that \( a_m = \alpha (m^3 - m)/2 \) for some \( \alpha \in \mathbb{C} \).

For \( u, v \in \text{Vect}(S^1) \), that is \( u_m = \overline{u}_m \) for all \( m \) and similarly for \( v \),

\[
\omega(u, v) = \frac{i\alpha}{2} \sum_{m \neq \pm 1, 0} (m^3 - m) u_m \overline{v}_{-m} = -\alpha \text{Im} \left( \sum_{m=2}^{\infty} (m^3 - m) u_m \overline{v}_m \right).
\]

We obtain \( \alpha > 0 \) from the assumption that

\[
g(u, v) := \omega(u, Jv) = \alpha \text{Re} \left( \sum_{m=2}^{\infty} (m^3 - m) u_m \overline{v}_m \right)
\]

is positive definite. \( \square \)

Takhtajan and Teo [13] defined rigorously the infinite-dimensional Kähler manifold structure and the Weil–Petersson metric on \( T(1) \). The subspace of \( u \in \text{Vect}(S^1) \) such that \( g(u, u) < \infty \) is the \( H^{3/2} \) Sobolev space of vector fields (which is strictly smaller than the tangent space of \( T(1) \) which are given by the space of Zygmund vector fields). The image of \( H^{3/2} \) under the right action by \( \text{QS}(S^1) \) on \( T(1) \) defines a tangent subbundle.

**Theorem 4** (See [13]). The connected component \( T_0(1) \) of the integral manifold containing \( [\text{Id}] \in T(1) \) — called the Weil–Petersson Teichmüller space — is a complete, infinite-dimensional Kähler-Einstein manifold with negative curvatures. Moreover, \( [\varphi] \in T_0(1) \) if and only if \( \varphi = g^{-1} \circ f|_{S^1} \) where \( \int_{S^1} |f''/f'|^2 \, d^2z < \infty \).
Therefore, $T_0(1)$ is the completion of $\text{Mob}(S^1) \setminus \text{Diff}^\infty(S^1)$. Moreover, a Jordan curve is associated with an element in $T_0(1)$ if and only if it is a Weil–Petersson quasicircle. Many equivalent descriptions of $T_0(1)$ are given, see, e.g. [2] for an extensive summary. In particular, Shen [12] showed that $[\varphi] \in T_0(1)$ if and only if $\log \varphi' \in H^{1/2}$. Furthermore, the unique homogeneous Kähler metric on $T_0(1)$ is tightly related to the universal Liouville action and can be derived from the latter in a simple way:

**Theorem 5** (See [13, Cor.II.4.2]). The universal Liouville action $S_1 : T_0(1) \to \mathbb{R}_+$ is a Kähler potential for the Weil–Petersson metric. In other words

$$\partial \bar{\partial} S_1 = -i \omega,$$

where $\omega$ is the symplectic form in Theorem 3 (up to a positive scaling factor).

**How do we come up with the identity?**

We now explain how we could guess the identity between the Loewner energy and the universal Liouville action using ideas from random conformal geometry. We will give the outline of the actual proof in the next section. The first step of the proof is to show the following identity for a Jordan curve passing through $\infty$.

**Theorem 6** (See [17, Thm.1.1]). If $\gamma$ is a Jordan curve passing through $\infty$, then

$$I^L(\gamma) = \frac{1}{\pi} \int_{\mathbb{H}} |\nabla \log |f'||^2 d^2 z + \frac{1}{\pi} \int_{\mathbb{H}^*} |\nabla \log |g'||^2 d^2 z = \frac{1}{\pi} \int_{\mathbb{H}} \left| \frac{f''}{f'} \right|^2 d^2 z + \frac{1}{\pi} \int_{\mathbb{H}^*} \left| \frac{g''}{g'} \right|^2 d^2 z$$

where $f$ and $g$ map conformally the upper half-plane $\mathbb{H}$ and the lower half-plane $\mathbb{H}^*$ onto, respectively, $H$ and $H^*$, the two components of $C \setminus \gamma$, while fixing $\infty$.

This theorem can be viewed as the finite energy analog of the quantum zipper coupling between SLE and Gaussian free field (GFF) [5,11] that we now explain. We do not make a rigorous statement and only argue heuristically here.

