MIT 18.675. THEORY OF PROBABILITY
FALL 2020, MW 11:00-12.30 ON ZOOM.

Updated: December 6, 2020

Aug. 31: Registration day

(1) **Sep. 02: Introduction to measure theory.**
Definition of measure space $(\Omega, \mathcal{F}, \mu)$, σ-algebra, measure on σ-algebra.
Immediate consequences of measure space: 1) $\mathcal{F}$ contains $\varnothing$ and Ω; 2) $\mathcal{F}$ closed under countable intersection; 3) monotonicity of μ; 4) countable subadditivity of μ; 5) continuity from below; 6) continuity from above.
Introduction to Lebesgue measure, Cantor set's measure.
Showed from the example of Vitali's set that Lebesgue measure cannot be defined on all subsets of $(0, 1]$.
Banach-Tarski Paradox.
[*Reference: Durrett, Chapter 1.1, A.2*]

Sep. 07: Labor day, no class

(2) **Sep. 09: Construction of Lebesgue measure**
Borel σ-algebra, algebras, construction of Lebesgue measure using Carathéodory extension theorem, Dynkin system (λ-system), π-system, proof of π-λ theorem.
[*Reference: Durrett, Chapter A.1*]

(3) **Sep. 14:** *PSET 1 due Sep. 14.*
Construction of Lebesgue measure: σ-additivity of Lebesgue measure on algebra.
[*Reference: Dembo, Section 1.1.3*]
Random variables: Measurable functions, random variables, distributions on $\mathbb{R}$, Lebesgue-Stieltjes theorem, density function.
[*Reference: Durrett, Chapter 1.2, 1.3*]

(4) **Sep. 16: Lebesgue integration, expectation**
P-almost sure convergence, σ-algebra generated by random variables, expected value, Monotone convergence theorem, Fatou's lemma, Hölder's inequality, Dominated convergence theorem.
[*Reference: Durrett, Chapter 1.4-1.6*]

(5) **Sep. 21: Expectation, product measure and independence**
Chebyshev inequality, push-forward measure and change of variable formula, expected value of functions of X, product space, Fubini-Tonelli Theorem, independence.
[*Reference: Durrett, Chapter 1.6.3, 1.7, 2.1.1, 2.1.2*]

(6) **Sep. 23: Infinite sequence of independent random variables**
Introduction to Strong law of Large numbers, infinite family of independent random variables, construction using Kolmogorov extension theorem, Kolmogorov's 0-1 law, convergence of series, weak law of large numbers, Borel Cantelli lemmas.
[*Reference: Durrett, Chapter 2.1.4, 2.2, 2.3, 2.5, A.3*]

(7) **Sep. 28 Strong law of large numbers SLLN** *PSET 2 due Sep. 29.*
Review density of a distribution, Fubini foiled*, proof of SLLN, renewal process.

[*Reference: Durrett, Chapter 2.4, 2.5*]

(8) **Sep. 30: Introduction to large deviations and central limit theorem CLT**
Empirical distribution function, Glivenko-Cantelli theorem. Introduction to large deviations, moment generating function, characteristic function, Cramér's theorem, introduction to CLT, De Moivre-Laplace theorem.
[*Reference: Durrett, Chapter 2.6, 3.1*]

(9) **Oct. 05: CLT and weak convergence**
Introduction to Central Limit Theorem, De Moivre-Laplace theorem. Weak convergence, equivalent statement. Examples: Geometric distribution converges to exponential distribution, Binomial distribution converges to Poisson distribution, uniform measure on Cantor set.
[*Reference: Durrett, Chapter 3.1, 3.2*]

(10) **Oct. 07: Tightness and continuity theorem**
Helly's selection theorem, vague convergence, functional analysis viewpoint* (Banach-Alaoglu theorem, weak*-convergence), tightness of a family of distributions, characteristic function, uniqueness property, Poisson distribution, compound Poisson distribution, continuity theorem.
[*Reference: Durrett, Chapter 3.2, 3.3*]

Oct. 12: Indigenous People's day

(11) **Oct. 13: Monday schedule** *PSET 3 due Oct. 13.*
Proof of continuity theorem, Prokhorov metric. Proof of central limit theorem, construction of symmetric stable distribution, Berry-Esseen theorem. Lindeberg-Feller theorem, Kolmogorov's three series theorem.
[*Reference: Durrett, Chapter 3.4, 3.8*]

(12) **Oct. 14: Applications of continuity theorem and higher dimension CLT**
Proof of Kolmogorov's three series theorem, Poisson convergence, comparison with Lindeberg-Feller theorem, higher-dimension CLT, convergence in distribution in higher dimension, Gaussian vector. Infinitely divisible law.
[*Reference: Durrett, Chapter 3.4, 3.6, 3.9*]

(13) **Oct. 19: Towards martingales**
Infinitely divisible law, Lévy-Khinchin theorem*, introduction to Lévy processes*. Random walk, Hewitt-Savage 0-1 law. Conditional probability, conditional expectation.
[*Reference: Durrett, Thm 2.5.4, Chapter 3.9, 4.1*]