A quantum surface is a domain $D$ equipped with a Liouville quantum gravity ($\sqrt{\kappa}$-LQG) measure, defined using a regularization of $e^{\sqrt{\kappa} \Phi} d^2 z$, where $\sqrt{\kappa} \in (0, 2)$, and $\Phi$ is a Gaussian field with the covariance of a free boundary GFF. GFF is a random real-valued Schwartz distribution defined on $D$, whose action functional is given by the Dirichlet energy on $D$ (therefore, GFF is the analog of Brownian motion by replacing the time interval by the two-dimensional domain $D$). The Schilder’s theorem for Gaussian measures shows that $\sqrt{\kappa} \Phi$, as $\kappa \to 0_+$ has the large deviation rate function the Dirichlet energy on $D$, defined for all $\varphi \in W^{1,2}_{\text{loc}}$ as

$$D_D(\varphi) := \frac{1}{4\pi} \int_D |\nabla \varphi|^2 d^2 z \in [0, \infty].$$

If $D_D(\varphi) < \infty$, we say that $\varphi \in \mathcal{E}(D)$.

We use the following dictionary illustrating the analogy between the concepts in random conformal geometry (left column) and their large deviation counterparts (right column).
We obtain the identity
\[ I^L(\gamma) < \infty \]
i.e., a Weil–Petersson quasicircle

On the other hand the independence between \( \Phi_1 \) and \( \Phi_2 \) gives
\[ -\kappa \log \mathbb{P}(\sqrt{\kappa} \Phi_1 \text{ stays close to } 2u, \sqrt{\kappa} \Phi_2 \text{ stays close to } 2v) = D_{\mathbb{H}}(2u) + D_{\mathbb{H}^*}(2v). \]

We obtain the identity \( I^L(\gamma) + D_{\mathbb{C}}(2\varphi) = D_{\mathbb{H}}(2u) + D_{\mathbb{H}^*}(2v) \) from (6). Theorem 6 follows by taking \( \varphi \equiv 0 \).

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### SLE/GFF with \( \kappa \to 0^+ \) vs. Finite energy

<table>
<thead>
<tr>
<th>SLE/( \kappa ) loop</th>
<th>Jordan curve ( \gamma ) with ( I^L(\gamma) &lt; \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free boundary GFF ( \sqrt{\kappa} \Phi ) on ( \mathbb{H} ) (on ( \mathbb{C} ))</td>
<td>( \mathbb{P}(\text{the measure on } \mathbb{H}) )</td>
</tr>
<tr>
<td>( \sqrt{\kappa} )-LQG on quantum plane ( \approx e^{\sqrt{\kappa} \Phi} d^2 z )</td>
<td>measure on ( \mathbb{C} ): ( e^{2\varphi} d^2 z, \varphi \in \mathcal{E}(\mathbb{C}) )</td>
</tr>
<tr>
<td>( \sqrt{\kappa} )-LQG on quantum half-plane on ( \mathbb{H} )</td>
<td>measure on ( \mathbb{H} ): ( e^{2u} d^2 z, u \in \mathcal{E}(\mathbb{H}) )</td>
</tr>
</tbody>
</table>

Quantum zipper coupling:
- \( \kappa \)-LQG cuts \( \gamma \) into two domains \( D \) and \( D' \) equipped with a measure are considered equivalent if there exist a conformal map \( D \to D' \) such that the measure on \( D' \) equals the pushforward of the measure on \( D \). In particular, if a Jordan curve \( \gamma \) cuts \( \mathbb{C} \) into two domains \( H \) and \( H^* \) as above and \( f \) and \( g \) are the conformal maps in Theorem 6, then \( e^{2\varphi} d^2 z \) and \( e^{2\nu} d^2 z \) where

\[ u = \varphi \circ f + \log |f'|, \quad v = \varphi \circ g + \log |g'|. \]

The identity \( I^L(\gamma) + D_{\mathbb{C}}(2\varphi) = D_{\mathbb{H}}(2u) + D_{\mathbb{H}^*}(2v) \) is more general than Theorem 6 (which corresponds to the case where \( \varphi \equiv 0 \)) and we argue heuristically as follows. From the quantum zipper coupling, one expects that under an appropriate choice of topology and for small \( \kappa \),

\[ \mathbb{P}(\text{SLE}_\kappa \text{ loop stays close to } \gamma, \sqrt{\kappa} \Phi \text{ stays close to } 2\varphi) = \mathbb{P}(\sqrt{\kappa} \Phi_1 \text{ stays close to } 2u, \sqrt{\kappa} \Phi_2 \text{ stays close to } 2v). \]