(14) **Oct. 21: Martingales**
Conditional expectation and its properties, conditional expectation for L^2 random variables. Discrete-time martingales, super-martingales, sub-martingales, filtration, adapted sequence.
[*Reference: Durrett, Chapter 4.1, 4.2*]

(15) **Oct. 26: Optional stopping theorem** *PSET 4 due Oct. 27.*
Conditional probability with density, asymmetric simple random walk, applying convex function to martingale, stopping time, Doob's decomposition of submartingale, predictable sequence, discrete stochastic integral, financial math interpretation, optional stopping theorem (version 1).
[*Reference: Durrett, Chapter 4.1, 4.2, 4.3.1*]

(16) **Oct. 28: P-a.s martingale convergence theorem and application**
Proof of optional stopping theorem, up-crossing inequality, P-a.s. submartingale convergence theorem. Galton-Watson chain, a.s extinction of the subcritical case ($m < 1$) and critical case ($m = 1$).

[*Reference: Durrett, Chapter 4.2, 4.3.4*]

(17) **Nov. 02: More applications and L^p convergence theorem** *Recording only*

Galton-Watson chain the supercritical case ($m > 1$): positive probability of non-extinction. Cox Ross Rubinstein model of call options. Doob's inequality, L^p-martingale convergence theorem ($p > 1$).

[*Reference: Durrett, Chapter 4.3.4, 4.4*]

(18) **Nov. 04: L^1 martingale convergence theorem** *Recording only*

L^p-martingale convergence theorem ($p > 1$). Uniform integrability, L^1 martingale convergence theorem. Applications of L^p convergence theorem: hitting probabilities of simple random walk, Galton-Watson chain with L^2 offspring distribution, Radon-Nikodym derivatives, Lévy's 0-1 law, alternative proof of Kolomogorov's 0-1 law.

[*Reference: Durrett, Chapter 4.4, 4.6*]

(19) **Nov. 09: More on optional stopping theorem and towards Markov chains**

Uniformly integrable stopped submartingales, optional stopping theorem (version 2). Applications to random walks: Wald identity. Strong Markov property of random walks, reflection principle, Ballot theorem, maximum of a simple random walk on $\mathbb{Z}$.

[*Reference: Durrett, Chapter 4.8, 4.9*]

Nov. 11: Veteran's day, no class *PSET 5 due Nov. 10.*

(20) **Nov. 16: Time-homogeneous Markov Chains with countable states**

Examples: Asymmetric random walk on $\mathbb{Z}$, random walks on graphs, Galton-Watson chain, Ehrenfest chain, card shuffling.

Transition matrix, stationary measure, stationary distribution, reversible measure, recurrent and transient states, irreducible MC, aperiodic states and MC.

[*Reference: Durrett, Chapter 5.1 - 5.6*]

(21) **Nov. 18: Construction of Markov Chains**

Stochastic kernels, semi-product of a probability measure and a stochastic kernel, construction of stochastic processes, Ionescu-Tulcea theorem, construction of time-homogeneous MC with general state space, shift operator, MC and martingales.

Reference: [Dembo, Chapter 6.1] [Ash, Real analysis and probability, Chapter 2.7.2]

Nov. 23: Thanks giving break

Nov. 25: Thanks giving break

(22) **Nov. 30: Introduction to Brownian motion** *PSET 6 due Dec. 01.*

Intuition: Isotropic and Markov continuous process. Definition: centered Gaussian process + independent stationary increments + a.s. continuity.

Construction of Brownian motion: Finite dimensional marginals, reminder of Gaussian vector, Kolmogorov extension theorem for uncountable family, continuous modification. Show isotropy from the definition.

Properties of Brownian motion, quadratic variation and relation to continuous martingales. Mandelbrot conjecture. Convergence of simple random walk, Donsker's theorem.

Reference: [Le Gall, Brownian Motion, Martingales, and Stochastic Calculus]

(23) **Dec. 02: Strong Markov property of BM and Dirichlet problems**

Brownian filtration, weak Markov property, stopping time, proof of strong Markov property, reflection principle, law of maximum over the interval $[0, t]$.

Dirichlet problem in discrete (recall from pset 5.10) and in continuum. Mean-value property of harmonic functions. Use strong Markov property to give a probabilistic expression of the solution

to Dirichlet problem, discussion about when does the solution exist, regularity of boundary of the domain. Other problems related to Dirichlet problem, Feynman-Kac formula.

(24) **Dec. 07: Large deviation principle of Brownian motion**

Recall large deviations of Gaussian distributions, contraction principle, LDP of the finite dimensional marginals of Brownian motion, Dirichlet energy, Schilder's theorem, Cameron-Martin theorem.

Reference: [Deuschel-Strook, Large deviations, Chapter 1.3]

(25) **Dec. 09: Schramm-Loewner evolution and Loewner energy** *Final project due Dec. 09.*

The last lecture is devoted to informal discussions on current research directions: Introduction to Schramm-Loewner evolution and Loewner energy.