We obtain from the independence between SLE and \( \Phi \), the large deviation principle of SLE (1), and the large deviation principle for GFF (Schilder’s theorem for Gaussian measure)

\[ -\kappa \log \mathbb{P}(\text{SLE}_\kappa \text{ stays close to } \gamma, \sqrt{\kappa} \Phi \text{ stays close to } 2\varphi) = \lim_{\kappa \to 0} -\kappa \log \mathbb{P}(\sqrt{\kappa} \Phi \text{ stays close to } 2\varphi) = I^L(\gamma) + D_{\mathbb{C}}(2\varphi). \]
One technical difficulty to make this argument rigorous lies in choosing the right topologies so that these three equations in quote marks hold (and the statement of the quantum zipper coupling here is only approximate). We did not try hard to make this approach work as the direct proof (without SLE or GFF) of Theorem 6 is elementary and straightforward. Therefore we shall rather view Theorem 6 as providing the intuition behind the quantum zipper coupling than the converse. For interested readers, more identities between the Dirichlet energy and the Loewner energy and their applications are explored in [14,15].

Outline of the proof of Theorem 1

To show Theorem 1 (where the Jordan curve does not pass through $\infty$), we prove first Theorem 6 without invoking SLE or GFF. For this,

- we show first the identity holds when the curve is of the form of $R_+ \cup \eta$, where $\eta$ is a chord in $(\mathbb{C} \setminus R_+; 0, \infty)$ with driving function $W : R_+ \to \mathbb{R}$. More specifically, we treat the following cases:
  - when $W_t = at$, for $t \in [0, T]$ and $W_t = aT$ for $t \geq T$ (the computation is slightly tedious in this case);
  - when $W$ is piecewise linear by concatenating linear driving functions;
  - when $W$ satisfies $I(W) < \infty$, we approximate $W$ by piecewise linear functions.
- We deduce the identity for curves of the form $[M, \infty] \cup \eta$ where $\eta$ is a chord in $(\mathbb{C} \setminus [M, \infty); M, \infty)$. Then we let $M \to \infty$.

The second step aims at giving a more symmetric description of the Loewner energy by viewing the curve as sitting on the sphere $S^2$, so that the point $\infty$ plays no special role. We equip $S^2$ with a Riemannian metric $g = e^{2\varphi}g_0$, conformally equivalent to the spherical metric $g_0$ (the metric induced from $S^2 \subset \mathbb{R}^3$ with the Euclidean metric). Let $\gamma \subset S^2$ be a smooth Jordan curve dividing $S^2$ into two components $D_1$ and $D_2$. Denote by $\Delta_{D_i,g}$ the Laplace-Beltrami operator with Dirichlet boundary condition on $(D_i,g)$. We introduce the functional $H(\cdot,g)$ on the space of smooth Jordan curves:

$$H(\gamma,g) := \log \det_\zeta(-\Delta_{S^2,g}) - \log \text{vol}_g(S^2) - \log \det_\zeta(-\Delta_{D_1,g}) - \log \det_\zeta(-\Delta_{D_2,g}),$$

where $\det_\zeta$ denotes the zeta-regularized determinant.

**Theorem 7** (See [17, Thm. 7.3]). We have the following results:

(i) The functional $H$ is conformally invariant, i.e. $H(\cdot,g) = H(\cdot,g_0)$;

(ii) Let $\gamma$ be a smooth Jordan curve on $S^2$. We have the identity

$$I^L(\gamma) = 12H(\gamma,g) - 12H(C,g) = 12\log \frac{\det_\zeta(-\Delta_{D_1,g})\det_\zeta(-\Delta_{D_2,g})}{\det_\zeta(-\Delta_{D_1,g})\det_\zeta(-\Delta_{D_2,g})},$$

where $C$ is any circle, and $D_1$ and $D_2$ are the two components of the complement of $C$.

Let us make some remarks:
• Since the Loewner energy is nonnegative, (ii) implies that circles minimize $H(\cdot, g)$ among all smooth Jordan curves. This result was proved previously by [4] using variational method.

• We assumed the curve $\gamma$ to be smooth so that $\det_\zeta(-\Delta)$ is well-defined. This assumption can possibly be weakened.

• The Polyakov-Alvarez formula compares $\det_\zeta(-\Delta_{D_i, g})$ to $\det_\zeta(-\Delta_{D_i, g})$ and involves conformal maps from $D_i$ to $D_i$. From this we deduce the result by comparing (8) to the expression in Theorem 6.

Finally, for a smooth Jordan curve which does not pass through $\infty$, we use Theorem 7 and Polyakov-Alvarez formula again to deduce the identity in Theorem 1. The identity for an arbitrary bounded Jordan curve follows from an approximation argument by smooth Jordan curves.

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References


